Multiplicity free and finite multiplicity indecomposable representations of the algebra su(1,1)

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A classification is given for the multiplicity free indecomposable representations of the simple Lie algebra su(1,1), which are unbounded on both sides. Formulas have been obtained for the *matrix elements* of the generators of su(1,1) for all these representations. Representations of su(1,1) are analyzed which have the property that all their weight subspaces are infinite dimensional. Subrepresentations and representations on quotient spaces of this infinite multiplicity representations are considered and their relationship to the multiplicity free indecomposable representations is determined (both, unbounded on both sides, and bounded on one side). *Finite multiplicity indecomposable* representations are obtained from the infinite multiplicity representation for special values of the Casimir operator. A decomposition of the infinite multiplicity representation into a direct sum of multiplicity free representations and finite multiplicity indecomposable respresentations is given in two *different* ways. Finally, formulas for the matrix elements of su(1,1) are given for the finite multiplicity indecomposable representations.

I. INTRODUCTION

Indecomposable representations of (semi)simple Lie algebras have recently found their way into physics, and their application has attracted considerable attention. Armstrong, in 1971, investigated properties of radial matrix elements and introduced a tensor operator that turned out to transform like an infinite-dimensional indecomposable representation of su(1, 1).[1] Crubellier in his analysis of two-body radial matrix elements generalized Armstrong's results.[2] In 1976 Chacon, Levi, and Moshinsky gave a full interpretation of the results obtained by Armstrong and Crubellier.[3] Other applications of indecomposable representations of semisimple (as well as non-semisimple) Lie algebras have been suggested by Barut in 1973 as a possibility for describing composite particles with internal degrees of freedom.[4]

It appears thus that a classification and an analysis of the properties of indecomposable representations of the simple Lie algebra su(1, 1) (A_1) might be of value. The algebra su(1, 1) is chosen since it provides the simplest case, and, moreover, figures so far prominently in applications of indecomposable representations of (simple) Lie algebras in physics.

Results on the classification of multiplicity free (the weight subspaces have dimension one) indecomposable representations of the algebra su(1, 1) have been obtained by Gel'fand, Graev, and Vilenkin.[5] This classification has been completed by Phillips in his thesis, but has remained unpublished.[6]

In this article we will give the *classification* of all infinite-dimensional multiplicity free indecomposable representations of the algebra su(1, 1), which are unbounded on both sides (for a classification of the infinite-dimensional multiplicity free indecomposable representations that are bounded on the side, see, for example, Ref. 7). The classification given here has been carried

out independently, and is achieved by methods which differ from those of other authors who have also investigated indecomposable representations of su(1, 1). The classification is given in Sec. II of this article.

Apart from the classification of the (multiplicity-free) indecomposable representations of su(1, 1), *formulas* are obtained for the *matrix elements* of the generators of su(1, 1) in these representations. They are also given in Sec. II of this article.

In Sec. III, a representation of su(1, 1) is analyzed, which has the property that all its weight subspaces are infinite dimensional. To our knowledge, no such analysis has been carried out before, and ours is not complete. Some of the subrepresentations of this infinite multiplicity representation, as well as representations induced on its quotient spaces, are shown to be equivalent to multiplicity free (infinite-dimensional) representations which are bounded on one side, or to the multiplicity free representations discussed in Sec. II of this article. From this infinite multiplicity representation *finite multiplicity indecomposable* representations are derived (the weight subspaces have finite dimension > 1) for special values of the Casimir operator (Sec. III). These representations have unusual properties, one of which might be described as "leakage" into an invariant subspace (see Fig. 11). They do also reflect properties of irreversible processes of a more general nature than the multiplicity free indecomposable representations of Sec. II. A decomposition of the infinite multiplicity representation is achieved in two *different* ways as direct sum of multiplicity free representations, all of which are either bounded above or below, and finite multiplicity indecomposable representations. An example is given of an infinite-dimensional, multiplicity free representation, unbounded on both sides, which is induced by the infinite multiplicity representation on one of its invariant subspaces. Again, as for all representations discussed in this article, the *matrix elements* of the
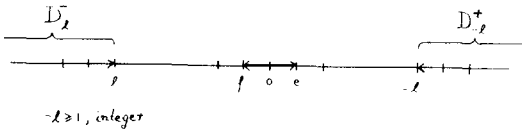
FIG. 1. Structure of representation $\pi^{l,h}$ for $l \leqslant 1$, l integer. The nonzero roots are indicated at the origin of the diagram. The other arrows indicate the points which cannot be crossed in that direction. Invariant subspaces which carry subrepresentations are indicated. Quotient representations are graphically described by simply deleting those invariant subspaces with respect to which the quotient space is formed. For half-integer values of l the structure is similar and no separate figure is given.

FIG. 2. Structure of representation $\pi^{l,h}$ for $l \geqslant 0$, l integer. See Fig. 1 caption.

generators of su(1, 1) are given for the finite multiplicity indecomposable representation as well as for all other representations derived from the infinite multiplicity representation.

II. MULTIPLICITY FREE SU (1,1) REPRESENTATIONS

The purpose of this section is to give a complete discussion of representations of the simple Lie algebra su(1, 1) which are multiplicity free, that is, of all representations of su(1, 1) whose weight subspaces have the dimensionality one.

The Lie products of the algebra su(1, 1) of the group SU(1, 1) are given by the commutation relations

$$[h, e] = e, \quad [h, f] = -f, \quad [e, f] = -h. \tag{1}$$

The Casimir operator (invariant) is

$$C_2 = ef + fe - h^2. \tag{2}$$

A. Representations of the principal nonunitary series of su (1,1)

Bergmann[8] constructed the principal unitary series of representations of the group SU(1, 1). The elements of this group act on a Hilbert space $L^2(T)$ of square integrable functions defined on the unit circle T in $\mathbb{C}$. The representations are labelled by two numbers h and l, where l is pure imaginary and h is equal to 0 or $\frac{1}{2}$. They act in $L^2(T)$ by means of the formula

$$U^{h,l}(g)f[\exp(i\theta)]$$

$$= \left(\frac{\exp(i\theta)\overline{\beta} + \overline{\alpha}}{|\exp(i\theta)\overline{\beta} + \overline{\alpha}|} \right)^{2h} |\exp(i\theta)\overline{\beta} + \overline{\alpha}|^{2l} f\left(\frac{\exp(i\theta)\alpha + \beta}{\exp(i\theta)\overline{\beta} + \overline{\alpha}} \right). \tag{3}$$

If Eq. (3) is analytically extended to arbitrary complex values of the parameter l, the principal nonunitary series of representations of SU(1, 1) is obtained. In the following, Eq. (3) will be considered for *arbitrary* complex l. On the other hand, the following discussion will be restricted to single valued representations of SU(1, 1). There exist many valued representations of SU(1, 1), which are single valued representations of its universal covering group SU(1, 1)'. The representations of the principal nonunitary series of SU(1, 1)' and its generators are discussed in Ref. 9.

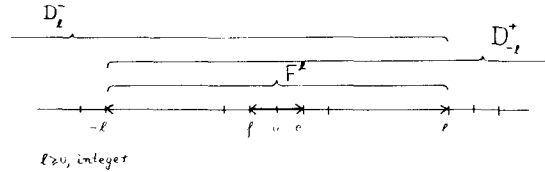In order to discuss conveniently the principal nonunitary series of representations in terms of the Lie

algebra su(1, 1), the following notation is introduced. As orthonormal basis for the space $L^2(T)$ the set of functions $\exp(-ik\theta)$, $k = 0, \pm 1, \pm 2, \cdots$, is chosen. Then, if $h = 0$, $\exp(-ik\theta) = |k\rangle$ and if $h = \frac{1}{2}$, $\exp(-ik\theta) = |k + \frac{1}{2}\rangle$.

The action of the differential operators h, e, f on the basis elements $|m\rangle$ is given by the formulas (the differential form of these operators is, for example, given in Ref. 10)

$$\rho(h)|m\rangle = m|m\rangle,$$

$$\rho(e)|m\rangle = \frac{1}{2}(m - l)|m + 1\rangle, \tag{4}$$

$$\rho(f)|m\rangle = -\frac{1}{2}(m + l)|m - 1\rangle.$$

These representations of the principal nonunitary series are denoted $\pi^{l,h}$. If $h = 0$, then $\pi^{l,h}$ is irreducible if and only if l is not an integer. If $h = \frac{1}{2}$, then $\pi^{l,h}$ is irreducible if and only if l is not half-integer.

In the following let l be integer if $h = 0$ and half-integer if $h = \frac{1}{2}$. Then for $l \leqslant -1$ the representation $\pi^{l,h}$ has two invariant subspaces on which the representations D_{-l}^+ and D_l^- of the discrete series are realized. The quotient representation $\pi^{l,h}/D_{-l}^+ \oplus D_l^-$ is a finite-dimensional representation with highest weight $-l - 1$. This representation is denoted by F^{-l-1} (Fig. 1).

If $l \geqslant 0$, then the representation $\pi^{l,h}$ has three invariant subspaces. One of the invariant subspaces is finite-dimensional, with highest weight l. This subspace carries a finite-dimensional representation, denoted by F^l. The other two invariant subspaces are infinite-dimensional, with highest weight l and lowest weight $-l$, respectively. (See *Note added in proof* at end of article.) On both of these invariant subspaces an indecomposable representation of su(1,1) is realized. Both contain a finite-dimensional invariant subspace with highest weight l and lowest $-l$ respectively, on which the representation F^l is realized. The infinite-dimensional orthogonal complements, however, are not invariant subspaces. The two indecomposable representations are denoted by D_l^- and D_{-l}^+ (note that here $l \geqslant 0$). It holds $D_{-l}^+/F^l \sim D_{l+1}^+$ and $D_l^-/F^l \sim D_{-l-1}^-$ (Fig. 2).

If $l = -\frac{1}{2}$, then the representation $\pi^{-1/2,h}$ decomposes into the direct sum of the two irreducible unitary representations $D_{1/2}^+$ and $D_{-1/2}^-$ (Fig. 3).

If l is neither integer or half-integer, then we obtain

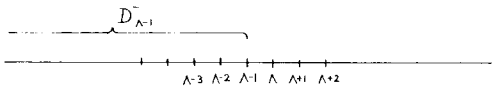FIG. 3. Structure of representation $\pi^{-1/2,h}$.

FIG. 4. Structure of representation σ_Λ, $\Lambda \neq n/2$.

infinite-dimensional irreducible representations which are neither bounded below or above.

B. Representations of su (1, 1) with highest (lowest) weight

The next type of representations of su(1, 1) to be considered are those which are bounded on one side, i.e., representations which have either a highest or a lowest weight.

Representations with highest weight (see prededing subsection A) are the representations D_l^-. For $l \leq -\frac{1}{2}$ they are irreducible; for $l > 0$ they are indecomposable. Representations with lowest weight are the representations D_l^+. For $l \geq \frac{1}{2}$ they are irreducible and for $l \leq 0$ they are indecomposable. The reducible representations D_l^- and D_{-l}^+ are not representations with highest or lowest weight as defined in Ref. 7. See *Note added in proof*.

If $x_0, x_1, x_2, \cdots$ denotes a basis of an infinite-dimensional vector space V and if Λ is a complex number, then the following relations hold for inifinite-dimensional representations with a highest weight (similar relations are obtained for infinite-dimensional representations with a lowest weight) of the type of Ref. 7:

$$\rho_\Lambda(h)x_i = (\Lambda - i)x_i, \quad i = 0, 1, 2, \cdots,$$

$$\rho_\Lambda(f)x_i = x_i + 1, \quad i = 0, 1, 2, \cdots,$$

$$\rho_\Lambda(e)x_0 = 0, \tag{5}$$

$$\rho_\Lambda(e)x_i = -i[\Lambda - \tfrac{1}{2}(i-1)]x_{i-1}, \quad i = 1, 2, 3, \cdots.$$

These representations are called elementary representations and are given the symbol d_Λ. For more details the reader is referred to Ref. 7.

C. New indecomposable multiplicity free representations of su (1, 1)

Let π denote a representation of su(1, 1) which is multiplicity free. Let $|\Lambda\rangle$ denote one of the eigenvectors of h such that $\pi(h)|\Lambda\rangle = \Lambda|\Lambda\rangle$. The operators $\pi(e^m)$ and $\pi(f^n)$, $m, n = 1, 2, 3, \cdots$ act on the vector $|\Lambda\rangle$. A sequence of vectors is obtained:

$$\cdots, \pi(f^2)|\Lambda\rangle, \pi(f)|\Lambda\rangle, |\Lambda\rangle, \pi(e)|\Lambda\rangle, \pi(e^2)|\Lambda\rangle, \cdots. \tag{6}$$

All vectors of this sequence are eigenvectors of $\pi(h)$ corresponding to the eigenvalues

$$\cdots, \Lambda - 2, \Lambda - 1, \Lambda, \Lambda + 1, \Lambda + 2, \cdots.$$

We denote the vectors of the sequence (6) by the symbols

$$\cdots, |\Lambda - 2\rangle, |\Lambda - 1\rangle, |\Lambda\rangle, |\Lambda + 1\rangle, |\Lambda + 2\rangle, \cdots. \tag{7}$$

In the following we proceed to analyze in general the types of representations that can be realized on the complex linear space for which (6) constitutes a basis. Some of the types of representations that will be found

represent nothing new. A resumé of these has been given in subsections A and B. Other types of representations will, however, be found which we believe to be unknown hitherto.[6]

Case A: None of the vectors $\pi(f^n)|\Lambda + m\rangle$, $n, m = 1, 2, 3, \cdots$, is equal to zero.

(a) No vector of the sequence $\pi(f)|\Lambda + m\rangle$, $m = 1, 2, 3, \cdots$, and of the sequence $\pi(e)|\Lambda - n\rangle$, $n = 1, 2, 3, \cdots$, is equal to zero.

Since the representations discussed are multiplicity free, it holds that $\pi(f)|\Lambda + m\rangle \sim |\Lambda + m - 1\rangle$ and $\pi(e)|\Lambda - n\rangle \sim |\Lambda - n + 1\rangle$. Therefore, the representation π is necessarily irreducible. Since every irreducible representation of a semisimple Lie algebra can be obtained from some representation of its principal nonunitary series (see, for example, Theorem 8.10 in Ref. 11), it follows that the set of representations π corresponds to the set of irreducible representations of the principle nonunitary series $\pi^{l,h}$, with l not an integer for $h = 0$ and l not a half-integer for $h = \frac{1}{2}$.

(b) For some integer m, $m > 0$, it holds that $\pi(e) \times |\Lambda - m\rangle = 0$. The sequence (7) is relabelled by taking the vector $|\Lambda - m + 1\rangle$ as the vector $|\Lambda\rangle$ of the sequence (7). In the new notation, then, $\pi(e)|\Lambda - 1\rangle = 0$ holds. Utilizing this equation, it can be proven, by means of induction, that the following equations hold:

$$\pi(e)|\Lambda - n\rangle = (n-1)\left(\frac{n}{2} - \Lambda\right)|\Lambda - n + 1\rangle, \quad n > 0, \tag{8}$$

$$\pi(f)|\Lambda + n'\rangle = n'\left(\frac{n'-1}{2} + \Lambda\right)|\Lambda + n' - 1\rangle, \quad n' > 0, \tag{9}$$

n, n' integers. A simple computation proves that the formulas (8) and (9) define representations of the algebra su(1, 1), i.e., that the commutation relations (1) are satisfied.

In Eq. (8) the coefficient on the right side becomes zero, for arbitrary Λ, if $n = 1$. This coefficient becomes zero also for $\Lambda = n/2$. The coefficient on the right side of Eq. (9) becomes zero for $\Lambda = -(n'-1)/2$.

In the case of $\Lambda = -(n'-1)/2$ the representation has two invariant subspaces with basis elements

$$|\Lambda - 1\rangle, |\Lambda - 2\rangle, |\Lambda - 3\rangle, \cdots$$

and

$$|-\Lambda + 1\rangle, |-\Lambda + 2\rangle, |-\Lambda + 3\rangle, \cdots,$$

respectively. Thus, this representation does not satisfy the condition of Case A which is considered at present. In fact, this representation is equivalent to one of the reducible representations of the principal nonunitary series of representations $\pi^{l,h}$.

Let $\Lambda \neq -(n'-1)/2$, $n' > 0$, integer. If Λ is complex and $\Lambda \neq n/2$, $n > 0$, integer, then the formulas (6)—(9) define a representation σ_Λ [in the space with the basis Eq. (6)] which has only one invariant subspace with basis elements

$$|\Lambda - 1\rangle, |\Lambda - 2\rangle, |\Lambda - 3\rangle, \cdots.$$

The subrepresentation of su(1, 1) on this subspace is a representation with highest weight $\Lambda - 1$. The represen-
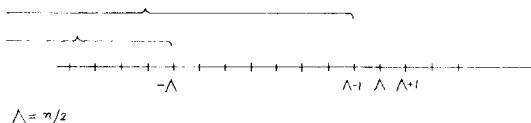
$\Lambda = n/2$

FIG. 5. Structure of representation σ_Λ, $\Lambda = n/2$.

tation which is carried by the quotient space is a representation with lowest weight Λ. The representation σ_Λ is an indecomposable one, i.e., it cannot be decomposed into a direct sum of the two representations with highest weight $\Lambda - 1$ and lowest weight Λ (Fig. 4).

The value of the Casimir operator for the representation σ_Λ is $\Lambda(1 - \Lambda)$.

Let $\Lambda = n/2$, $n > 0$, integer. Equations (6)—(9) give now, on the space with basis (6), a representation σ_Λ with two invariant subspaces. One of the invariant subspaces has the basis

$$|-\Lambda\rangle, \ |-\Lambda - 1\rangle, \ |-\Lambda - 2\rangle, \cdots.$$

This subspace is irreducible. The other invariant subspace has the basis

$$|\Lambda - 1\rangle, \ |\Lambda - 2\rangle, \ |\Lambda - 3\rangle, \cdots.$$

This subspace is reducible and has the first invariant subspace as a subspace. The representation which is realized on this second invariant subspace is indecomposable (Fig. 5).

The eigenvalue of the Casimir operator in the representation σ_Λ is again $\Lambda(1 - \Lambda)$.

It may be worthwhile to point out that from the representation σ_Λ, $\Lambda = n/2$, $n = 1, 2, 3, \cdots$, the same representations can be obtained, defined on its invariant subspaces and quotient spaces, as from the representations $\pi^{l,h}$, $l = \Lambda - 1$, and $\pi^{l,h}$, $l = -\Lambda$, even though their structure is distinct.

Case B: None of the $\pi(e^n)|\Lambda - m\rangle$, $n, m = 1, 2, 3, \cdots$, is equal to zero.

(a) None of the $\pi(e)|\Lambda - m\rangle$, $m = 1, 2, 3, \cdots$, and none of the $\pi(f)|\Lambda + n\rangle$, $n = 1, 2, 3, \cdots$, is equal to zero.

This yields the same set of representations as case A(a).

(b) For some m, $m > 0$, it holds that $\pi(f)|\Lambda + m\rangle = 0$. The sequence (6) is relabelled by taking $|\Lambda + m - 1\rangle$ as the vector $|\Lambda\rangle$. In the new notation it then holds that $\pi(f)|\Lambda + 1\rangle = 0$. With the help of this equation and using induction, it can be proven that the following equations hold:

$$\pi(e)|\Lambda - n\rangle = n((n - 1)/2 - \Lambda)|\Lambda - n + 1\rangle, \quad n > 0, \quad (10)$$

$$\pi(f)|\Lambda + n'\rangle = (n' - 1)(n'/2 + \Lambda)|\Lambda + n' - 1\rangle, \quad n' > 0, \quad (11)$$
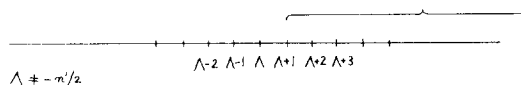
n, n' integers. A direct evaluation shows that the formulas (6), (7), (10), and (11) satisfy the commutation relations (1).

The coefficient on the right side of Eq. (10) is equal to zero if $\Lambda = (n - 1)/2$. The coefficient on the right side of Eq. (11) is equal to zero if $n' = 1$, for arbitrary Λ, and for $\Lambda = -n'/2$.

If $\Lambda = (n - 1)/2$, then the representation has two invariant subspaces. The first one has as basis elements

$$|-\Lambda - 1\rangle, \ |-\Lambda - 2\rangle, \ |-\Lambda - 3\rangle, \cdots,$$

while the second one has as basis elements

$$|\Lambda + 1\rangle, \ |\Lambda + 2\rangle, \ |\Lambda + 3\rangle, \cdots.$$

This representation does not satisfy the condition for Case B. It corresponds to one of the reducible representations $\pi^{l,h}$.

Now consider $\Lambda \neq (n - 1)/2$, $n > 0$, integer. If Λ is complex, $\Lambda \neq -n'/2$, $n' > 0$, integer, then a representation τ_Λ with one invariant subspace is obtained. The basis elements of the invariant subspace are given as

$$|\Lambda + 1\rangle, \ |\Lambda + 2\rangle, \ |\Lambda + 3\rangle, \cdots.$$

This subspace carries an irreducible representation with lowest weight $\Lambda + 1$. The representation induced on the quotient space with respect to the invariant subspace is a representation with highest weight Λ. The representation τ_Λ is indecomposable (Fig. 6).

The value of the Casimir operator for the representation τ_Λ is $\Lambda(1 - \Lambda)$.

Now consider $\Lambda = -n'/2$, $n' > 0$, integer. Then the formulas (6), (7), (10), (11) define a representation τ_Λ with two invariant subspaces. One of the invariant subspaces has the basis vectors

$$|\Lambda + 1\rangle, \ |\Lambda + 2\rangle, \ |\Lambda + 3\rangle, \cdots$$

and is irreducible. The other invariant subspace has the basis

$$|-\Lambda\rangle, \ |-\Lambda + 1\rangle, \ |-\Lambda + 2\rangle, \cdots$$

and is reducible, with the first invariant subspace as an invariant subspace. In fact this second invariant subspace is indecomposable (Fig. 7).

The value of the Casimir operator on τ_Λ is $\Lambda(1 - \Lambda)$.

The representations τ_Λ, $\Lambda \neq (n - 1)/2$, $n > 0$, integer, and $\sigma_{\Lambda-1}$ yield the same set of irreducible representations, defined on subspaces of their quotient spaces.
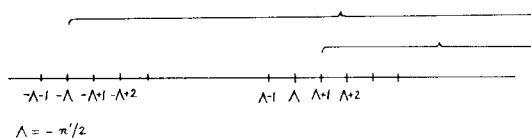

$\Lambda \neq -n'/2$

FIG. 6. Structure of representation τ_Λ, $\Lambda \neq -n'/2$.

$\Lambda = -n'/2$

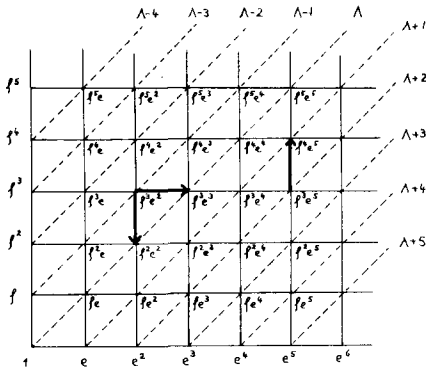FIG. 7. Structure of representation τ_Λ, $\Lambda = -n'/2$.

FIG. 8. Graphical representation of basis of vector space V $= \Omega/I$. The arrows indicate the action of the shift operators on the basis elements. Basis elements along each dashed line correspond to the same weight.

III. REPRESENTATIONS OF SU (1, 1) ON THE SPACE OF ITS UNIVERSAL ENVELOPING ALGEBRA

Let Ω denote the universal enveloping algebra of the complexification of su(1, 1). In an earlier article the universal enveloping algebra was used to construct representations with a highest (lowest) weight.[7] Here the universal algebra Ω is used to construct other types of representations of su(1, 1), namely representations whose weight subspaces are infinite-dimensional. Properties of these types of representations of su(1, 1) are analyzed below and their intimate relationship to the multiplicity-free representations of Sec. II is demonstrated.

In the space of the enveloping algebra Ω the left multiplication by elements of the algebra su(1, 1) defines a representation denoted by π. Consider in Ω the left ideal I that is generated by the element $h - \Lambda$ of Ω, where Λ is some fixed complex number. Thus the elements $x(h - \Lambda)$, $x \in \Omega$, form the ideal I. Therefore, π can be reduced to a representation on the quotient space $V = \Omega/I$. This representation is denoted by ρ_Λ.

In the following a basis is chosen in the space V and the representation ρ_Λ is determined with respect to this basis. A basis for V can be selected in different ways. The elements

$$1, f^n e^m, \quad n = 0, 1, 2, \cdots, m = 0, 1, 2, \cdots,$$

excluding the case $n = m = 0$, can be chosen as a basis for V. It is then easy to evaluate that $\rho_\Lambda(h), \rho_\Lambda(f), \rho_\Lambda(e)$ act on those basis elements as

$$\rho_\Lambda(h)1 = \Lambda 1, \quad \rho_\Lambda(e)1 = e, \quad \rho_\Lambda(f)1 = f, \tag{12}$$

$$\rho_\Lambda(h)f^n e^m = (\Lambda - n + m)f^n e^m, \tag{13}$$

$$\rho_\Lambda(f)f^n e^m = f^{n+1} e^m, \tag{14}$$

$$\rho_\Lambda(e)f^n e^m = f^n e^{m+1} - n(\Lambda + m - (n-1)/2)f^{n-1} e^m, \tag{15}$$

$$\rho_\Lambda(C)f^n e^m = 2f^{n+1} e^{m+1} - (\Lambda + m)(\Lambda + m + 1)f^n e^m.$$

The basis elements of V are shown in Fig. 8. The dotted lines in this figure connect elements that correspond to the same weight. The operator $\rho_\Lambda(f)$, acting on a basis element, causes a vertical upwards shift in Fig. 8, while the operator $\rho_\Lambda(e)$ acting on a basis element

causes a simultaneous shift veritically downwards and to the right. Therefore, as is seen from Fig. 8, the operators $\rho_\Lambda(f)$, $\rho_\Lambda(e)$, acting on the basis elements cause upward shifts, downward shifts, and shifts to the right only. This implies that for every fixed positive integer s the subspace V_s of V with basis elements

$$f^n e^m, \quad n = 0, 1, 2, \cdots, \quad m = s, s + 1, s + 2, \cdots,$$

is invariant under ρ_Λ. The corresponding subrepresentation is denoted ρ_Λ^s. The representation on the quotient space V/V_s is denoted by ${}^s\rho_\Lambda$. The representation ${}^1\rho_\Lambda$ is then the so-called elementary representation d_Λ with highest weight Λ (for the definition see Ref. 7; the d_Λ of this article is, in fact, the $d_{2\Lambda}$ of Ref. 7). In general, the elementary representation $d_{\Lambda+s}$ with highest weight $\Lambda + s$ is realized on the quotient space V_s/V_{s+1}. Thus, the representation ρ_Λ consists of the elementary representations

$$d_\Lambda, d_{\Lambda+1}, d_{\Lambda+2}, \cdots. \tag{16}$$

It is now possible that for some of the representations of the sequence (16) Casimir operator C_2 has the same eigenvalue. This is the case for two representations $d_{\Lambda'}$ and $d_{\Lambda''}$ if $\Lambda'(\Lambda' + 1) = \Lambda''(\Lambda'' + 1)$, $\Lambda' \neq \Lambda''$. The only solution to this equation is $\Lambda'' = -\Lambda - 1$. Therefore, representations of (16) can have the same eigenvalue of C_2 if Λ is equal to $(-m)/2$, $m > 1$, integer. Then every pair of representations

$$(d_{-m/2}, d_{m/2-1}), (d_{-m/2+1}, d_{m/2-2}), \cdots \tag{17}$$

corresponds to the same eigenvalue of C_2.

Let us now consider the case that all of the representations of (16) have a different eigenvalue of C_2, i.e., that $\Lambda \neq -n/2$, $n > 1$, integer. In this case the representation ${}^s\rho_\Lambda$ can be decomposed into a direct sum of representations with highest weights. This decomposition is achieved by means of the following extremal vectors (a vector y is called extremal if $\rho(e)y = 0$):

$$x^0 = 1 + \sum_{i=1}^{s-1}\left\{\prod_{j=1}^{i} j[\Lambda + \tfrac{1}{2}(j+1)]\right\}^{-1} f^i e^i, \tag{18}$$

$$x^m = e^m + \sum_{i=1}^{s-m-1}\left\{\prod_{j=1}^{i} j[\Lambda + m + \tfrac{1}{2}(j+1)]\right\}^{-1} f^i e^{m+i},$$

$$m = 1, 2, \ldots, s - 1. \tag{19}$$

The s subspaces obtained from these s extremal vectors, with the basis elements
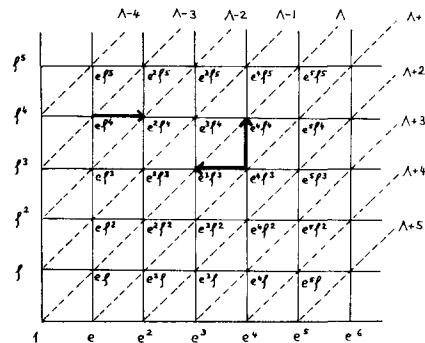

FIG. 9. Same as Fig. 8, except that a different basis has been chosen for the vector space V. The arrows indicate again the action of the shift operators on the (new) basis elements.
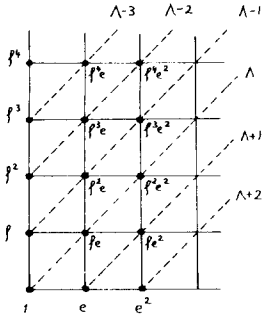
FIG. 10. Representation $^3\rho_\Lambda$ on quotient space V/V_3. The dots correspond to representative basis vectors f^n, $f^n e$, $f^n e^2$, $n = 0$, 1, 2, $\cdots$, of the quotient space. For $\Lambda \neq -3/2$ this representation can be decomposed into the direct sum of the elementary representations d_Λ, $d_{\Lambda+1}$, $d_{\Lambda+2}$ with basis x_n^0, x_n^1, and x_n^2, respectively.

$$x^0, fx^0, f^2 x^0, \cdots \qquad (20)$$

$$x^m, fx^m, f^2 x^m, \cdots, \quad m = 1, 2, \ldots, s - 1, \qquad (21)$$

are invariant under $^s\rho_\Lambda$ (in the space V/V_s of the representation $^s\rho_\Lambda$ it holds $f^i e^n = 0$ if $n \geq s$), as can be verified by direct evaluation. In fact in each of the s distinct bases, (20) and (21), the representation obtained is given by Eq. (5). Thus, on the subspaces defined by (20) and (21) the representations $d_\Lambda, d_{\Lambda+1}, \ldots, d_{\Lambda+s-1}$ are realized, with Eqs. (18) and (19) as the vectors corresponding to their highest weight.

Next we consider the case of $\Lambda = -n/2$, $n > 1$, integer. In this case pairs of representations exist, Eq. (17), for which the Casimir operator C_2 has the same eigenvalue. In this case the representation $^s\rho_\Lambda$ can be decomposed into a direct sum among which those of the representation $d_\Lambda, d_{\Lambda+1}, d_{\Lambda+2}, \ldots, d_{\Lambda+s-1}$ occur for which C_2 has different eigenvalues. For these representations the vector corresponding to their highest weight is obtained from Eqs. (18) and (19). Representations for which the Casimir operator has the same eigenvalue occur in pairs, as was shown before in (17). These pairs then occur as direct summands in the decomposition of the representation $^s\rho_\Lambda$, while each pair itself cannot be decomposed into a direct sum. For these undecomposable pairs of representations Eqs. (18) and (19) fail to determine an extremal vector which belongs to the highest weight. This is best demonstrated by means of an example. Let $\Lambda = -3/2$ and consider the representation $^3\rho_\Lambda$. This representation consists of the representations $d_{-3/2}$, $d_{-1/2}$, $d_{1/2}$. The Casimir operator has the same eigenvalue for the representations $d_{-3/2}$, $d_{1/2}$. The representation $d_{1/2}$ acts invariantly on the subspace that is spanned by the basis elements $f^m e^2$, $m = 0, 1, 2, \cdots$. Its extremal vector belonging to its highest weight is obtained from Eq. (19) and is $x^2 = e^2$. The extremal vector belonging to the highest weight of the representation $d_{-1/2}$ is again obtained from Eq. (19) and is $x^1 = e + 2fe^2$. The extremal vector belonging to the highest weight of the representation $d_{-3/2}$, if it exists in $^3\rho_\Lambda$, $\Lambda = -3/2$, is to be of the form $x^0 = 1 + a_1 fe + a_2 f^2 e^2$. This vector is to satisfy $^3\rho_\Lambda(e)x^0 = 0$. This implies

$$e + \tfrac{1}{2} a_1 e + a_1 fe^2 = 0,$$

a condition which cannot be satisfied. If, however, the vector $\overline{x}^0 = 1 - 2fe + f^2 e^2$ is chosen, then it follows that $^3\rho_\Lambda(e)\overline{x}^0 = -2fe^2$ is a vector in $d_{1/2}$. It follows therefore that the subrepresentation generated by $\overline{x}^0$ is really indecomposable, as it is impossible to construct an ex-

tremal vector for the highest weight of $d_{-3/2}$, while a vector $\overline{x}^0$ of weight $\Lambda = -3/2$ can be found which has components in $d_{-3/2}$ and $d_{1/2}$ only. The representation $^3\rho_\Lambda$, $\Lambda = -3/2$, can thus be decomposed into the direct sum of the representation $d_{-1/2}$ and an indecomposable representation d which consists of the representations $d_{-3/2}$ and $d_{1/2}$. The representation $d_{1/2}$ is a subrepresentation of d, while $d_{-3/2}$ is realized on the quotient space of d with respect to $d_{1/2}$.

In the general case of paried representations analogous properties hold. Whenever in Eq. (18) or (19) one of the expressions $[\Lambda + m + \tfrac{1}{2}(j + 1)]$ vanishes, it is to be replaced by a constant different from zero. Then the resultant vector generates a basis for an undecomposable representation consisting of the pair.

In the following the indecomposable representation d of $^3\rho_\Lambda$, $\Lambda = -3/2$, is discussed explicitly (see Fig. 10). As can be verified by direct computation the following relations hold:

$$\rho(h)x_n^m = (\Lambda + m - n)x_n^m,$$

$$\rho(f)x_n^m = x_{n+1}^m,$$

$$\rho(e)x_n^m = -n\left(\Lambda + m - \frac{n-1}{2}\right)x_{n-1}^m, \qquad (22)$$

$$\rho(C)x_n^m = -(\Lambda + m)(\Lambda + m + 1)x_n^m,$$

with $f^{i+n} e^s - 0$, $x_n^m = f^n x^m$, $m = 0, 1, 2, \ldots, s - 1$ and $n = 0, 1, 2, \cdots$. These equations hold as long as none of the coefficients in Eqs. (18) and (19) becomes infinite. Then a direct sum of elementary representations d_Λ, $d_{\Lambda+1}, \ldots, d_{\Lambda+s-1}$, is obtained. For the representation $^3\rho_\Lambda$, $\Lambda = -3/2$, however, not all of Eqs. (18) and (19) hold. In fact x^0 had to be changed to $\overline{x}^0$. For the new basis elements $x_n^1 = f^n x^1$ and $x_n^2 = f^n x^2$, $n = 0, 1, 2, \cdots$, Eqs. (19) hold. Thus, both x_n^1, $n = 0, 1, 2, \cdots$, and x_n^2, $n = 0, 1, 2, \cdots$, form a basis for a (multiplicity-free) elementary representation, namely $d_{1/2}$ and $d_{-1/2}$, induced by $^3\rho_\Lambda$ on the respective subspaces. If the action of the elements of the algebra on the basis elements $\overline{x}_n^0$, $n = 0, 1, 2, \cdots$, is studied, it turns out that they do not form an invariant subspace. The basis elements $\overline{x}_n^0 = f^n \overline{x}^0$ are connected, through the action of the algebra, with the basis elements x_n^2, $n = 0, 1, 2, \cdots$. In fact, it can be shown that the following relations hold $(\overline{x}^0 = 1 - 2fe + f^2 e^2$, $x^2 = e^2)$:
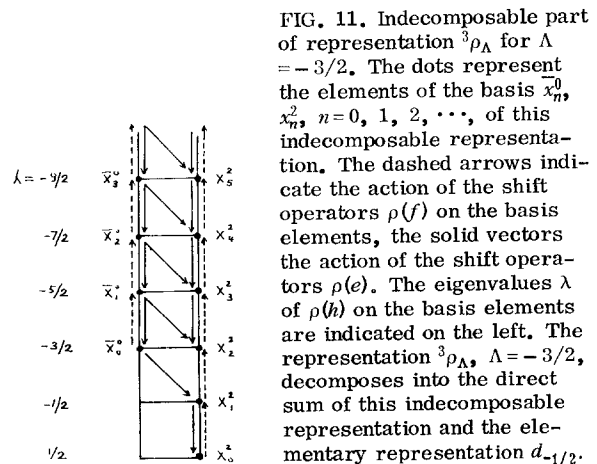
FIG. 11. Indecomposable part of representation $^3\rho_\Lambda$ for $\Lambda = -3/2$. The dots represent the elements of the basis $\overline{x}_n^0$, x_n^2, $n = 0$, 1, 2, $\cdots$, of this indecomposable representation. The dashed arrows indicate the action of the shift operators $\rho(f)$ on the basis elements, the solid vectors the action of the shift operators $\rho(e)$. The eigenvalues λ of $\rho(h)$ on the basis elements are indicated on the left. The representation $^3\rho_\Lambda$, $\Lambda = -3/2$, decomposes into the direct sum of this indecomposable representation and the elementary representation $d_{-1/2}$.

$$\rho(h)\bar{x}_n^0 = -(\tfrac{3}{2} + n)\bar{x}_n^0, \quad \rho(h)x_n^2 = (\tfrac{1}{2} - n)x_n^2,$$

$$\rho(f)\bar{x}_n^0 = \bar{x}_{n+1}^0, \quad \rho(f)x_n^2 = x_{n+1}^2,$$

$$\rho(e)\bar{x}_n^0 = \tfrac{1}{2}n(n+2)\bar{x}_{n-1}^0 - 2x_{n+1}^2, \qquad (23)$$

$$\rho(e)x_n^2 = \tfrac{1}{2}n(n-2)x_{n-1}^2,$$

$$\rho(C)\bar{x}_n^0 = -\tfrac{3}{4}\bar{x}_n^0 - 4x_{n+2}^2, \quad \rho(C)x_n^2 = -\tfrac{3}{4}x_n^2.$$

Thus Eqs. (23) define an indecomposable representation d (see Fig. 11). Ordering the basis elements in the manner $\bar{x}_0^0, x_0^2, \bar{x}_1^0, x_1^2, \bar{x}_2^0, x_2^2, \cdots$, the matrices $\rho(h), \rho(e), \rho(f)$, and $\rho(C)$ take on the following form for this indecomposable representation d:

$$\rho(h) = \begin{bmatrix} -\tfrac{3}{2} & & & & & & & \\ & \tfrac{1}{2} & & & & & & \\ & & -\tfrac{5}{2} & & & & & \\ & & & -\tfrac{1}{2} & & & & \\ & & & & -\tfrac{7}{2} & & & \\ & & & & & -\tfrac{3}{2} & & \\ & & & & & & -\tfrac{9}{2} & \\ & & & & & & & -\tfrac{5}{2} \\ & & & & & & & & -\tfrac{11}{2} \\ & & & & & & & & & -\tfrac{7}{2} \end{bmatrix}$$

$$\rho(f) = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 & 0 \end{bmatrix},$$

$$\rho(e) = \begin{bmatrix} 0 & \tfrac{1}{2} & & & & & \\ & 0 & -\tfrac{1}{2} & & & & \\ & & 0 & 4 & & & \\ -2 & & 0 & & 0 & & \\ & & & 0 & \tfrac{15}{2} & & \\ & -2 & & 0 & & \tfrac{3}{2} & \\ & & & & 0 & & 12 \\ & & & -2 & & 0 & & 4 \\ & & & & & & 0 & \\ & & & & & -2 & & 0 \end{bmatrix},$$

$$\rho(C) = \begin{bmatrix} -\tfrac{3}{4} & & & & & & & & \\ & -\tfrac{3}{4} & & & & & & & \\ & & -\tfrac{3}{4} & & & & & & \\ & & & -\tfrac{3}{4} & & & & & \\ & & & & -\tfrac{3}{4} & & & & \\ -4 & & & & & -\tfrac{3}{4} & & & \\ & & & & & & -\tfrac{3}{4} & & \\ & -4 & & & & & & -\tfrac{3}{4} & \\ & & & & & & & & -\tfrac{3}{4} \\ & & -4 & & & & & & & -\tfrac{3}{4} \end{bmatrix}.$$

Let us now consider the representation ρ_Λ. The representations $d_\Lambda, d_{\Lambda+1}, d_{\Lambda+2}, \cdots$ are coupled in ρ_Λ in the following manner:

$$\begin{bmatrix} d_\Lambda & & & \\ * & d_{\Lambda+1} & & 0 \\ * & * & d_{\Lambda+2} & \\ * & * & * & \ddots \\ & & & & \ddots \\ & & & & & \circ \end{bmatrix}$$

The representation ρ_Λ has no subrepresentation with highest weight. If it did have, then there would exist a vector of the form

$$x^0 = \sum_{i=0}^{k} a_i f^{m+i} e^{n+i},$$

such that $\rho_\Lambda(e)x^0 = 0$ for some finite integer $k \geq 0$ (according to the definition of Ω). From the last of the Eq. (15) it follows readily that $\rho_\Lambda(e)x^0 \neq 0$.

The representation ρ_Λ contains also no subrepresentation which consists of a finite number of representations with highest weight (these representations may be coupled into indecomposable representations). Such subrepresentations contain invariant subrepresentations with highest weights.

The representation ρ_Λ can, however, be decomposed into a direct sum of representations with highest weights, or at least into a direct sum of pairs of representations with highest weights which are coupled in an indecomposable manner, if the space V of ρ_Λ is extended to include (formal) infinite sums, and the action of ρ_Λ on these infinite sums is defined by linearity. The resultant representation is denoted by $\bar{\rho}_\Lambda$. If $\Lambda \neq -m/2$, $m > 1$, integer, then $\bar{\rho}_\Lambda$ can be decomposed into a direct sum of the representations $d_\Lambda, d_{\Lambda+1}, d_{\Lambda+2}, \cdots$. The vectors

$$x^0 = 1 + \sum_{i=1}^{\infty} \left\{ \prod_{j=1}^{i} j[\Lambda + \tfrac{1}{2}(j+1)] \right\}^{-1} f^i e^i,$$

$$x^m = e^m + \sum_{i=1}^{\infty} \left\{ \prod_{j=1}^{i} j[\Lambda + m + \tfrac{1}{2}(j+1)] \right\}^{-1} f^i e^{m+i},$$

$$m = 1, 2, 3, \cdots,$$

are the extremal vectors belonging to the highest

weights $\Lambda, \Lambda + 1, \Lambda + 2, \cdots$ of these representations. The proof is simple, and we omit it.

If $\Lambda = -m/2$, $m > 1$, integer, then $\bar{\rho}_\Lambda$ can be decomposed into a direct sum of the representations $d_{\Lambda+s}$, $d_{\Lambda+s+1}, \cdots$ (with s an integer such that $\Lambda + s \geq m/2$) and the representations into which ${}^s\rho_\Lambda$ decomposes.

Instead of the basis introduced in the space $V = \Omega/I$ that was underlying the discussion so far, another basis can be chosen. Namely, the basis 1, $e^m f^n$, $m = 0, 1, 2, \ldots$, $n = 0, 1, 2, \ldots$, excluding the case $m = n = 0$. It is easy to show that in this basis the following relations hold for the representation ρ_Λ:

$$\rho_\Lambda(h)1 = \Lambda 1, \quad \rho_\Lambda(e)1 = e, \quad \rho_\Lambda(f)1 = f,$$

$$\rho_\Lambda(h)e^m f^n = (\Lambda + m - n)e^m f^n,$$

$$\rho_\Lambda(e)e^m f^n = e^{m+1} f^n,$$

$$\rho_\Lambda(f)e^m f^n = e^m f^{n+1} + n[\Lambda - n + (m-1)/2]e^{m-1} f^n.$$

A graphical description of this basis is given in Fig. 9. The action of the shift operators $\rho_\Lambda(e)$ and $\rho_\Lambda(f)$ on the basis elements is also indicated in this figure. It is easy to realize that for every fixed integer $s > 0$ the subspace H_s of V with the basis

$$e^m f^n, \quad m = 0, 1, 2, \cdots, \quad n = s, s+1, s+2, \cdots,$$

is invariant under ρ_Λ. The representation induced by ρ_Λ on the subspace H_s is denoted by π_Λ^s. The representation on the quotient space $V|H_s$ will be denoted by ${}^s\pi_\Lambda$. It is easy to observe that each of the representations on the quotient spaces $H_s|H_{s+1}$ is a representation with a *lowest* weight $\Lambda - S$. We denote this representation by $d'_{\Lambda-s}$. Thus the representation ρ_Λ is of the form

$$\begin{bmatrix} d'_\Lambda & & & & \\ * & d'_{\Lambda-1} & & & 0 \\ * & * & d'_{\Lambda-2} & & \\ * & * & * & \cdot & \\ & & & & \cdot \\ & & & & & \cdot \\ & & & \cdot & \\ & & \cdot & & \\ & & & & \circ \end{bmatrix}$$

Therefore, the representation ρ_Λ admits two different structures. It can be represented either as a semidirect sum of representations $d_{\Lambda+s}$ with highest weight, or as a semidirect sum of representations $d'_{\Lambda-s}$ with lowest weights. This is contrary to what could happen for representations with finite-dimensional weight spaces.

The detailed structure of the representation ρ_Λ with respect to this new basis is completely analogous to the one in the old basis, except that now the representations $d'_{\Lambda-s}$ are representations with lowest weights. Thus, ρ_Λ decomposes into a direct sum of representations $d'_{\Lambda-s}$, for which the eigenvalue of C_2 is different, and a direct sum of pairs of $d'_{\Lambda-s}$, the eigenvalue of C_2 being the same

for the two members of a pair, but being different for the various pairs. (Pairs are obtained for $\Lambda = m/2$, $m > 1$, integer). The extended representation $\bar{\rho}_\Lambda$ decomposes in the same manner as before, the $d'_{\Lambda-s}$ replacing the $d_{\Lambda+s}$.

Two different decompositions of the representations ρ_Λ and $\bar{\rho}_\Lambda$ have thus been obtained. It may be that there exist still other decompositions. An indication of this is given by the following observation for the representation $\bar{\rho}_\Lambda$. The vector (formal sum)

$$x = 1 + \sum_{s=1}^{\infty} a_s f^s e^s$$

with

$$a_s = \binom{\Lambda + s + 1}{\Lambda}^{-1} \binom{\Lambda + s}{\Lambda + 1}^{-1} \sum_{i=1}^{s} 2^i \binom{\Lambda + s - i}{\Lambda - 1} \binom{\Lambda + s - 1 - i}{\Lambda}$$

is an eigenvector of the Casimir operator C_2 with eigenvalue $-\Lambda(\Lambda + 1)$ (for m, n positive integers $\binom{m}{n} = m!/n!$; this notation is extended in the usual manner for other values of m and n). Acting on x by $\bar{\rho}_\Lambda(y)$, $y \in \Omega$, an invariant subspace is obtained which is an eigenspace of C_2. The representation $\bar{\rho}_\Lambda$ induces on this subspace a representation of su(1, 1) which does not have a highest or lowest weight.

Note added in proof: A vector x is called extremal, if $\rho(h)x = \Lambda(h)x$ for $h \in G$, $\Lambda(h)$ a linear form on the Cartan subalgebra, and $\rho(e)x = 0$ or $\rho(f)x = 0$. A representation may have several extremal weights. If a representation is bounded, either on one side or on both sides, then it has at least one extremal weight. Representations bounded either on one side or on both sides, are referred to as representations with a lowest or highest (extremal) weight. This differs from the definition of a representation with highest or lowest weight as given in Ref. 7. Condition (3) of Ref. 7, namely, $V = \{\rho(a)x \mid a \in \Omega\}$ may not be satisfied. This is, for example, indeed the case for the representations D_l^- and D_{-l}^+ of Fig. 2. If the definition of Ref. 7 for a representation with highest or lowest weight is used, then it will be explicitly stated in the text.

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On the interaction of the type $\lambda x^2/(1+gx^2)$

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The ground state and the first two excited state energy levels for the interaction of the type $\lambda x^2/(1+gx^2)$ have been calculated nonperturbatively as the eigenvalues of the one-dimensional Schrödinger operator defined by $Au = -u'' + x^2u + \lambda x^2 u/(1+gx^2)$. The Ritz variational method in combination with the Givens–Householder algorithm has been used for numerical computations.

1. INTRODUCTION

The purpose of this paper is to study the eigenvalue problem $Au = \epsilon u$, where A is the one-dimensional Schrödinger operator defined by

$$Au = -u'' + x^2 u + \frac{\lambda x^2}{1+gx^2}\,u, \quad -\infty < x < \infty, \quad \lambda, \; g > 0 \qquad (1)$$

and $u(\pm \infty) = 0$. Here the dimensionless parameter ϵ is related to the energy E in conventional units by $E = \frac{1}{2}\hbar\sqrt{k/m}\,\epsilon$, where k is the strength of the harmonic oscillator potential, and λ and g are dimensionless parameters.

Interest in the above type of interaction $\lambda x^2/(1 + gx^2)$, is derived from several contexts. In one application, as pointed out by Biswas $et\ al.$,[1] the Schrödinger equation with such an interaction Lagrangian is the analogue of a zero-dimensional field theory with a nonlinear Lagrangian, which is of current interest in particle physics. On the other hand, as pointed out by Haken,[2] such a Schrödinger equation may be related indirectly to certain specific models in laser theory in a manner originally treated by Risken and Vollmer.[3] These authors have reduced the Fokker—Planck equation of a single mode laser to the Schrödinger equation. Subsequently, the general reduction of the Fokker—Planck equation to the Schrödinger equation under certain conditions has been treated in other works.[4] The expression of the type $\lambda x^2/(1+gx^2)$ occurs in this context if the gain saturation is taken into account for arbitrary high photon numbers, where x^2 denotes the photon number.[5]

To obtain the eigenvalues for the problem of the type,[1] which is not exactly solvable, one has to use some approximation scheme, such as one based on a variational or perturbation method. Among these, the variational method, in some form or other, is known to be one of the most effective techniques for the approximate calculation of the first few eigenvalues of a semibounded operator in a Hilbert space.[6,7] Among the commonly used variational schemes, the Ritz method usually finds an upper bound for the eigenvalues, whereas the Weinstein method, in its original form, finds a corresponding lower bound. This technique has been used by Bazley and Fox[7] in connection with the helium atom and the anharmonic oscillator. Because of its simplicity and because of its success in finding eigenvalues in several other contexts considered earlier by the author,[8-10] We have chosen to use the Ritz method in the present context too.

The Ritz method[6] essentially considers the representation, in terms of a symmetric infinite matrix, of a positive operator A in an underlying Hilbert space H. The basis $\{u_n\} \subset D_A$ for the representation is chosen in such a way that the sequence $\{u_n\}$ is complete with respect to the energy norm defined by

$$\|u\|_A^2 = \langle Au, u \rangle. \qquad (2)$$

Then the eigenvalues of the truncated $N \times N$ principal minors of the infinite representation matrix of A with respect to the so-chosen basis approximate the exact ones closer and closer as the size N of the truncated form is made larger and larger. That is, if ϵ_k and $\epsilon_k^{(N)}$ are respectively the kth exact and approximate eigenvalues (as obtained from the above mentioned truncated $N \times N$ form), then $\epsilon_k^{(N)} \geq \epsilon_k^{(N+1)} \to \epsilon_k$ from above as $N \to \infty$. This is the numerical scheme which will be adopted in the present paper. Experience shows that one of the main difficulties in this kind of approach is that of finding an efficient algorithm for the computation of the elements of the corresponding $N \times N$ matrix to a desired accuracy for sufficiently large N, a subject which will be discussed in Sec. 2. Another difficulty, namely the difficulty in computing the eigenvalues of a large matrix may arise in this context. However, the frequently used Givens—Householder algorithm for finding the eigenvalues of a symmetric $N \times N$ matrix[11,12] does not seem, according to our previous experience, to present any significant difficulty, particularly in such problems where the matrix is not drastically ill-defined. Therefore, the present scheme has been based on the Ritz method in combination with Given—Householder algorithm.

In Sec. 2 we discuss the numerical schemes for the computation of the elements of the representation matrix. It turns out that the Gauss—Laguerre quadrature used to evaluate the integrals representing the matrix elements is effective only for small g. As an alternative, a recursive evaluation of the matrix elements has been used for large g.

In Sec. 3 we discuss a scale transformation of the type $y = \alpha x$ which transforms (1) to a form which is particularly used for perturbative treatment for the cases with small g and large λ. On the other hand, for the variational method we use in this paper, the matrix elements of the corresponding transformed operator for small g and large λ can be more effectively computed by the quadrature method, and therefore this serves also as a check for the results, as obtained starting from Eq. (1).

Finally we present the approximate eigenvalues for several values of the parameters λ and g in Sec. 4.

2. ANALYTIC DERIVATIONS AND NUMERICAL SCHEMES FOR THE MATRIX ELEMENTS

We begin with the obvious choice of $L_2(-\infty, \infty)$ as the underlying Hilbert space with a dense subset D_A as the domain of definition of A in $L_2(-\infty, \infty)$, from which one can derive a lower bound for the eigenvalues as indicated below. With the usual definition of the inner product

$$\langle u, v \rangle = \int_{-\infty}^{\infty} uv \, dx, \qquad (3)$$

in $L_2(-\infty, \infty)$, it follows from (1) that

$$\langle Au, u \rangle = \int_{-\infty}^{\infty} \left(u'^2 + x^2 u^2 + \frac{\lambda x^2 u^2}{1 + gx^2} \right) dx. \qquad (4)$$

By virtue of Eq. (4) and the minimum criteria for the eigenvalues, and since λ and g are both positive, it is easy to derive that A is positive definite. In fact, $\langle Au, u \rangle > 1$ for all $u \in D_A$ with $\|u\| = 1$; more generally, $\epsilon_k > \epsilon_k^{(h)}$, $k = 1, 2, 3, \cdots$, where $\epsilon_k^{(h)} = 2k - 1$ are the corresponding eigenvalues of the harmonic oscillator.

It is therefore natural to choose the corresponding orthonormal set, $\{u_n\}$, of the harmonic oscillator eigenfunctions defined by

$$u_n = [2^{n-1}(n-1)! \, \pi^{1/2}]^{-1/2} \exp(-x^2/2) H_{n-1}(x),$$
$$n = 1, 2, 3, \cdots, \quad (5)$$

as the basis for the representation of A, where the H_n's are the Hermite polynomials.

In the above, the indexing of the harmonic oscillator functions u_n has been chosen to start at $n = 1$ (instead of conventional $n = 0$) merely for computational convenience.

The purpose of this section is to evaluate the expressions of the type $\langle Au_n, u_m \rangle$ for the elements of the representation matrix of A with respect to the orthonormal basis $\{u_n\}$ defined by (5).

Since $-u_n'' + x^2 u_n = (2n - 1)u_n$, $n = 1, 2, 3, \cdots$, we obtain from (1), (3), and (5)

$$S_{mn} = \langle Au_n, u_m \rangle = (2n - 1)\delta_{mn} + \lambda G_{mn}, \qquad (6)$$

where

$$G_{mn} = \frac{1}{g} (\delta_{mn} - H_{mn}) \qquad (7)$$

and

$$H_{mn} = \int_{-\infty}^{\infty} \frac{u_m u_n}{1 + gx^2} \, dx. \qquad (8)$$

Also since $H_n(-x) = (-1)^n H_n(x)$, it is clear from (8) that $H_{mn} = 0$ unless $m + n$ is an even integer. In this case, we need to evaluate only those elements for which $m + n$ is even, and then

$$H_{mn} = 2 \int_0^{\infty} \frac{u_n u_m}{1 + gx^2} \, dx, \quad m + n = \text{even}. \qquad (9)$$

Clearly the integrals of the type (9) cannot be evaluated exactly and therefore we have used two separate

numerical schemes as discussed in the following subsections.

A. Quadrature method

Since the generalized Gaussian quadrature formulas are frequently used for the approximate calculation of integrals[13,14] we have used one such quadrature for the evaluation of the integrals (9). Making change of the integration variables by the substitution $y = x^2$, we can express the integral (9), by the generalized Gauss—Laguerre formula, in the form

$$H_{mn} = \int_0^{\infty} \frac{\exp(-y)F(y) \, dy}{y^{1/2}} = \sum_{k=1}^{N} A_k^{(N)} F(y_k^{(N)}),$$
$$N = 2, 3, \cdots, \quad (10)$$

where

$$F(y) = C_{mn} \frac{H_{m-1}(y^{1/2}) H_{n-1}(y^{1/2})}{1 + gy} \qquad (11)$$

and C_{mn} is a constant. Here the weights $A_k^{(N)}$ and the nodes $y_k^{(N)}$ [i.e., roots of the associated Laguerre polynomial $L_N^{(-1/2)}(y)$ of degree N] are readily available from standard monographs.[14] The quadrature formula is exact whenever $F(y)$ is polynomial up to degree $2N - 1$. Otherwise, some error is involved, which has been estimated in Ref. 14. We have settled on the accuracy in a different way to be explained below.

We used the 32-point generalized Gauss—Laguerre quadrature as available from the IBM package. Unfortunately, the results obtained were too inaccurate (even in the second decimal places) for large g, say $g = 100$, even for the first element H_{11}. This may be due to the fact that the graph of $\exp(-x^2)/(1 + gx^2)$, for large g, is almost entirely concentrated near $x = 0$, and falls rapidly, whereas the quadrature points do not mesh fine enough near $x = 0$. Therefore, as an alternative, this quadrature up to 64-point has been generated by the author using Sturm's separation theorem and Sturm sequence method. The improvement was significant, but not significant enough to ensure sufficient accuracy to serve our purpose. This quadrature, beyond the 64-point, has not been generated here or elsewhere to the knowledge of the author.

On the other hand, for small g (i.e., $g < 1$) this quadrature yields fairly accurate results, as indicated by the error estimates obtained as the difference of the results given by the 48-point and 64-point quadrature formulas. In this way matrices, up to 30×30, accurate to about five decimal places, have been obtained for small g. For higher values of g, we have used, as an alternative, a method of recursive evaluation of the integrals, which should, in principle, be capable of generating the matrices of arbitrary size. This will be discussed in the following subsection.

B. A recursive evaluation of the integrals

Using the standard recurrence formula

$$H_{n+1}(x) = 2xH_n - 2nH_{n-1}$$

for the Hermite polynomials, we obtain for the corresponding u_n, defined by (5),

$$u_n = [2/(n-1)]^{1/2} x u_{n-1} - [(n-2)/(n-1)]^{1/2} u_{n-2}, \quad (12)$$

where

$$u_1 = \frac{1}{\pi^{1/4}} \exp(-x^2/2), \quad u_2 = \frac{\sqrt{2}x}{\pi^{1/4}} \exp(-x^2/2). \quad (13)$$

By repeated use of (12) and a little rearrangement it may be shown that

$$u_n = \frac{2}{g} \frac{(1+gx^2)}{[(n-1)(n-2)]^{1/2}} u_{n-2} - c_n u_{n-2} - d_n u_{n-4}, \quad (14)$$

where

$$c_n = \frac{2/g + 2n - 5}{[(n-1)(n-2)]^{1/2}}, \quad d_n = \left(\frac{(n-3)(n-4)}{(n-1)(n-2)} \right)^{1/2}. \quad (15)$$

From (14) and the definition (9) of H_{mn}, we obtain our basic recursive formula,

$$H_{mn} = \frac{2}{a[(n-1)(n-2)]^{1/2}} H_{m,n-2} - c_n H_{m,n-2} - d_n H_{m,n-4}, \quad (16)$$

where $H_{mn} = 0$ unless $m + n =$ even. Equation (1) implies that the mth row of the matrix (H_{mn}), $n > m$, can be generated from the very first two nonzero elements in the upper triangle of (H_{mn}). Also the second nonzero element in the mth row $(m \geqslant 3)$ of the upper triangular form of H_{mn} can be obtained from the previous row by setting $n = m + 2$ in (1) and using the symmetry of H_{mn}, in the form

$$H_{m,m+2} = \frac{2}{g[m(m+1)]^{1/2}} - c_{m+2} H_{mm} - d_{m+2} H_{m-2,m}, \quad m \geqslant 3. \quad (17)$$

On the other hand, the diagonal elements can be computed from

$$H_{mm} = \left(\frac{m}{m-1} \right)^{1/2} H_{m-1,m+1}$$

$$+ H_{m-1,m-1} - \left(\frac{m-2}{m-1} \right)^{1/2} H_{m-2,m}, \quad (18)$$

To derive (20), we multiply (12) by u_n and obtain

$$u_n^2 = \left(\frac{2}{n-1} \right)^{1/2} u_{n-1}(x u_n) - \left(\frac{n-2}{n-1} \right)^{1/2} u_{n-2} u_n. \quad (19)$$

On the other hand, replacing n by $n+1$ in (12) we can express the factor $x u_n$ in (21) in the form

$$x u_n = \left(\frac{n}{2} \right)^{1/2} \left[u_{n+1} + \left(\frac{n-1}{n} \right)^{1/2} u_{n-1} \right]. \quad (20)$$

Substituting the above expression for $x u_n$ in (21), we get

$$u_n^2 = \left(\frac{n}{n-1} \right)^{1/2} u_{n-1} u_{n+1} + u_{n-1}^2 - \left(\frac{n-2}{n-1} \right)^{1/2} u_{n-2} u_n. \quad (21)$$

Then by definition (9) of H_{mn} and (23), one can easily obtain the relation (20).

The above three recursive formulas, namely (16), (17), and (18), indicate that we in fact need only the four elements H_{11}, H_{13}, H_{22}, H_{24} to generate the matrix (H_{mn}) of arbitrary size through the use of this recursive procedure. On the other hand, using the explicit forms of the corresponding Hermite polynomials, the matrix elements H_{13}, H_{22}, H_{24} can be easily expressed in terms of H_{11}, and therefore the recursive evaluation

of the matrix elements depends only on the accurate knowledge of H_{11}.

As mentioned before, this element H_{11} cannot be computed accurately enough for large g by Gauss—Laguerre quadrature. However, there is an alternative way to compute H_{11} with arbitrary accuracy by using standard Gaussian quadrature for finite intervals. That is, we replace the infinite interval in (9) by a finite number k of subintervals, each of length h, and evaluate the integral from 0 to h, h to $2h$, and so on. The process is terminated when the contribution from the integral $((k-1)h, kh)$ is less than the degree δ of accuracy required. We have chosen $h = 0.5$ as an initial guess and $\delta = 10^{-10}$. We have used standard 48-point Gaussian quadrature[13] to evaluate the integrals corresponding to each subinterval. An obvious way to check or improve the accuracy would be either to choose a smaller value of h or a higher quadrature (keeping h fixed), say, 64-point and comparing the result with the previous one. In this way values of H_{11} accurate to ten decimal places have been obtained.

It may be noted that the above method for obtaining H_{11}, though very accurate and stable, is quite expensive in computer time and therefore has not been applied to find all the elements of the matrix. On the other hand, the recursive procedure has been very effective, especially for $g > 2$, except for some possible round off errors inherent in many recursive procedures. (Since this recursive method incorporates only a finite number of arithmetical operations, there is no convergence error in this method, except for round off errors in arithmetic operations.) On the other hand, for small g, due to the factor $2/g$ in (15), the round off errors become severe. In such a case one has to rely more on Gauss—Laguerre quadrature, as discussed before.

This is essentially the numerical scheme for obtaining the matrix elements. In this way we have generated matrices up to 30×30 for small g's by Gauss—Laguerre method and up to 100×100 for large g by the recursive procedure. Also the accuracy of the recursive procedure has been checked by comparing the values of the integrals for a few of the matrix elements which can be evaluated otherwise, say, by Gaussian quadrature.

Another alternative is to use a scale transformation, particularly useful for a perturbative treatment. This is presented in the next section. Our purpose, however, is to provide another way to check the accuracy of the results presented in this paper based on a nonperturbative method.

3. A SCALE TRANSFORMATION

It is clear that for $g = 0$, the eigenvalues of (1) are given by $(2n-1)(1+\lambda)^{1/2}$ in terms of the harmonic oscillator values. Therefore, we may choose a scaling of the type

$$y = \alpha x, \quad \text{where } \alpha^4 = 1 + \lambda, \quad (22)$$

to transform the original Eq. (1) into the following form:

$$A_1 u = -\frac{d^2 u}{dy^2} + y^2 u - \lambda_1 \frac{g_1 y^4}{1 + g_1 y^2} u = \epsilon_1 u, \quad (23)$$

TABLE I. The first three energy levels for different values of λ and g are given in increasing order of excitations. Each column designates the energy levels for a fixed λ and varying g, whereas each row designates the same for fixed g and variable λ.

g \\ λ	0.1	0.2	0.5	1	2	5	10	20	50	100
0.1	1.04317	1.08495	1.20303	1.38053	1.68561	2.38954	3.25026	4.51242	7.06869	9.97618
	3.12008	3.23700	3.57080	4.0798	4.96859	7.05096	9.61906	13.39736	21.06073	29.78119
	5.18109	5.35866	5.87158	6.667	8.08680	11.48480	15.72933	22.00557	34.76383	49.29269
0.5	1.03121	1.06196	1.15156	1.29295	1.55104	2.19211	3.01685	4.25506	6.79278	9.69215
	3.07389	3.14722	3.36380	3.71390	4.37658	6.12105	8.482	12.12361	19.68503	28.8362
	5.09305	5.18591	5.46320	5.92063	6.81529	9.32076	12.948	18.79614	31.23804	45.636
1	1.02410	1.04801	1.11854	1.23235	1.44732	2.01300	2.78233	3.97769	6.47811	9.3594
	3.05149	3.10281	3.25577	3.50738	3.99840	5.37944	7.41751	10.79063	18.12871	26.705
	5.03444	5.11793	5.29488	5.58977	6.17848	7.92192	10.701	15.698	27.375	41.441
2	1.01718	1.03429	1.08519	1.16867	1.33072	1.78213	2.44250	3.53492	5.93198	8.7582
	3.03276	3.06549	3.16346	3.32602	3.64837	4.59337	6.09510	8.83870	15.497	23.743
	5.03444	5.06890	5.17240	5.34524	5.69212	6.73922	8.49042	11.94154	21.395	34.257
5	1.00978	1.01956	1.04886	1.09729	1.19331	1.47402	1.91890	2.73391	4.75570	7.34216
	3.01608	3.03215	3.08037	3.16066	3.32099	3.80001	4.59156	6.14795	10.58520	17.182
	5.01560	5.03120	5.07804	5.15620	5.31285	5.78531	6.58033	8.19247	13.1059	21.207
10	1.00594	1.01188	1.02968	1.05929	1.11830	1.29358	1.58002	2.13243	3.64441	5.794
	3.00881	3.01762	3.04405	3.08809	3.17614	3.44004	3.87903	4.75378	7.35010	11.572
	5.00828	5.01656	5.04141	5.08284	5.16579	5.41520	5.83277	6.67468	9.24640	13.628
20	1.00343	1.00686	1.01716	1.03430	1.06855	1.17096	1.34047	1.67518	2.64547	4.157
	3.00465	3.00931	3.02328	3.04656	3.09312	3.23277	3.46544	3.93043	5.32264	7.633
	5.00432	5.00865	5.02163	5.04328	5.08658	5.21657	5.43357	5.86882	7.18462	9.409
50	1.00156	1.00313	1.00784	1.01569	1.03138	1.07840	1.15667	1.31275	1.77748	2.5401
	3.00193	3.00387	3.00968	3.01937	3.03874	3.04916	3.19371	3.38740	3.96837	4.9362
	5.00180	5.00361	5.00904	5.01808	5.03617	5.04638	5.18094	5.36202	5.90604	6.8154
100	1.00084	1.00168	1.00420	1.00841	1.01682	1.04204	1.08406	1.16803	1.41938	1.8364
	3.00098	3.00200	3.00492	3.00983	3.01966	3.04916	3.09832	3.19663	3.4915	3.9831
	5.00093	5.00185	5.00464	5.00928	5.01855	5.04638	5.09276	5.18554	5.4639	5.928

where

$$\lambda_1 = \frac{\lambda}{1+\lambda}, \quad g_1 = \frac{g}{\sqrt{1+\lambda}}, \quad \epsilon_1 = \frac{\epsilon}{\sqrt{1+\lambda}}.$$

The form (23) is particularly convenient for small g and large λ, since $0 < \lambda_1 < 1$ and $g_1 < g$ for any positive λ. In this case, perturbation methods near $g = 0$ can be conveniently applied since $\lambda_1 < 1$.

In the line of variational method used in this paper, the matrix elements have been calculated by using Gauss—Laguerre quadrature for small g and for the whole range of values of λ given in the next section. Alternatively, one can as well develop a recursive procedure in a manner similar to the one presented in the last section. The eigenvalues obtained thereafter agree fairly well with those obtained from the matrices generated by the methods given in Sec. 3 [it may be emphasized that the eigenvalues ϵ_1 are related to ϵ, those of the original problem (1), by $\epsilon = (1 + \lambda)^{1/2}\epsilon_1$]. the results are presented in Sec. 4.

4. NUMERICAL RESULTS

Once the matrix elements are computed to generate the representation matrices of adequate size, the approximate eigenvalues can be computed from the corresponding $N \times N$ principal minors using the standard Givens—Householder algorithm and studying the convergences of these eigenvalues as N increases. This procedure has been already explained in Sec. 1. We start with $N = 10$, compute the first three eigenvalues

by the Givens—Householder method. Then we increase N to 20, compute the eigenvalues and compare them with those obtained from the 10×10 matrix. If the
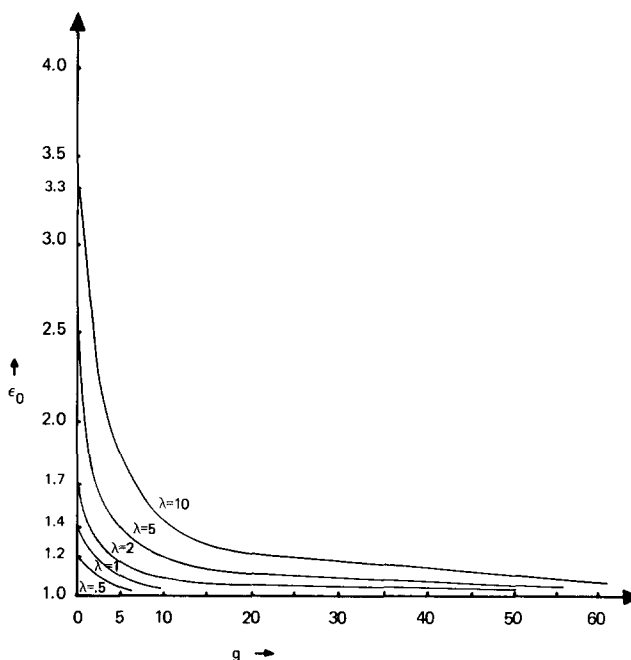

FIG. 1. The ground state energy of the harmonic oscillator with the added interaction $\lambda x^2/(1 + gx^2)$ as a function of g for different parametric values of λ.

agreement is up to three decimal places, we stop. Otherwise, continue the process by increasing N by a step of 10, until the desired accuracies (agreement to about three decimal places) are obtained. In this way, the first three eigenvalues for a range of values of λ and g have been obtained from the matrices as described in Secs. 2 and 3. The results are presented in Table I.

A plot of the ground state energy against g for certain fixed values of λ is shown in Fig. 1. Similar plots also hold for the higher energy levels. As is expected, the eigenvalues decrease steadily with increasing g and fixed λ and approach the eigenvalues $(2n-1)$ $(n=1,2,\cdots)$ of the harmonic oscillator asymptotically for large g. Finally, we note that the size N of the matrix needed to find the above eigenvalues has been well within 100, whereas for small g (i.e., $g=0.1$, 0.5, 1) good approximations have been obtained by using only 30×30 matrices.

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Average force and force correlation formulas for momentum transfer cross section

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Gerjuoy has related the momentum transfer cross section to the average force. The force correlation function formula of Rousseau, Stoddart, and March is here shown to be transformable into Gerjouy's result for a spherically symmetrical scatterer.

Gerjuoy[1] considered the scattering of a particle by a potential $V(\mathbf{r})$, which need not be spherically symmetrical. In terms of the scattering solutions ψ of the Schrödinger equation

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) - E\right)\psi = 0, \tag{1}$$

he calculated the momentum transfer cross section as

$$\sigma_A = \frac{1}{2E}\int d\mathbf{r}\,\psi^*\frac{\partial V}{\partial z}\psi, \tag{2}$$

which we shall refer to as the average force formula.

In the different context of the electrical resistivity of a metal with scattering centers present, Rousseau et al.[2] have expressed the same weighted cross section appropriate to the momentum transfer associated with resistivity in terms of a force correlation function. Stripped of unimportant multiplying factors, this is

$$F = \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{\partial V(r_1)}{\partial \mathbf{r}_1}\cdot\frac{\partial V(r_2)}{\partial \mathbf{r}_2}\,|\sigma(\mathbf{r}_1,\mathbf{r}_2)|^2$$
$$\equiv \int d\mathbf{r}_1 d\mathbf{r}_2\, f, \tag{3}$$

where $\sigma(\mathbf{r}_1, \mathbf{r}_2)$ is the energy derivative of the Dirac density matrix. In Eq. (3), we note that (a) we have already specialized to the case this note treats, a spherical potential $V(r)$, and (b) σ is to be evaluated at the Fermi energy $E_f = k_f^2/2$, where all the scattering takes place.

The manipulation of the force correlation function below now follows earlier work by the writer[3]; at the time that work was done the result of Gerjuoy was not known to him. First one utilizes the central field form of σ as given by March and Murray[4]

$$\sigma(\mathbf{r}_1, \mathbf{r}_2) = \sum_l (2l+1)\sigma_l(r_1, r_2) P_l(\cos\gamma), \tag{4}$$

γ being the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$. Performing the angular integration implied in Eq. (3) the result is[3]

$$\int_0^\pi f\sin\gamma\,d\gamma = 4\sum_{l=1}^\infty l\sigma_{l-1}(r_1, r_2)\sigma_l(r_1, r_2)\frac{\partial V(r_1)}{\partial r_1}\frac{\partial V(r_2)}{\partial r_2}. \tag{5}$$

But now[4-6]

$$\sigma_l(r_1, r_2) \propto R_l(r_1) R_l(r_2), \tag{6}$$

where R_l is the radial wavefunction for potential $V(r)$ at $E = E_f$ for the lth partial wave.

As Gerjuoy[1] and independently Gaspari and Gyorffy[7] have shown, with suitable normalization

$$\int_0^\infty dr\, r^2 R_{l-1}(r)\frac{\partial V}{\partial r}R_l(r) = \sin(\delta_{l-1} - \delta_l), \tag{7}$$

where δ_l are the phase shifts at energy E_f. Hence one obtains

$$F = \text{const}\sum_{l=1}^\infty l\int dr\,\frac{\partial V(r)}{\partial r}R_{l-1}(r)R_l(r)\sin(\delta_{l-1} - \delta_l). \tag{8}$$

Let us turn finally to the average force formula (1). For a spherical potential, the scattering solution ψ takes the form

$$\psi = \frac{1}{k_f}\sum_l (2l+1)i^l\exp(i\delta_l)P_l(\cos\theta)R_l(r, k_f) \tag{9}$$

at energy $E = E_f = k_f^2/2m$. Substituting this into Eq. (2), Gerjuoy[1] shows that, apart from constants[8]

$$\int d\mathbf{r}\,\psi^*\frac{\partial V}{\partial z}\psi$$
$$= \text{const}\int_0^\infty dr\,r^2\frac{dV}{dr}\sum_{l=0}^\infty (l+1)R_l R_{l+1}\sin(\delta_l - \delta_{l+1}) \tag{10}$$

which, noting that the summation now ranges from $l = 0$, is trivially different from Eq. (8).

In summary, the average force and the force correlation formula are quite equivalent for spherical scatterers. In practical applications, formula (3) has the merit that if one substitutes $\sigma = \sigma_0$, its free electron limit, one immediately obtains the Born approximation result. Evidently, in the average force formula (2), substituting for ψ as a plane wave would be inadequate, leading to zero. But Gerjuoy points out that his formula may have merit for treating the scattering of plane waves off nonspherical potentials. One example which comes to mind, in the present context, is conduction electron scattering off two vacancies in association, but it is not our purpose here to go into any further detail on that possible application.[9]

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Complex *pp* waves and the nonlinear graviton construction

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We show how to construct all of the complex *pp* waves using the nonlinear graviton construction of Penrose.

INTRODUCTION

Several years ago, Penrose[1] showed that one could construct generic, half-flat solutions of the complex Einstein equations through the use of deformation theory. His idea is to deform the complex structure of a neighborhood of a projective line in $P_3(C)$. The original undeformed neighborhood contains a four-complex-parameter family of lines which are identified with the points of an open subset of complex Minkowski space. For small deformations, Kodaira's theorems[2,3] guarantee the continued existence of a four-parameter family of "lines" (i.e., compact holomorphic curves) in the deformed space; these are identified as the points of a new complex manifold $\mathcal{G}$. A holomorphic metric is then introduced on $\mathcal{G}$ in a natural way and the resulting complex spacetime is shown to be half-flat, that is, its Ricci tensor vanishes and its conformal curvature tensor is anti-self-dual.

While it is a relatively straightforward matter to construct deformations, the task of finding the four-parameter family of lines in the deformed space is usually very difficult. Because of this, only a few isolated solutions have actually been explicitly constructed, only one of which, to our knowledge, has appeared in print.[4] The purpose of this paper is to show that the simplest half-flat spacetimes, known as complex pp waves or Plebański plane waves,[5] can all be obtained explicitly using the Penrose construction.

1. THE NONLINEAR GRAVITON CONSTRUCTION

In this section we summarize the Penrose construction. For more details, in particular for the proof that $\mathcal{G}$ is half-flat, we refer the reader to Penrose's original article.[1]

Denote a point of $C^4 - (0)$ (a twistor) by $Z^\alpha = (\omega^A, \pi_{A'})$, and let $[\omega^A, \pi_{A'}]$ denote the corresponding point in $P_3(C)$. If $x^{AA'}$ is any point in complex Minkowski space, CM, we may associate with it the projective line $L(x)$:
$= \{[ix^{AA'} \pi_{A'}, \pi_{A'}] \mid [\pi_{A'}] \in P_1(C)\}$. If W is a connected open neighborhood of x in CM, the set $PT(W) := \{L(y) \mid y \in W\}$ is a connected open neighborhood of $L(x)$ in $P_3(C)$. In this paper we shall consider only the case $W = CM$ and we set $PT := PT(CM)$. Notice that PT is just $P_3(C)$ with one projective line removed (namely, all points of the

form $[\omega^A, 0]$). T is the corresponding set of points in $C^4 - (0)$. Then we have:

(a) PT is a holomorphic fiber space over $P_1(C)$ with projection $[\omega^A, \pi_{A'}] \to [\pi_{A'}]$. Similarly we have a fiber space $T \to C^2 - (0)$ given by $(\omega^A, \pi_{A'}) \to \pi_{A'}$ and the following diagram commutes.

$$
\begin{array}{ccc}
T & \longrightarrow & PT \\
\downarrow & & \downarrow \\
C^2 - (0) & \longrightarrow & P_1(C)
\end{array}
\tag{1}
$$

(b) The points of CM are in 1—1 correspondence with the global holomorphic cross sections of $PT \to P_1(C)$: Given $x^{AA'}$, define a section by $[\pi_{A'}] \to [ix^{AA'} \pi_{A'}, \pi_{A'}]$. Alternately points of CM may be put in correspondence with global holomorphic cross sections of $T \to C^2 - (0)$ which are homogeneous of degree 1 in $\pi_{A'}$.

(c) The conformal structure of CM is obtained by observing that points x and y in CM are null-separated iff $L(x)$ and $L(y)$ intersect. If $y = x + \Delta x$ and if $\omega^A(\Delta x, \pi_{A'})$ is the section of $T \to C^2 - (0)$ corresponding to Δx, then $\omega^A(\Delta x, \overset{\circ}{\pi}_{A'}) = 0$ for some $\overset{\circ}{\pi}_{A'}$ [and hence for $\lambda \overset{\circ}{\pi}_{A'}$ for $\lambda \in C - (0)$]. Thus null vectors in CM correspond to global holomorphic sections of $T \to C^2 - (0)$ which vanish somewhere. In order to pin down the conformal factor, one makes use of the 2-form $d\omega_A \wedge d\omega^A = \mu$. This will be considered in more detail later.

If we let $\hat{U}$ (resp. $\hat{\hat{U}}$) be the subset of $C^2 - (0)$ given by $\pi_{0'} \neq 0$ (resp. $\pi_{1'} \neq 0$), then we get a decomposition of T as the union $U \cup \hat{U}$, where $U = \{(\omega^A, \pi_{A'}) \mid \pi_{A'} \in \hat{U}\}$ and $\hat{U} = \{(\hat{\omega}^A, \hat{\pi}_{A'}) \mid \hat{\pi}_{A'} \in \hat{\hat{U}}\}$. We may view T as being formed by glueing together U and $\hat{U}$ by the trivial equations $\hat{\omega}^A = \omega^A$, $\hat{\pi}_{A'} = \pi_{A'}$. To deform T, we consider a one-parameter family of patchings of the form

$$
\hat{\omega}^A = f^A(\omega, \pi, \lambda), \qquad \hat{\pi}_{A'} = \pi_{A'},
\tag{2}
$$

satisfying

$$
f^A(\alpha\omega, \alpha\pi, \lambda) = \alpha f^A(\omega, \pi, \lambda), \qquad \alpha \in C - (0),
$$
$$
f^A(\omega, \pi, 0) = \omega^A.
\tag{3}
$$

Here λ ranges over a neighborhood B of $0 \in C$ and the

FIG. 1.

$\xi_z \in T_{z_0}$ to be null if and only if $\xi(z, \pi)$ has a zero at some $\overset{\circ}{\pi}$ (and hence for any nonzero multiple of $\overset{\circ}{\pi}$). See (c) above.

In order to define the actual metric on $\mathcal{G}$ we use the 2-form, $\mu = \epsilon_{AB} d\omega^A \wedge d\omega^B = \epsilon_{AB} d\hat{\omega}^A \wedge d\hat{\omega}^B \pmod{d\pi_{A'}}$, which is well defined on the fibers of $\mathcal{T}$. If $\xi \in T_z \mathcal{G}$ the relation (4) shows that we may regard $\xi(z, \pi)$ as a vector field along $\underline{\mathcal{L}}(z)$ which is everywhere tangent to the fibers. Suppose ξ and η are null vectors at z and that $\xi(z, \pi)$, $\eta(z, \pi)$ vanish at $\alpha_{A'}$, $\beta_{A'}$ respectively. Then Penrose defines

$$g_z(\xi, \eta) := (\alpha_B \cdot \beta^{B'}) \mu(\xi, \eta) / (\alpha_B \cdot \pi^{B'})(\beta_B \cdot \pi^{B'}). \tag{5}$$

The right side of (5) is symmetric in (ξ, η). As a function of $\pi_{A'}$, it is homogeneous of degree 0, and holomorphic on $C^2 - (0)$. Thus it is constant and so $g_z(\xi, \eta)$ is a well-defined complex number.

2. CONSTRUCTION OF THE COMPLEX pp WAVES

We choose the patching (2) to have the form

$$\hat{\omega}^0 = \omega^0,$$

$$\hat{\omega}^1 = \omega^1 + \lambda h(\omega^0, \pi_{A'}), \tag{6}$$

$$\hat{\pi}_{A'} = \pi_{A'},$$

where h is homogeneous of degree 1 and holomorphic on $C \times (\underline{D} \cap \hat{\underline{D}})$. We set $\lambda = 1$ in what follows. We shall write down all global holomorphic sections of the bundle $\mathcal{T} \to C^2 - (0)$ obtained using (6) which are also homogeneous of degree 1 in $\pi_{A'}$. Such a section is given by a pair $\{\omega^A(\pi), \hat{\omega}^A(\pi)\}$ satisfying (6) with $\omega^A(\pi)$, $\hat{\omega}^A(\pi)$ homogeneous of degree 1 and holomorphic in $\underline{D}$, $\hat{\underline{D}}$ respectively. Thus $\omega^0(\pi) = \hat{\omega}^0(\pi)$, so these give a holomorphic function on $C^2 - (0)$, homogeneous of degree 1. Therefore, there exist $u, \zeta \in C$, constants, such that

$$\hat{\omega}^0(\pi) = \omega^0(\pi) = u\pi_{0'} + \zeta\pi_{1'}. \tag{7}$$

$\omega^1(\pi)$ and $\hat{\omega}^1(\pi)$ are related as follows:

$$\hat{\omega}^1(\pi) = \omega^1(\pi) + h(\omega^0(\pi), \pi), \quad \pi \in \underline{D} \cap \hat{\underline{D}}. \tag{8}$$

For fixed u and ζ, $h(u\pi_{0'} + \zeta\pi_{1'}, \pi_{0'}, \pi_{1'})$ is a holomorphic function $\underline{D} \cap \hat{\underline{D}}$. To construct a section of our bundle, we must express this function as a difference $\hat{\omega}^1(\pi) - \omega^1(\pi)$ where $\omega^1(\hat{\omega}^1)$ is holomorphic in $\underline{D}(\hat{\underline{D}})$ and homogeneous of degree 1. Each distinct way of splitting h will give a pair $\{\hat{\omega}^1, \omega^1\}$ and hence a section. We claim that for each pair (u, ζ) there is a two complex parameter family of splittings. Thus we get a four-parameter family of sections of $\mathcal{T} \to C^2(0)$ as desired.

To see this, let $\xi \neq 0$. Then setting $(\pi_{0'}, \pi_{1'}) = (1/\xi, 1)$ in (8),

$$\hat{\omega}^1(\xi^{-1}, 1) = \xi^{-1}\omega^1(1, \xi) + \xi^{-1}h(u + \xi\zeta, 1, \xi). \tag{8'}$$

Let $h(\xi) := h(u + \xi\zeta, 1, \xi)$ and expand $h(\xi)$ in a Laurent series, $h = \sum_{n=-\infty}^{\infty} b_n(u, \zeta)\xi^n$. Put $\omega^1(1, \xi) = \sum_{n=0}^{\infty} a_n\xi^n$. This series is convergent for all $\xi \in C$, and $\xi^{-1}\omega^1(1, \xi) + \xi^{-1}h(\xi)$ is to be entire in ξ^{-1}. We conclude that in the series expansion of the right side of (8') all positive

---

functions $\hat{f}^A$ are holomorphic in $C^2 \times (\underline{D} \cap \hat{\underline{D}}) \times B$. For each fixed $\lambda \in B$ the patching (2) gives a fiber space $\mathcal{T}(\lambda) \to C^2 - (0)$ (a deformed twistor space). Since the transition functions are homogeneous of degree 1, the identification $(\omega^A, \pi_{A'}) \sim (\alpha\omega^A, \alpha\pi_{A'})$, $\alpha \in C - (0)$, is consistent over $\underline{D} \cap \hat{\underline{D}}$ and gives rise to a deformed projective twistor space $P\mathcal{T}(\lambda) \to P_1(C)$. An important aspect of these deformed twistor spaces is the existence of a preferred 2-form on the fibers of $\mathcal{T}(\lambda) \to C^2 - (0)$; the transformations (2) are required to satisfy $d\hat{\omega}^0 \wedge d\hat{\omega}^1 = d\omega^0 \wedge d\omega^1 + $ (terms involving $d\pi_{A'}$).

Kodaira's stability theorem[2] ensures that for $|\lambda|$ sufficiently small, $P\mathcal{T}(\lambda) \to P_1(C)$ still has a global holomorphic section. His completeness theorem[3] then guarantees the existence of a four-parameter family of global holomorphic sections of $P\mathcal{T}(\lambda) \to P_1(C)$. So for λ fixed and sufficiently small we have an open set $\mathcal{G} \subset C^4$ and for each $z \in \mathcal{G}$ a global holomorphic section, $L_\lambda(z)$, of $P\mathcal{T}(\lambda) \to P_1(C)$, distinct z's giving rise to distinct sections. Each such $L_\lambda(z)$ pulls back to a global holomorphic section of $\mathcal{T}(\lambda) \to C^2 - (0)$ which is homogeneous of degree 1. This section, $\underline{\mathcal{L}}_\lambda(z)$, is represented by a pair $\{\omega^A(z, \pi, \lambda), \hat{\omega}^A(z, \pi, \lambda)\}$ satisfying the transition relation (2). Each function $\omega^A(z, \pi, \lambda)$, $\hat{\omega}^A(z, \pi, \lambda)$ is holomorphic on its domain and homogeneous of degree 1 in $\pi_{A'}$.

Henceforth we shall consider a fixed value of λ and shall omit λ from the notation.

Now let $z_0 \in \mathcal{G}$ and let $\delta z = \xi^a(\partial/\partial z^a)(z_0)$ be tangent to $\mathcal{G}$ at z_0. Define $\xi^A(z_0, \pi) := \xi^a(\partial\omega^A/\partial z^a)(z_0, \pi)$ and $\hat{\xi}^A(z_0, \pi) := \xi^a(\partial\hat{\omega}^A/\partial z^a)(z_0, \pi)$. Then

$$\hat{\xi}^A(z_0, \pi) = (\partial\hat{f}^A/\partial\omega^B)(\omega(z_0, \pi), \pi)\xi^B(z_0, \pi). \tag{4}$$

Thus the pair $\{\xi^A, \hat{\xi}^A\}$ is a section of the normal bundle of $\underline{\mathcal{L}}(z_0)$ in $\mathcal{T}$. Intuitively the situation is as follows: $\underline{\mathcal{L}}(z_0)$ is a section of $\mathcal{T} \to C^2 - (0)$ which we view as a submanifold of $\mathcal{T}$. We write (to first order) $\underline{\mathcal{L}}(z_0 + \delta z) := \{\omega^A(z_0, \pi) + \xi^A(z_0, \pi); \hat{\omega}^A(z_0, \pi) + \hat{\xi}^A(z_0, \pi)\}$.

That is, we have a "nearby" section $\underline{\mathcal{L}}(z_0 + \delta z)$ and the "difference" between the two is a section of the normal bundle of $\underline{\mathcal{L}}(z_0)$ (see Fig. 1). It follows from Kodaira's completeness theorem[3] that the map $\delta z = \xi^a(\partial/\partial z^a)(z_0) \to \xi(z_0, \pi)$ is an isomorphism from $T_{z_0}\mathcal{G}$ onto the space of global holomorphic sections of the normal bundle of $\underline{\mathcal{L}}(z_0)$ which are homogeneous of degree 1 in $\pi_{A'}$.

The conformal metric on $\mathcal{G}$ is obtained by defining

powers of ξ disappear. We conclude $a_n = -b_n$ for $n \geq 2$, while $\tilde{\zeta} := a_0$ and $v := a_1$ are free parameters. Thus in the first chart the sections are given by

$$\omega^0(u, v, \zeta, \tilde{\zeta}, \pi_{A'}) = u\pi_{0'} + \zeta\pi_{1'},$$

$$\omega^1(u, v, \zeta, \tilde{\zeta}, \pi_{A'}) = \tilde{\zeta}\pi_{0'} + v\pi_{1'} - \sum_{n=2}^{\infty} b_n(u, \zeta)\frac{(\pi_{1'})^n}{(\pi_{0'})^{n-1}} \quad (9)$$

and in the second chart by

$$\hat{\omega}^0(u, v, \zeta, \tilde{\zeta}, \pi_{A'}) = u\pi_{0'} + \zeta\pi_{1'}$$

$$\hat{\omega}^1(u, v, \zeta, \tilde{\zeta}, \pi_{A'}) = \tilde{\zeta}\pi_{0'} + v\pi_{1'} + \sum_{n=-1}^{\infty} b_{-n}(u, \zeta)\frac{(\pi_{0'})^{n+1}}{(\pi_{1'})^n} \quad (\hat{9})$$

Computing the metric: Consider $(u, v, \zeta, \tilde{\zeta})$ as co-ordinates of a point in $\mathcal{G}$. Let $(du, dv, d\zeta, d\tilde{\zeta})$ be components of a tangent vector at $(u, v, \zeta, \tilde{\zeta})$. According to the discussion in Sec. 1, we get a section of the normal bundle to the section of $\mathcal{T}$ labelled by $(u, v, \zeta, \tilde{\zeta})$ by writing, in un-hatted coordinates,

$$V^0 = \pi_{0'}\, du + \pi_{1'}\, d\zeta$$

$$V^1 = \pi_{0'}\, d\tilde{\zeta} + \pi_{1'}\, dv$$

$$- \sum_{n=2}^{\infty} \left(\frac{\partial b_n}{\partial u}\, du + \frac{\partial b_n}{\partial \zeta}\, d\zeta \right) \frac{(\pi_{1'})^n}{(\pi_{0'})^{n-1}}. \quad (10)$$

Assume $d\zeta \neq 0$. Then, for a null vector, $(V^0(\pi_{A'}), V^1(\pi_{A'})) = (0, 0)$ for some $\pi_{A'}$. We must in fact have a zero at $(\pi_{0'}, \pi_{1'}) = (-d\zeta, du)$. But then $V^1 = 0$ gives

$$0 = -d\zeta d\tilde{\zeta} + du\, dv - \sum_{n=2}^{\infty} \frac{\partial b_n}{\partial u}\frac{(du)^{n+1}}{(d\zeta)^{n-1}}(-1)^{n-1}$$

$$- \sum_{n=2}^{\infty} \frac{\partial b_n}{\partial \zeta}\frac{(du)^n}{(d\zeta)^{n-2}}(-1)^{n-1},$$

$$0 = du\, dv - d\zeta d\tilde{\zeta} + \frac{\partial b_2}{\partial \zeta}\, du^2$$

$$+ \sum_{n=2}^{\infty}(-1)^n \left(\frac{\partial b_n}{\partial u} - \frac{\partial b_{n+1}}{\partial \zeta} \right) \frac{(du)^{n+1}}{(d\zeta)^{n-1}}.$$

But, recalling that $h(u + \xi\zeta, 1, \xi) = \sum_{n=-\infty}^{\infty} b_n(u, \zeta)\xi^n$, we conclude $\partial b_n/\partial u = \partial b_{n+1}/\partial \zeta$. Thus the power series vanishes and the conformal metric is given by

$$ds^2 = \kappa \left(du\, dv - d\zeta d\tilde{\zeta} + \frac{\partial b_2}{\partial \zeta}\, du^2 \right), \quad (11)$$

where κ is an arbitrary nonzero holomorphic function. We now show that the actual metric on $\mathcal{G}$ as defined by Penrose is obtained by taking $\kappa = 2$. Let

$$X := \frac{\partial}{\partial u} - \frac{\partial b_2}{\partial \zeta}\frac{\partial}{\partial v}, \quad Y := \frac{\partial}{\partial v}. \quad (12)$$

X and Y define sections as in (10). For X we have

$$X^0 = \pi_{0'},$$

$$X' = -\pi_{1'}\frac{\partial b_2}{\partial \zeta} - \sum_{n=2}^{\infty}\frac{\partial b_n}{\partial u}(u, \zeta)\frac{(\pi_{1'})^n}{(\pi_{0'})^{n-1}}. \quad (13)$$

For Y we have

$$Y^0 = 0, \quad Y^1 = \pi_{1'}. \quad (14)$$

The section X^A vanishes at $\pi_{A'} = (0, 1)$ while Y^A vanishes at $\pi_{A'} = (1, 0)$. That $X^A(0, 1) = 0$ is not evident in (13), but one must remember "unhatted" coordinates are not valid for $\pi_{0'} = 0$.

Let $x_{A'} = (0, 1)$, $y_{A'} = (1, 0)$. Now $x^{A'} = (1, 0)$, $y^{A'} = (0, -1)$. Then the Penrose inner product of X and Y is

$$g(X, Y) = \frac{x_{A'}y^{A'}\mu(X^A, Y^A)}{(x_{A'}\pi^{A'})(y_{A'}\pi^{A'})},$$

$$x_{A'}y^{A'} = -1, \quad x_{A'}\pi^{A'} = -\pi_{0'}, \quad y_{A'}\pi^{A'} = \pi_{1'}$$

and

$$\mu(X^A(\pi), Y^A(\pi)) = X^0(\pi)Y^1(\pi) - X^1(\pi)Y^0(\pi) = \pi_{0'}\pi_{1'}.$$

So $g(X, Y) = (-1)\pi_{0'}\pi_{1'}/-\pi_{0'}\lambda_{1'} = 1$. On the other hand, if we simply substitute X and Y into (11), to obtain $g(X, Y) = 1$ we find that we must take $\kappa = 2$, as asserted.

Now let $f(u, \zeta)$ be any entire function on C^2. We claim we can choose h so that $\partial b_2(u, \zeta)/\partial \zeta = f(u, \zeta)$. If so, then we will have generated all metrics of the form

$$ds^2 = du\, dv - d\zeta d\tilde{\zeta} + f(u, \zeta)du^2. \quad (15)$$

It is enough to show we can choose h so as to make $b_2(u, \zeta) = g(u, \zeta)$, where g is a given entire function. Write

$$g(u, \zeta) = \sum_{n, m \geq 0} a_{mn}u^m\zeta^n,$$

where the series converges everywhere. Define

$$\phi(x, y) := \sum_{n, m \geq 0} \frac{a_{mn}}{\binom{m+n}{n}}\, x^m y^n.$$

Then ϕ is an entire function of x, y. We then write,

$$h(\omega^0, \pi_{0'}, \pi_{1'}) := \phi\left(\frac{\omega^0}{\pi_{0'}}, \frac{\omega^0}{\pi_{1'}} \right)\frac{(\pi_{1'})^2}{\pi_{0'}}.$$

Clearly h is holomorphic on $C \times (D \cap \hat{D})$, and h is homogeneous of degree 1:

$$h(u + \zeta\xi, 1, \xi) = \phi\left(\frac{u + \zeta\xi}{1}, \frac{u + \zeta\xi}{\xi} \right)\xi^2$$

$$= \sum_{n, m \geq 0} \frac{a_{mn}}{\binom{m+n}{n}}\frac{(u + \zeta\xi)^{m+n}}{\xi^{n-2}},$$

$$\frac{(u + \zeta\xi)^{m+n}}{\xi^{n-2}} = \sum_{k=0}^{m+n}\binom{m+n}{k}u^{m+n-k}\zeta^k\xi^{k-n+2}$$

so when everything is expanded in powers of ξ for fixed u and ζ the coefficient of ξ^2 is $\sum_{n, m \geq 0}a_{mn}u^m\zeta^n = g(u, \zeta)$ as desired. For the metric (15) one can directly show that the Ricci tensor vanishes; whether the Weyl tensor is self-dual or anti-self-dual depends on the choice of complex volume element ϵ_{abcd}. There is on $\mathcal{G}$ a natural

choice of ϵ due to the existence of a natural spinor structure on $\mathcal{G}$. In the coordinates $(z^\mu) = (u, v, \xi, \tilde{\xi})$ used above, ϵ is specified by $\epsilon_{0123} = -i$. With this choice the space is right-flat, i.e., $*C_{abcd} = -iC_{abcd}$.

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On the computation formulas of the SO(n-1,1) representation matrix elements

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The formulas for computing the boost matrix elements are found for all classes of the unitary irreducible representations of $SO(n-1,1)$ by defining the invariant scalar product in the space consisting of the D functions of $SO(n-1)$ and assuming that functions with some property exist for the complementary series. The normalization constants of the bases are completely determined by requiring that the boost matrix elements in the finite transformations agree with those obtained by the method of the infinitesimal operators in the infinitesimal transformations.

1. INTRODUCTION

The representation theory of $SO(n)$ is important for mathematical physics and used in many branches of physics. The matrix elements of the infinitesimal generators of the representation of $SO(n)$ have been known since Gel'fand and Tsetlin,[1] whose bases are classified by the group chain $SO(n) \supset SO(n-1) \supset \cdots \supset SO(2)$. The D matrix elements[2] are given in terms of the raising and the lowering operators of Pang and Hecht[3] but in general it is hard to give explicit expressions and study their properties from these. On the other hand, the formulas[4-6] for computing the d matrix elements is given in a form with a single integral, over a compact domain, of two matrix elements of $SO(n-1)$ and a multiplier. This formula is convenient for computing the matrix elements explicitly and examining their properties.

The representation theory of $SO(n-1,1)$ is also important for mathematical physics and for clarifying the unitary representations of a noncompact group. The classification of the unitary irreducible representations (UIR's) of $SO(n-1,1)$ is made by means of the infinitesimal method by many authors.[7-9] For the $SO(n-1,1)$ finite transformations, the computation formula for the matrix elements of the UIR is known only for the principal series[4-6] and seems not to be known for many other classes, which will be hereafter called "complementary series," except for low n cases.[10,11]

In the case of the principal series, the scalar product of two functions in the representation space is defined by the integral of the product of these functions, one of which is taken to be complex conjugate, and then the computation formula is obtained. For the complementary series, we must find the scalar product relative to which the representations become unitary. Many possibilities are considered to define the scalar product. However, in the case of the Lorentz group $(n=4)$, we know the scalar product[11] such that the representation is unitary for the complementary series. This suggests that even in the general case a scalar product can be defined in a similar way but will contain a function by which the complementary series are classified. If we assume the existence of such a function subjected to some condition, it is possible to define the scalar product relative to which the representations of $SO(n-1,1)$ is unitary and to construct the computation formulas for the ma-

trix elements of the UIR's. The scalar product defined in this article is to be considered as a generalization of that in Ref. 11.

In Sec. 2, we summarize the results for the groups $SO(n)$ and $SO(n-1,1)$. In Sec. 3, it is shown that we can construct the computation formulas for the matrix elements of the UIR's of $SO(n-1,1)$ by defining the scalar product in the space of the D functions of $SO(n-1)$. In Sec. 4, it is shown that the complex numbers ρ_n introduced in the infinitesimal and finite methods denote the same one. In Sec. 5, the matrix elements for $n=3$ and 4 are given as special cases of our formulas.

2. GROUPS SO (n) AND SO (n - 1, 1)

In this section, the results for the compact and noncompact groups $SO(n)$ and $SO(n-1,1)$ are summa rized.[1,5,9] Hereafter, we use the same symbol $SO(n)$ $[SO(n-1,1)]$ for both the group $SO(n)$ $[SO(n-1,1)]$ and its twofold covering group.

The generators D_{jk} $(=-D_{kj})$ of the representation of $SO(n)$ obey the commutation relations

$$[D_{jk}, D_{lm}] = i(\delta_{jl}D_{km} + \delta_{km}D_{jl} - \delta_{jm}D_{kl} - \delta_{kl}D_{jm}), \quad (2.1)$$

where the D_{jk}'s are Hermitian. The corresponding generators fo $SO(n-1,1)$ satisfy the relations (2.1) in which we replace the Kronecker δ's by the metric tensor g's which take the values, $g_{11} = g_{22} = \cdots = g_{n-1\,n-1} = -g_{nn} = 1$ and $g_{jk} = 0$ for $j \neq k$.

We introduce the Gel'fand and Tsetlin bases[1] of the UIR's of $SO(n)$ which are classified by the canonical chain of subgroups $SO(n) \supset SO(n-1) \supset \cdots \supset SO(2)$. They are given by

$$|m_{jk}\rangle = |\lambda_n, \lambda_{n-1}, \ldots, \lambda_2\rangle, \quad (2.2)$$

where λ_j stands for the row $(m_{j1}, m_{j2}, \ldots, m_{j[j/2]})$, all λ_j are written in a row, and $[j/2]$ is the largest integral part of $j/2$. The UIR of $SO(n)$ is characterized by λ_n. The numbers m_{jk} are simultaneously integers or half-integers and are restricted by the conditions

$$m_{2j+1\,i+1} \leq m_{2j\,i} \leq m_{2j+1\,i} \quad (i=1,2,\ldots,j-1),$$

$$m_{2j\,i+1} \leq m_{2j-1\,i} \leq m_{2j\,i} \quad (i=2,3,\ldots,j-1),$$

$$(2.3)$$

$$|m_{2j\ j}| \le m_{2j-1\ j-1} \le m_{2j\ j-1},$$

$$|m_{2j\ j}| \le m_{2j+1\ j}.$$

The action of the generators $D_{j\ j+1}$ $(1 \le j \le n-1)$ on the bases (2.2) is given by[1]

$$D_{2k\ 2k+1}|m_{ij}\rangle = \sum_{j=1}^{k} A(m_{2k\ j})|m_{2k\ j}+1\rangle$$

$$-\sum_{j=1}^{k} A(m_{2k\ j}-1)|m_{2k\ j}-1\rangle, \qquad (2.4a)$$

$$D_{2k-1\ 2k}|m_{ij}\rangle = \sum_{j=1}^{k-1} B(m_{2k-1\ j})|m_{2k-1\ j}+1\rangle$$

$$-\sum_{j=1}^{k-1} B(m_{2k-1\ j}-1)|m_{2k-1\ j}-1\rangle + C_{2k}|m_{ij}\rangle. \qquad (2.4b)$$

In these equations, the matrix elements A, B, and C are given by

$$A(m_{2k\ j}) = -\frac{i}{2}\Big\{\prod_{i=1}^{k-1}[(l_{2k-1\ i}-\tfrac{1}{2})^2 - (l_{2k\ j}+\tfrac{1}{2})^2]$$

$$\times\prod_{i=1}^{k}[(l_{2k+1\ i}-\tfrac{1}{2})^2 - (l_{2k\ j}+\tfrac{1}{2})^2]\Big\}^{1/2}$$

$$\times\Big\{\prod_{i=1}^{k}{}'(l_{2k\ i}^2 - l_{2k\ j}^2)[l_{2k\ i}^2 - (l_{2k\ j}+1)^2]\Big\}^{-1/2}, \qquad (2.5a)$$

$$B(m_{2k-1\ j}) = -i\Big[\prod_{i=1}^{k-1}(l_{2k-2\ i}^2 - l_{2k-1\ j}^2)\prod_{i=1}^{k}(l_{2k\ i}^2 - l_{2k-1\ j}^2)\Big]^{1/2}$$

$$\times\Big\{l_{2k-1\ j}^2(4l_{2k-1\ j}^2 - 1)\prod_{i=1}^{k-1}{}'(l_{2k-1\ i}^2 - l_{2k-1\ j}^2)$$

$$\times[(l_{2k-1\ i}-1)^2 - l_{2k-1\ j}^2]\Big\}^{-1/2}, \qquad (2.5b)$$

$$C_{2k} = \prod_{i=1}^{k-1}l_{2k-2\ i}\prod_{i=1}^{k}l_{2k\ i}\Big[\prod_{j=1}^{k-1}l_{2k-1\ j}(l_{2k-1\ j}-1)\Big]^{-1}, \qquad (2.5c)$$

where the prime on $\prod'$ means the product of factors with $i\ne j$ and l_{jk} is defined by

$$l_{jk} = m_{jk} + [(j+1)/2] - k. \qquad (2.6)$$

Similarly, the bases for $SO(n-1, 1)$ are given by the canonical chain $SO(n-1, 1) \supset SO(n-1) \supset \cdots SO(2)$[1][2] and can be written in the form (2.2) in which all the numbers m_{jk} except for m_{n1} $(=\rho_n)$ are subject to the conditions (2.3) (note that for some classes other conditions will be imposed[8][9]). The UIR's of $SO(n-1,1)$ are classified by the values of the numbers m_{nj} $(j \ge 2)$ and the complex number ρ_n which will also be written as $\rho_n = (2-n)/2 + \sigma_n + i\nu_n$, σ_n and ν_n real. The classification of the UIR's of $SO(n-1, 1)$ can be tabulated,[7-9] but we do not do that here.

The action of the generators $D_{j\ j+1}(1 \le j \le n-2)$ on the bases of $SO(n-1, 1)$ and their matrix elements are the same as in $SO(n)$, i.e., they are given by (2.4) and (2.5). The action of $D_{n-1\ n}$ on the bases of $SO(n-1, 1)$ is given by the same form as (2.4) and their matrix elements A, B, and C are obtained from (2.5) by the following replacements:

A, B, $C \to -iA$, $-iB$, $-iC$,

$l_{n1} - \tfrac{1}{2} \to \sigma_n + i\nu_n$ in (2.5a) for n odd, $\qquad (2.7)$

$l_{n1} \to \sigma_n + i\nu_n$ in (2.5b), (2.5c) for n even,

and the other quantities are not changed. It is noted that the matrix elements A and B contain $\sigma_n + i\nu_n$ as squared, but C depends linearly on $\sigma_n + i\nu_n$ and $l_{n\ n/2}$ $(= m_{n\ n/2})$.

An element of $SO(n)$ can be parametrized by the Euler angles as follows[2][5][6]:

$$g^{(n)} = g^{(n-1)}S_n,$$

$$S_n = \Big\{\prod_{j=n}^{4} t_{jj-1}^{(n)}(\theta_{n\ n-j+1})\Big\} t_{31}^{(n)}(\theta_{n\ n-2}) t_{12}^{(n)}(\theta_{n\ n-1}), \qquad (2.8)$$

$$g^{(2)} = t_{12}^{(2)}(\theta_{21}),$$

where

$$0 \le \theta_{jk} \le \pi, \quad k = 1, 2, \ldots, j-2,$$

$$0 \le \theta_{j\ j-1} < 2\pi, \quad j = 2, 3, \ldots, n,$$

and $t_{jk}^{(p)}(\theta)$ is a rotation through θ in the (j, k) plane and can be expressed with a $n \times n$ matrix whose elements are given as follows:

$$(t_{q\ q-1}^{(p)}(\theta))_{q\ q-1} = -(t_{q\ q-1}^{(p)}(\theta))_{q-1\ q} = \sin\theta,$$

$$(t_{q\ q-1}^{(p)}(\theta))_{qq} = (t_{q\ q-1}^{(p)}(\theta))_{q-1\ q-1} = \cos\theta, \qquad (2.9)$$

$$(t_{q\ q-1}^{(p)}(\theta))_{rs} = \delta_{rs} \quad \text{for } r,\ s \text{ besides the above cases.}$$

It is noted that the following relations hold from (2.8) and (2.9):

$$(g^{(n)\prime} g^{(n)-1})_{nn} = \cos\theta'_{n1}\cos\theta_{n1}$$

$$+ \sin\theta'_{n1}\sin\theta_{n1}(g^{(n-1)\prime} g^{(n-1)-1})_{n-1\ n-1},$$

$$(g^{(n)})_{nn} = \cos\theta_{n1}, \qquad (2.10)$$

and the bracket term on the right-hand side does not depend on θ_{n1} and θ'_{n1}.

Similarly, an element of $SO(n-1, 1)$ is parametrized by (2.8) in which $t_{n\ n-1}^{(n)}(\theta_{n1})$ is replaced by ${}^b t_{n\ n-1}^{(n)}(\zeta)$, where ${}^b t_{nn-1}^{(n)}(\zeta)$ is the boost in the $(n-1)$th direction through ζ $(0 \le \zeta < \infty)$ and has elements

$$({}^b t_{n\ n-1}^{(n)}(\zeta))_{n\ n-1} = ({}^b t_{n\ n-1}^{(n)}(\zeta))_{n-1\ n} = -\sinh\zeta,$$

$$({}^b t_{n\ n-1}^{(n)}(\zeta))_{n-1\ n-1} = ({}^b t_{n\ n-1}^{(n)}(\zeta))_{nn} = \cosh\zeta, \qquad (2.11)$$

$$({}^b t_{n\ n-1}^{(n)}(\zeta))_{rs} = \delta_{rs} \quad \text{for } r,\ s \text{ besides the above cases.}$$

Another important parametrization of any element of $SO(n-1, 1)$ is due to Iwasawa,[13] i.e., the element of $SO(n-1, 1)$ can be written uniquely as $g^{(n)} = n(\xi)a(\eta)k(\{\theta\})$, that is,

$$g^{(n)} = \begin{pmatrix} 1 & \xi & \xi \\ -\xi^t & 1-\Xi & -\Xi \\ \xi^t & \Xi & 1+\Xi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\eta & -\sinh\eta \\ 0 & -\sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} & & & 0 \\ & k(\{\theta\}) & & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(2.12)$$

where: $k(\{\theta\})$ $(=g^{(n-1)}) \in SO(n-1)$ is the maximal compact subgroup of $SO(n-1, 1)$ parametrized as in (2.8); $a(\eta)$ $[={}^b t_{n\ n-1}^{(n)}(\eta)] \in A$, the Abelian subgroup of $SO(n-1, 1)$; $n(\xi) \in N$, the nilpotent subgroup of $SO(n-1, 1)$, where ξ is the column vector $(\xi_1, \xi_2, \ldots, \xi_{n-2})$, ξ^t its transpose, and $\Xi = (\xi_1^2 + \cdots + \xi_{n-2}^2)/2$. By considering the transformation induced by

$$g^{(n)} \xrightarrow{t} g^{(n)\prime} = g^{(n)}a(\zeta) = n(\xi')a(\eta')k(\{\theta'\}), \qquad (2.13)$$

we obtain

$$\cos\theta'_{n-1\ 1} = \frac{\cos\theta_{n-11}\cosh\zeta - \sinh\zeta}{\cosh\zeta - \cos\theta_{n-11}\sinh\zeta}, \qquad (2.14)$$

$$\theta'_{n-1\ j} = \theta_{n-1\ j} \quad (j \neq 1),$$

$$\exp\eta' = \exp\eta(\cosh\zeta - \cos\theta_{n-1\ 1}\sinh\zeta).$$

The invariant measure of $SO(n)$ is given as follows:

$$dV_n = dV_{n-1}dS_n, \quad dV_2 = d\theta_{21},$$

$$dS_n = \prod_{j=1}^{n-1}\{(\sin\theta_{nj})^{n-j-1}d\theta_{nj}\}, \qquad (2.15)$$

where dS_n is the surface element of a sphere in an n-dimensional space. The volume V_n of $SO(n)$ is given by $V_n = V_{n-1}2\pi^{n/2}/\Gamma(n/2)$ and $V_2 = 2\pi$.

The representation D matrix of $SO(n)$ corresponding to the rotation (2.8) is given by[2,5,6]

$$D(g^{(n)}) = D(\{\theta_n\}) = D(\{\theta_{n-1}\})H(S_n), \qquad (2.16)$$

where

$$H(S_n) = \{\prod_{j=n}^{4} R_{j\ j-1}(\theta_{n\ n-j+1})\}R_{31}(\theta_{n\ n-2})R_{12}(\theta_{n\ n-1}),$$

and the notation $\{\theta_n\} = (\theta_{21}, \ldots, \theta_{n\ n-1})$ will also be used instead of $g^{(n)}$ in order to show the arguments explicitly and $R_{j\ j-1}(\theta)$ is the representation matrix corresponding to the rotation $t_{j\ j-1}(\theta)$. The representation D matrix elements of $SO(n)$ are defined through

$$D_{\{\lambda'_{n-1}\}\{\lambda_{n-1}\}}^{(\lambda_n)}(\{\theta_n\}) = \langle \lambda_n\{\lambda'_{n-1}\} | D(g^{(n)}) | \lambda_n\{\lambda_{n-1}\}\rangle, \qquad (2.17)$$

where the notations such as $\{\lambda_{n-1}\} = (\lambda_{n-1}, \lambda_{n-2}, \ldots, \lambda_2)$ are used. The corresponding D matrix of $SO(n-1, 1)$ is obtained from (2.16) by substituting $R_{n\ n-1}^b(\zeta)$ for $R_{n\ n-1}(\theta)$, where $R_{n\ n-1}^b(\zeta)$ is the representation matrix corresponding to ${}^b t_n^{(n)}{}_{n-1}(\zeta)$, and its matrix elements are defined by sandwiching the D matrix between the bases of $SO(n-1, 1)$. It, therefore, follows that in order to calculate the D matrix elements it is sufficient for us to know the d matrix elements (the boost matrix elements) provided that those of $SO(n-1)$ are known. The d matrix elements of $SO(n)$ and the boost matrix elements of $SO(n-1, 1)$ are defined as follows:

$$d_{\lambda_{n-1}^\prime(\lambda_{n-2})\lambda_{n-1}}^{(\lambda_n)}(\theta) = \langle \lambda_n\lambda'_{n-1}\{\lambda_{n-2}\} | R_{n\ n-1}(\theta) | \lambda_n\lambda_{n-1}\{\lambda_{n-2}\}\rangle, \qquad (2.18)$$

$${}^b d_{\lambda_{n-1}^\prime(\lambda_{n-2},\rho_n)\lambda_{n-1}}^{(\Lambda_{n-2},\rho_n)}(\zeta) = \langle \Lambda_{n-2}, \rho_n; \lambda'_{n-1}\{\lambda_{n-2}\} | R_{n\ n-1}^b$$

$$\times (\zeta) | \Lambda_{n-2}, \rho_n; \lambda_{n-1}\{\lambda_{n-2}\}\rangle, \qquad (2.19)$$

where Λ_{n-2} stands for $(m_{n2}, m_{n3}, \ldots, m_{n[n/2]})$.

The orthogonality and the completeness relations for the D matrix elements of $SO(n)$ are given as follows[5,14]:

$$\int_{SO(n)} dV_n \overline{D_{\{\lambda_{n-1}\}\{\lambda'_{n-1}\}}^{(\lambda_n)}(g^{(n)})} D_{\{\lambda''_{n-1}\}\{\lambda'''_{n-1}\}}^{(\lambda'_n)}(g^{(n)})$$

$$= \delta_{\{\lambda_n\}\{\lambda'_n\}}\delta_{\{\lambda''_{n-1}\}\{\lambda'''_{n-1}\}} V_n/N(\lambda_n), \qquad (2.20a)$$

$$\sum_{\lambda_n}\frac{N(\lambda_n)}{V_n}\sum_{\{\lambda_{n-1}\}\{\lambda'_{n-1}\}} \overline{D_{\{\lambda_{n-1}\}\{\lambda'_{n-1}\}}^{(\lambda_n)}(g^{(n)})} D_{\{\lambda_{n-1}\}\{\lambda'_{n-1}\}}^{(\lambda_n)}(g^{(n)\prime})$$

$$= \delta_{(n)}(\{\theta_n\}, \{\theta'_n\}), \qquad (2.20b)$$

where $\delta_{\{\lambda\}\{\lambda'\}}$ stands for a product of Kronecker δ's in the individual indices and $N(\lambda_n)$ is the dimension of the UIR of $SO(n)$.[4] The expression for $\delta_{(n)}(\cdot,\cdot)$ is given by

$$\delta_{(n)}(\{\theta_n\}, \{\theta'_n\}) = \delta_{(n-1)}(\{\theta_{n-1}\}, \{\theta'_{n-1}\})\{\prod_{j=1}^{n-1}(\sin\theta_{nj})^{j-n+1}\delta(\theta_{nj}-\theta'_{nj})\},$$

$$(2.21)$$

$$\delta_{(2)}(\{\theta_2\}, \{\theta'_2\}) = \delta(\theta_{21} - \theta'_{21}).$$

The following relations are obtained from (2.20) by taking into account (2.16), (2.17), and (2.18)[5]:

$$\sum_{\{\lambda_{n-2}\}}\int_{S_n} dS_n \overline{H_{\{\lambda_{n-1}\}\{\lambda'_{n-1}\}}^{(\lambda_n)}(S_n)} H_{\{\lambda_{n-1}\}\{\lambda''_{n-1}\}}^{(\lambda'_n)}(S_n)$$

$$= \delta_{\{\lambda'_{n-1}\}\{\lambda''_{n-1}\}}\delta_{\lambda_n\lambda'_n}\frac{V_nN(\lambda_{n-1})}{V_{n-1}N(\lambda_n)}, \qquad (2.22)$$

$$\sum_{\lambda_{n-2}}N(\lambda_{n-2})\int_0^\pi d\theta \sin^{n-2}\theta \overline{d_{\lambda_{n-1}^\prime(\lambda_{n-2})\lambda'_{n-1}}^{(\lambda_n)}(\theta)} d_{\lambda_{n-1}^\prime(\lambda_{n-2})\lambda'_{n-1}}^{(\lambda'_n)}(\theta)$$

$$= \delta_{\lambda_n\lambda'_n}\frac{\sqrt{\pi}\ \Gamma((n-1)/2)N(\lambda_{n-1})N(\lambda'_{n-1})}{\Gamma(n/2)N(\lambda_n)}, \qquad (2.23)$$

$$\sum_{\lambda_n}N(\lambda_n) \overline{d_{\lambda_{n-1}^\prime(\lambda_{n-2})\lambda'_{n-1}}^{(\lambda_n)}(\theta)} d_{\lambda_{n-1}^\prime(\lambda_{n-2})\lambda'_{n-1}}^{(\lambda_n)}(\theta')$$

$$= \delta_{\lambda_{n-2}\lambda'_{n-2}}\frac{\delta(\theta - \theta')\sqrt{\pi}\ \Gamma((n-1)/2)N(\lambda_{n-1})N(\lambda'_{n-1})}{(\sin\theta)^{n-2}\Gamma(n/2)N(\lambda_{n-2})}. \quad (2.24)$$

3. COMPUTATION FORMULAS FOR BOOST MATRIX ELEMENTS

In this section, the formulas for computing the boost matrix elements are constructed for all classes of the UIR's of $SO(n-1, 1)$ by defining the invariant scalar product in the space of the representation matrix elements of $SO(n-1)$. We discuss the cases of the principal and the complementary series separately.

A. Principal series

The formula for the boost matrix elements is given in terms of the integral of the d matrix elements of $SO(n-1)$ and that for the d matrix elements is obtained

by analytic continuation.[4-6] We explain briefly the derivation of the formula because some results are needed for the derivation of the formulas for the complementary series.

We consider a linear space consisting of the D matrix elements of the UIR of SO$(n-1)$ and define the scalar product in the space as follows:

$$\langle \Phi_1, \Phi_2 \rangle = \int_{SO(n-1)} dV_{n-1} \, \overline{\Phi_1(g^{(n-1)})} \Phi_2(g^{(n-1)}), \qquad (3.1)$$

where Φ_1 and Φ_2 belong to the space. The orthogonal bases are given by

$$\Phi_{\{\lambda_{n-1}\}}^{(\Lambda_{n-2}, \rho_n)}(g^{(n-1)}) = \sqrt{N(\lambda_{n-1})/V_{n-1}} \, N(\rho_n; \lambda_{n-1}) D_{\{\Lambda_{n-2}\}\{\lambda_{n-2}\}}^{(\lambda_{n-1})}(g^{(n-1)}), \qquad (3.2)$$

where $\{\Lambda_{n-2}\}$ denotes $(\Lambda_{n-2}, \Lambda_{n-3}, \ldots, \Lambda_2)$ in which Λ_{n-2} is defined in (2.19) and Λ_j $(2 \le j \le n-3)$ stand for some fixed $(m_{j1}, \ldots, m_{j[j/2]})$. The magnitude of the constant $N(\rho_n; \lambda_{n-1})$ is determined by the normalization of the base (3.2). That is, we obtain from (2.2a)

$$|N(\rho_n; \lambda_{n-1})|^2 = 1. \qquad (3.3)$$

It is noted that we must take into account the phase factor in (3.3) as shown below.

If we consider (2.13) and (2.14), it is expected that the action of the representation operator $R_{n\ n-1}^b(\zeta)$, which corresponds to the boost $^b t_{n\ n-1}^{(n)}(\zeta)$, on the bases (3.2) can be put into the following form:

$$R_{n\ n-1}^b(\zeta)\Phi_{\{\lambda_{n-1}\}}^{(\Lambda_{n-2}, \rho_n)}(g^{(n-1)})$$

$$= (\cosh\zeta - \cos\theta_{n-1\ 1}\sinh\zeta)^{\rho_n}$$

$$\times \Phi_{\{\lambda_{n-1}\}}^{(\Lambda_{n-2}, \rho_n)}(\{\theta_{n-2}\}, (\theta'_{n-1\ 1}, \theta_{n-1\ 2}, \ldots, \theta_{n-1\ n-2})),$$

$$\qquad (3.4)$$

where

$$\cos\theta'_{n-1\ 1} = \frac{\cos\theta_{n-1\ 1}\cosh\zeta - \sinh\zeta}{\cosh\zeta - \cos\theta_{n-1\ 1}\sinh\zeta}.$$

Then it is easy to show that the condition of the representation is satisfied for any complex number ρ_n, i.e.,

$$R_{n\ n-1}^b(\zeta_1 + \zeta_2)\Phi_{\{\lambda_{n-1}\}}^{(\Lambda_{n-2}, \rho_n)}(g^{(n-1)})$$

$$= R_{n\ n-1}^b(\zeta_1) R_{n\ n-1}^b(\zeta_2)\Phi_{\{\lambda_{n-1}\}}^{(\Lambda_{n-2}, \rho_n)}(g^{(n-1)}). \qquad (3.5)$$

However, the unitarity condition relative to the scalar product (3.1) is satisfied only for $\rho_n = (2-n)/2 + i\nu_n$, ν_n real, i.e.,

$$\langle R_{n\ n-1}^b(\zeta)\Phi_{\{\lambda_{n-1}\}}^{(\Lambda_{n-2}, \rho_n)}, R_{n\ n-1}^b(\zeta)\Phi_{\{\lambda_{n-1}\}}^{(\Lambda_{n-2}, \rho_n)} \rangle$$

$$= \langle \Phi_{\{\lambda_{n-1}\}}^{(\Lambda_{n-2}, \rho_n)}, \Phi_{\{\lambda_{n-1}\}}^{(\Lambda_{n-2}, \rho_n)} \rangle. \qquad (3.6)$$

It, therefore, follows that the unitary representation of SO$(n-1, 1)$ is realized only for $\rho_n = (2-n)/2 + i\nu_n$ and this is the principal series of the UIR.[4,5]

The boost matrix elements of the representation are calculated through

$$^b d_{\lambda'_{n-1}\{\lambda_{n-2}\}\lambda_{n-1}}^{(\Lambda_{n-2}, \nu_n)}(\zeta) = \langle \Phi_{\lambda'_{n-1}\{\lambda_{n-2}\}}^{(\Lambda_{n-2}, \nu_n)}, R_{n\ n-1}^b(\zeta)\Phi_{\{\lambda_{n-1}\}}^{(\Lambda_{n-2}, \nu_n)} \rangle. \qquad (3.7)$$

Making use of the relations (2.8), (2.17), (2.20a), and (2.22), we reduce (3.7) to the form

$$^b d_{\lambda'_{n-1}\{\lambda_{n-2}\}\lambda_{n-1}}^{(\Lambda_{n-2}, \nu_n)}(\zeta) = \frac{\Gamma((n-1)/2)}{\sqrt{\pi}\Gamma((n-2)/2)} \frac{\sqrt{N(\lambda_{n-1})N(\lambda'_{n-1})}}{N(\Lambda_{n-2})N(\Lambda_{n-2})}$$

$$\times \frac{N(\nu_n; \lambda_{n-1})}{N(\nu_n; \lambda'_{n-1})} \sum_{\lambda_{n-3}} N(\lambda_{n-3})$$

$$\times \int_0^\pi d\theta \sin^{n-3}\theta \, \overline{d_{\Lambda_{n-2}(\lambda_{n-3})\lambda_{n-2}}^{(\lambda_{n-1})}(\theta)}$$

$$\times (\cosh\zeta - \cos\theta\sinh\zeta)^{(2-n)/2 + i\nu_n}$$

$$\times d_{\Lambda_{n-2}(\lambda_{n-3})\lambda_{n-2}}^{(\lambda'_{n-1})}(\theta'). \qquad (3.8)$$

In order to determine the constant $N(\nu_n; \lambda_{n-1})$, we consider the infinitesimal transformation with respect to ζ, i.e.,

$$^b d_{\lambda'_{n-1}\{\lambda_{n-2}\}\lambda_{n-1}}^{(\Lambda_{n-2}, \nu_n)} = \delta_{\lambda'_{n-1}\lambda_{n-1}}$$

$$+ i\zeta \langle \Lambda_{n-2}, \nu_n; \lambda'_{n-1}\{\lambda_{n-2}\} | D_{n\ n-1} | \Lambda_{n-2}, \nu_n; \{\lambda_{n-1}\} \rangle$$

$$+ \cdots,$$

$$(\cosh\zeta - \cos\theta\sinh\zeta)^{(2-n)/2 + i\nu_n} \, i\nu_n \, d_{\Lambda_{n-2}(\lambda_{n-3})\lambda_{n-2}}^{(\lambda_{n-1})}(\theta')$$

$$= d_{\Lambda_{n-2}(\lambda_{n-3})\lambda_{n-2}}^{(\lambda_{n-1})}(\theta) - \zeta\left\{\left(\frac{2-n}{2} + i\nu_n\right)\cos\theta + \sin\theta \frac{d}{d\theta}\right\}$$

$$\times d_{\Lambda_{n-2}(\lambda_{n-3})\lambda_{n-2}}^{(\lambda_{n-1})}(\theta) + \cdots.$$

Substituting these into (3.8) and taking into account (2.23), we obtain the relation from the coefficient of ζ,

$$i\langle \Lambda_{n-2}, \nu_n; \lambda'_{n-1}\{\lambda_{n-2}\} | D_{n\ n-1} | \Lambda_{n-2}, \nu_n; \{\lambda_{n-1}\} \rangle$$

$$= -\frac{\Gamma((n-1)/2)}{\sqrt{\pi}\Gamma((n-2)/2)} \frac{\sqrt{N(\lambda_{n-1})N(\lambda'_{n-1})}}{N(\Lambda_{n-2})N(\Lambda_{n-2})} \qquad (3.9)$$

$$\times \frac{N(\nu_n; \lambda_{n-1})}{N(\nu_n; \lambda'_{n-1})}\left[\left(\frac{2-n}{2} + i\nu_n\right)I_{\Lambda_{n-2}\lambda_{n-2}}^{(\lambda'_{n-1}, \lambda_{n-1})} + T_{\Lambda_{n-2}\lambda_{n-2}}^{(\lambda'_{n-1}, \lambda_{n-1})}\right],$$

where

$$I_{\Lambda_{n-2}\lambda_{n-2}}^{(\lambda'_{n-1}, \lambda_{n-1})} = \sum_{\lambda_{n-3}} N(\lambda_{n-3})\int_0^\pi d\theta \sin^{n-3}\theta \cos\theta \, \overline{d_{\Lambda_{n-2}(\lambda_{n-3})\lambda_{n-2}}^{(\lambda'_{n-1})}(\theta)}$$

$$\times d_{\Lambda_{n-2}(\lambda_{n-3})\lambda_{n-2}}^{(\lambda_{n-1})}(\theta),$$

$$T_{\Lambda_{n-2}\lambda_{n-2}}^{(\lambda'_{n-1}, \lambda_{n-1})} = \sum_{\lambda_{n-3}} N(\lambda_{n-3})\int_0^\pi d\theta \sin^{n-3}\theta \sin\theta \, \overline{d_{\Lambda_{n-2}(\lambda_{n-3})\lambda_{n-2}}^{(\lambda'_{n-1})}(\theta)}$$

$$\times \frac{d}{d\theta} d_{\Lambda_{n-2}(\lambda_{n-3})\lambda_{n-2}}^{(\lambda_{n-1})}(\theta).$$

We first consider the case of n odd. Taking into account the matrix elements (2.7) with (2.5a), we obtain the relation from (3.9) for $\lambda'_{n-1} = (m_{n-1\ 1}, \ldots, m_{n-1\ j-1},$

$$m_{n-1\ j} + 1,\ m_{n-1\ j+1}, \ldots, m_{n-1\ (n-1)/2}),$$

$$\frac{N(\nu_n; \lambda'_{n-1})}{N(\nu_n; \lambda_{n-1})}$$

$$= \frac{(2-n)/2 + i\nu_n + l_1}{[(m_{n-1\ j} - i\nu_n + n/2 - j)(m_{n-1\ j} + i\nu_n + n/2 - j)]^{1/2}} I_2,$$

$$(3.10)$$

where I_1 and I_2 are some quantities which are independent of ν_n. As the magnitude on the left-hand side of (3.10) is unity due to (3.3) and the two quantities in the denominator on the right-hand side are complex conjugate to each other, we obtain the two cases of the solution for $m'_{n-1\ j} = m_{n-1\ j} + 1$ and $m'_{n-1\ l} = m_{n-1\ l}$ $(l \neq j)$

Case 1:

$$\frac{N(\nu_n; \lambda'_{n-1})}{N(\nu_n; \lambda_{n-1})} = \epsilon \left(\frac{m_{n-1\ j} - i\nu_n + n/2 - j}{m_{n-1\ j} + i\nu_n + n/2 - j} \right)^{1/2} \qquad (3.11)$$

Case 2:

$$\frac{N(\nu_n; \lambda'_{n-1})}{N(\nu_n; \lambda_{n-1})} = \epsilon' \left(\frac{m_{n-1\ j} + i\nu_n + n/2 - j}{m_{n-1\ j} - i\nu_n + n/2 - j} \right)^{1/2} \qquad (3.12)$$

Here ϵ and ϵ' are some constants of the unit magnitude which do not depend on ν_n. Taking into account the restrictions (2.3), we obtain from (3.11) and (3.12) for any $m_{n-1\ j}$ and $m'_{n-1\ j}$,

$$\frac{N(\nu_n; \lambda_{n-1})}{N(\nu_n; \lambda'_{n-1})}$$

$$= \epsilon \left[\prod_{j=1}^{[(n-1)/2]} \frac{\Gamma(m_{n-1\ j} - i\nu_n + n/2 - j)\Gamma(m'_{n-1\ j} + i\nu_n + n/2 - j)}{\Gamma(m_{n-1\ j} + i\nu_n + n/2 - j)\Gamma(m'_{n-1\ j} - i\nu_n + n/2 - j)} \right]^{1/2},$$

$$(3.13)$$

for Case 1, and

$$\frac{N(\nu_n; \lambda_{n-1})}{N(\nu_n; \lambda'_{n-1})} = \epsilon' [\text{complex conjugate of } (3.13)], \qquad (3.14)$$

for Case 2. It follows that (3.13) and (3.14) are complex conjugate to each other up to a phase factor. We may, therefore, choose one of them without loss of generality because we can start with (3.2) or its complex conjugate, and the representations with (Λ_{n-2}, ν_n) and $(\Lambda'_{n-2}, -\nu_n)$ are unitarily equivalent, where $\Lambda'_{n-2} = (m_{n2}, \ldots, m_{n\ (n-1)/2})$ for n odd and $\Lambda'_{n-2} = (m_{n2}, \ldots, m_{n\ (n-2)/2}, -m_{nn/2})$ for n even. In what follows, we adopt (3.13) which agrees with that in Ref. 4. Similarly, for n even we can obtain the same solution as in (3.13). Thus the normalization constant of (3.2) has been determined up to a phase factor which is equated to unity without loss of generality.

As a by-product of the above result, we obtain useful formulas for the integral containing the d matrix elements. Substituting (3.13) with $\epsilon = 1$ into (3.9) and taking into account the matrix elements (2.7) with (2.5), we obtain the following results:

(i) $n = 2k+1$ odd, $m'_{n-1\ j} = m_{n-1\ j} + 1$ and $m'_{n-1\ l} = m_{n-1\ l}$ for $l \neq j$,

$$I^{(\lambda'_{n-1}, \lambda_{n-1})}_{\Lambda_{n-2}\lambda_{n-2}} = \frac{1}{2} \frac{\sqrt{\pi}\,\Gamma((n-2)/2)}{\Gamma((n-1)/2)} \frac{N(\Lambda_{n-2})N(\lambda_{n-2})}{[N(\lambda_{n-1})N(\lambda'_{n-1})]^{1/2}}$$

$$(3.15)$$

$$\times \left[\frac{\prod_{r=1}^{k-1}\{(l_{n-2\ r} - \tfrac{1}{2})^2 - (l_{n-1\ j} + \tfrac{1}{2})^2\}}{\prod_r^{k'}(l_{n-1\ r}^2 - l_{n-1\ j}^2)\{l_{n-1\ r}^2 - (l_{n-1\ j} + 1)^2\}} \right.$$

$$\left. \times \prod_{s=2}^{k}\{(l_{ns} - \tfrac{1}{2})^2 - (l_{n-1\ j} + \tfrac{1}{2})^2\} \right]^{1/2},$$

$$- T^{(\lambda'_{n-1}, \lambda_{n-1})}_{\Lambda_{n-2}\lambda_{n-2}} = (m'_{n-1\ j} - j)I^{(\lambda'_{n-1}, \lambda_{n-1})}_{\Lambda_{n-2}\lambda_{n-2}}. \qquad (3.16)$$

(ii) $n = 2k$ even,

(a) $m'_{n-1\ j} = m_{n-1\ j}$ for all j,

$$I^{(\lambda_{n-1}, \lambda_{n-1})}_{\Lambda_{n-2}\lambda_{n-2}} = \frac{\sqrt{\pi}\,\Gamma((n-2)/2)N(\Lambda_{n-2})N(\lambda_{n-2})}{\Gamma((n-1)/2)N(\lambda_{n-1})}$$

$$\times \frac{\prod_{r=1}^{k-1} l_{n-2\ r}\prod_{s=2}^{k} l_{ns}}{\prod_{r=1}^{k-1} l_{n-1\ r}(l_{n-1\ r} - 1)}, \qquad (3.17)$$

$$T^{(\lambda_{n-1}, \lambda_{n-1})}_{\Lambda_{n-2}\lambda_{n-2}} = (k-1)I^{(\lambda_{n-1}, \lambda_{n-1})}_{\Lambda_{n-2}\lambda_{n-2}}, \qquad (3.18)$$

(b) $m'_{n-1\ j} = m_{n-1\ j} + 1$ and $m'_{n-1\ l} = m_{n-1\ l}$ for $l \neq j$,

$$I^{(\lambda'_{n-1}, \lambda_{n-1})}_{\Lambda_{n-2}\lambda_{n-2}} = \frac{\sqrt{\pi}\,\Gamma((n-2)/2)}{\Gamma((n-1)/2)} \frac{N(\Lambda_{n-2})N(\lambda_{n-2})}{\sqrt{N(\lambda_{n-1})N(\lambda'_{n-1})}}$$

$$\times \left[\frac{\prod_{r=1}^{k-1}(l_{n-2\ r}^2 - l_{n-1\ j}^2)\prod_{s=2}^{k}(l_{ns}^2 - l_{n-1\ j}^2)}{l_{n-1\ j}^2(4l_{n-1\ j}^2 - 1)\prod_{r=1}^{k-1}{}'(l_{n-1\ r}^2 - l_{n-1\ j}^2)\{(l_{n-1\ r} - 1)^2 - l_{n-1\ j}^2\}} \right]^{1/2},$$

$$(3.19)$$

$$- T^{(\lambda'_{n-1}, \lambda_{n-1})}_{\Lambda_{n-2}\lambda_{n-2}} = (m'_{n-1\ j} - j)I^{(\lambda'_{n-1}, \lambda_{n-1})}_{\Lambda_{n-2}\lambda_{n-2}}. \qquad (3.20)$$

In this way, we can give many integral formulas with respect to the d matrix elements by considering the higher power in ζ. Their expressions are not given for they are not needed in the following discussion.

B. Complementary series

We consider the UIR in the space consisting of the D matrix elements of $SO(n-1)$ as in (A). The bases and the action of the representation $R^b_{n\ n-1}(\zeta)$ on the bases are the same as in (A), i.e.,

$$\Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda_{n-1}\}}(g^{(n-1)}) = [N(\lambda_{n-1})/V_{n-1}]^{1/2}$$

$$\times N(\sigma_n, \lambda_{n-1})D^{(\lambda_{n-1})}_{\{\Lambda_{n-2}\}\{\lambda_{n-2}\}}(g^{(n-1)}), \qquad (3.21)$$

$$R^b_{n\ n-1}(\zeta)\Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda_{n-1}\}}(g^{(n-1)})$$

$$= (\cosh\zeta - \cos\theta_{n-1\ 1}\sinh\zeta)^{(2-n)/2 + \sigma_n}$$

$$\times \Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda_{n-1}\}}(\{\theta_{n-2}\}, (\theta'_{n-1\ 1}, \theta_{n-1\ 2}, \ldots, \theta_{n-1\ n-2})).$$

$$(3.22)$$

It is easily seen that the condition of the representation is satisfied, but the unitarity condition is not satisfied under the scalar product (3.1) defined at (A). We must, therefore, find a scalar product under which the unitarity condition holds. We know the scalar product for the complementary series of the Lorentz group.[11] Contrary to the Lorentz group, we have many classes of

the UIR's for the complementary series. This suggests that there exist some functions by which the UIR's are classified in the general case.

By assuming the existence of such functions, we define a scalar product as follows:

$$\langle \Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda'_{n-1}\}}, \ \Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda_{n-1}\}}\rangle_c$$

$$= \int_{SO(n-1)} \int_{SO(n-1)} dV_{n-1} dV'_{n-1}$$

$$\times [1 - (g^{(n-1)'} g^{(n-1)^{-1}})_{n-1\ n-1}]^{(2-n)/2 - \sigma_n} \ K(g^{(n-1)'} g^{(n-1)^{-1}})$$

$$\times \overline{\Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda'_{n-1}\}}(g^{(n-1)'})} \Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda_{n-1}\}}(g^{(n-1)}). \qquad (3.23)$$

The quantity $[1 - (g^{(n-1)'} g^{(n-1)^{-1}})_{n-1\ n-1}]$ is considered to be a generalization of the kernel in the case of the Lorentz group[11] and depends on the Euler angles as given in (2.8). The function K is new as stated above. We assume that the function exists and does not depend on the angles $\theta_{n-1\,1}$ and $\theta'_{n-1\,1}$. It will, always, be possible to construct such functions for special cases, because we have $(n-1)^2$ elements of $g^{(n-1)'} g^{(n-1)^{-1}}$ and only $(n-1)(n-2)$ Euler angles. It, however, is sufficient for us to assume the existence of such functions in what follows.

The normalization of the bases (3.21) under the scalar product (3.23) is given by

$$\langle \Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda'_{n-1}\}}, \ \Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda_{n-1}\}}\rangle_c = \delta_{\{\lambda'_{n-1}\}\{\lambda_{n-1}\}}, \qquad (3.24)$$

which gives the condition

$$\int_{SO(n-1)} dV_{n-1} [1 - \cos\theta_{n-1\,1}]^{(2-n)/2 - \sigma_n} K(g^{(n-1)}) \overline{D^{(\lambda_{n-1})}_{\{\Lambda_{n-2}\}\{\Lambda_{n-2}\}}(g^{(n-1)})}$$

$$= |N(\sigma_n; \lambda_{n-1})|^{-2},$$

$$(g^{(n-1)})_{n-1\ n-1} = \cos\theta_{n-1\,1}. \qquad (3.25)$$

Here we use the condition of the representation of $SO(n-1)$ and the relation (2.20a) to rewrite (3.24) into (3.25). As the function K does not depend on the angles $\theta_{n-1\,1}$ and $\theta'_{n-1\,1}$, it is easy to see that the unitarity condition relative to the scalar product (3.23) is satisfied, i.e.,

$$\langle R^b_{n\ n-1}(\zeta)\Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda'_{n-1}\}}, \ R^b_{n\ n-1}(\zeta)\Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda_{n-1}\}}\rangle_c$$

$$= \langle \Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda'_{n-1}\}}, \ \Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda_{n-1}\}}\rangle_c. \qquad (3.26)$$

Thus the representation becomes unitary relative to the scalar product (3.23).

The boost matrix elements of the representation are given by

$$^b d^{(\Lambda_{n-2}, \sigma_n)}_{\lambda_{n-1}(\Lambda_{n-2})\lambda_{n-1}}(\zeta) = \langle \Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda'_{n-1}\{\lambda_{n-2}\}\}}, \ R^b_{n\ n-1}(\zeta)\Phi^{(\Lambda_{n-2}, \sigma_n)}_{\{\lambda_{n-1}\}}\rangle_c.$$

$$(3.27)$$

As in the case of (A), this reduces to

$$'d^{(\Lambda_{n-2}, \sigma_n)}_{\lambda'_{n-1}(\Lambda_{n-2})\lambda_{n-1}}(\zeta) = \frac{\Gamma((n-1)/2}{\sqrt{\pi}\,\Gamma((n-2)/2)} \frac{\sqrt{N(\lambda_{n-1})N(\lambda'_{n-1})}}{N(\lambda_{n-2})N(\Lambda_{n-2})}$$

$$\times \frac{N(\sigma_n; \lambda_{n-1})}{N(\sigma_n; \lambda'_{n-1})} \sum_{\lambda_{n-3}} N(\lambda_{n-3})$$

$$\times \int_0^\pi d\theta \sin^{n-3}\theta \ \overline{d^{(\lambda'_{n-1})}_{\Lambda_{n-2}(\lambda_{n-3})\lambda_{n-2}}(\theta)}$$

$$\times (\cosh\zeta - \cos\theta \sinh\zeta)^{(2-n)/2 + \sigma_n}$$

$$\times d^{(\lambda_{n-1})}_{\Lambda_{n-2}(\lambda_{n-3})\lambda_{n-2}}(\theta'), \qquad (3.28)$$

where

$$\cos\theta' = \frac{\cos\theta \cosh\zeta - \sinh\zeta}{\cosh\zeta - \cos\theta \sinh\zeta}.$$

By considering the infinitesimal transformation as in (A), we obtain

$$i\langle \Lambda_{n-2}, \sigma_n; \lambda'_{n-1}\{\lambda_{n-1}\}| D_{n\ n-1} |\Lambda_{n-2}, \sigma_n; \{\lambda_{n-1}\}\rangle$$

$$= -\frac{\Gamma((n-1)/2)}{\sqrt{\pi}\,\Gamma((n-2)/2)}$$

$$\times \frac{\sqrt{N(\lambda_{n-1})N(\lambda'_{n-1})}}{N(\lambda_{n-2})N(\Lambda_{n-2})} \frac{N(\sigma_n; \lambda_{n-1})}{N(\sigma_n; \lambda'_{n-1})} \qquad (3.29)$$

$$\times \left[\left(\frac{2-n}{2} + \sigma_n \right) I^{(\lambda'_{n-1}, \lambda_{n-1})}_{\Lambda_{n-2}, \lambda_{n-2}} + T^{(\lambda'_{n-1}, \ \lambda_{n-1})}_{\Lambda_{n-2}\lambda_{n-2}} \right].$$

Taking into account (2.7), (3.15)—(3.20), we obtain for $m'_{n-1\ j} = m_{n-1\ j} + 1$ and $m'_{n-1\ l} = m_{n-1\ l}$ ($l \neq j$)

$$\frac{N(\sigma_n; \lambda'_{n-1})}{N(\sigma_n; \lambda_{n-1})} = \left(\frac{m_{n-1\ j} - \sigma_n + n/2 - j}{m_{n-1\ j} + \sigma_n + n/2 - j} \right)^{1/2}, \qquad (3.30)$$

for n odd. For n even, we must have $\sigma_n = 0$ or $m_{n\ n/2} = 0$ due to the C term. The first case is contained in the principal series and we adopt the second case. Then we obtain the same expression as (3.30) for n even. We, therefore, obtain the following expression for any λ_{n-1} and λ'_{n-1} as in (A), i.e.,

$$\frac{N(\sigma_n; \lambda_{n-1})}{N(\sigma_n; \lambda'_{n-1})}$$

$$= \left(\prod_{j=1}^{[(n-1)/2]} \frac{\Gamma(m_{n-1\ j} - \sigma_n + n/2 - j)\Gamma(m'_{n-1\ j} + \sigma_n + n/2 - j)}{\Gamma(m_{n-1\ j} + \sigma_n + n/2 - j)\Gamma(m'_{n-1\ j} - \sigma_n + n/2 - j)} \right)^{1/2}.$$

$$(3.31)$$

It is noted that the expression (3.31) is the analytic continuation of (3.13) in ν_n. For the complementary series contrary to the principal series, we cannot determine $N(\sigma_n; \lambda_{n-1})$ uniquely by (3.24) and (3.31), because the scalar product contain the function K and the constant is determined up to a constant factor from (3.31). We may, however, choose the factor as follows:

$$N(\sigma_n; \lambda_{n-1}) = \left[\prod_{j=1}^{[(n-1)/2]} \frac{\Gamma(m_{n-1\ j} - \sigma_n + n/2 - j)}{\Gamma(m_{n-1\ j} + \sigma_n + n/2 - j)} \right]^{1/2}. \qquad (3.32)$$

Then it follows that the function K is determined from (3.25).

Thus we have determined the constants which may be written in a form

$$W^{(\rho_n)}(\lambda'_{n-1}, \lambda_{n-1})$$

$$= \frac{N(\rho_n; \lambda_{n-1})}{N(\rho_n; \lambda'_{n-1})}$$

$$= \left[\prod_{j=1}^{[(n-1)/2]} \frac{\Gamma(m_{n-1\,j} - \rho_n - j + 1)\Gamma(m'_{n-1\,j} + \rho_n + n - j - 1)}{\Gamma(m_{n-1\,j} + \rho_n + n - j - 1)\Gamma(m'_{n-1\,j} - \rho_n - j + 1)} \right]^{1/2},$$

(3.33)

where $\rho_n = (2 - n)/2 + \sigma_n + i\nu_n$, $\sigma_n = 0$ for the principal series, and $\nu_n = 0$ for the complementary series.

As we have obtained the general formulas for the bases and the boost matrix elements for the UIR's of SO$(n-1, 1)$ by defining the scalar products, it is easy to give their expressions for each of the UIR's. We, however, point out one special case. For $K = $const, it follows that the representation of $\Lambda_{n-2} = 0$ [$0 \leqslant -\sigma_n < (n-2)/2$] is realized, i.e., $m_{nj} = 0$ for $j = 2, 3, \ldots, [n/2]$. In this case, the expressions (3.21), (3.25), and (3.28) become as follows:

$$\Phi^{(0,\sigma_n)}_{\{\lambda_{n-1}\}}(g^{(n-1)}) = \left[\frac{N(\lambda_{n-1})}{V_{n-1}} \frac{\Gamma(m_{n-1\,1} - \sigma_n + n/2 - 1)}{\Gamma(m_{n-1\,1} + \sigma_n + n/2 - 1)} \right]^{1/2}$$

$$\times D^{(\lambda_{n-1})}_{\{0\}\{\lambda_{n-2}\}}(g^{(n-1)}),$$

(3.34)

$$c \int_0^\pi d\theta \sin^{n-3}\theta \, (1 - \cos\theta)^{(2-n)/2 - \sigma_n} \, d^{(\lambda_{n-1})}_{0(0)0}(\theta)$$

$$= \frac{V_{n-2}\Gamma(n-2)/2}{2\pi^{(n-2)/2}} \frac{\Gamma(m_{n-1\,1} + \sigma_n + n/2 - 1)}{\Gamma(m_{n\,1\,1} - \sigma_n + n/2 - 1)},$$

(3.35)

$${}^b d^{(0,\sigma_n)}_{m_{n-1}\,(m_{n-2\,1})m_{n-1\,1}}(\zeta)$$

$$= \frac{\Gamma((n-1)/2)}{\sqrt{\pi}\,\Gamma((n-2)/2)} \frac{\sqrt{N(\lambda_{n-1})N(\lambda'_{n-1})}}{N(\lambda_{n-2})}$$

$$\times \left[\frac{\Gamma(m_{n-1\,1} - \sigma_n + n/2 - 1)\Gamma(m'_{n-1\,1} + \sigma_n + n/2 - 1)}{\Gamma(m_{n-1\,1} + \sigma_n + n/2 - 1)\Gamma(m'_{n-1\,1} - \sigma_n + n/2 - 1)} \right]^{1/2}$$

$$\times \sum_{\lambda_{n-3}} N(\lambda_{n-3}) \int_0^\pi d\theta \sin^{n-3}\theta \, \times$$

$$d^{(m_{n-1\,1})}_{0(m_{n-3\,1})m_{n-2\,1}}(\theta)\,(\cosh\zeta$$

$$- \cos\theta \sinh\zeta)^{(2-n)/2 + \sigma_n}$$

$$\times d^{(m_{n-1\,1})}_{0(m_{n-3\,1})m_{n-2\,1}}(\theta'),$$

(3.36)

where $\lambda_j = (m_{j1}, 0, \ldots, 0)$ for $j = n-1, n-2, n-3,$

$$N(\lambda_j) = \frac{(m_{j1} + j - 3)!\,(2m_{j1} + j - 2)}{(j-2)!\,m_{j1}!} \quad (j = n-1, n-2, n-3),$$

and $d^{(m_{n-1\,1})}_{0(m_{n-3\,1})m_{n-2\,1}}(\theta)$'s are the d matrix elements with the special values of the numbers m_{jk} of SO$(n-1)$.

The d matrix elements of SO(n) and those of the inhomogeneous orthogonal group (Euclid motion) are obtained from (3.8) by the continuation and the contraction.[4-6]

4. A CHARACTERISTIC NUMBER ρ_n

In Sec. 3., a complex number ρ_n is regarded as the same as the corresponding one in Sec. 2. It will, however, be necessary to show that they may be considered as the same one.

Let us start with the infinitesimal method of Sec. 2. It is known that the bases of SO$(n-1, 1)$ are classified by the group chain SO$(n-1, 1) \supset$ SO$(n-1) \supset \cdots \supset$ SO(2)[12] and the second-order Casimir operator $F^{(n-1,1)}$ of SO$(n-1, 1)$ has the eigenvalue with respect to the bases[15]

$$F^{(n-1,\,1)} = \rho_n(\rho_n + n - 2) + \sum_{j=2}^{[n/2]} m_{nj}(m_{nj} + n - 2j),$$

(4.1)

where $F^{(n-1,1)}$ is defined by

$$F^{(n-1,\,1)} = \sum_{j>k}^{n-1} D^2_{jk} - \sum_j^{n-1} D^2_{nj}.$$

The characteristic number ρ_n takes a value in each class of the UIR of SO$(n-1, 1)$.[7-9]

Now, we consider the Casimir operator in the case of the finite method in Sec. 3. It is known that the action of $D_{jk}(j, k \leqslant n-1)$ on the $D^{(n-1)}$ of (2.16) can be expressed in terms of the parameters (Euler angles) of the first and second parameter groups as follows

$$D_{jk}D^{(n-1)} = \bar{J}_{jk}D^{(n-1)}, \qquad D^{(n-1)}D_{jk} = J_{jk}D^{(n-1)},$$

(4.2)

where $\bar{J}_{jk}$ and J_{jk} are differential operators of the first and second parameter groups corresponding to D_{jk}. These $\bar{J}_{jk}$ and J_{jk} can be expressed in terms of the Euler angles (θ_{jk}) and differential operators $(p_{jk} = -i\partial/\partial\theta_{jk})$ and their expressions may be summarized as follows[2]:

$$\bar{J}_{k+1\,k} = \cos\theta_{k1}\,P_{k+1\,1}$$

$$- \frac{\cos\theta_{k+1\,1}}{\sin\theta_{k+1\,1}}\sin\theta_{k1}P_{k1} + \frac{\sin\theta_{k1}}{\sin\theta_{k+1\,1}}\bar{J}'_{k\,k-1},$$

(4.3)

$$J_{k+1\,k} = \cos\theta_{n-1\,n-k}P_{n-1\,n-k-1}$$

$$- \frac{\cos\theta_{n-1\,n-k-1}}{\sin\theta_{n-1\,n-k-1}}\sin\theta_{n-1\,n-k}P_{n-1\,n-k}$$

$$+ \frac{\sin\theta_{n-1\,n-k}}{\sin\theta_{n-1\,n-k-1}}J'_{k\,k-1} \quad (3 \leqslant k \leqslant n-2),$$

(4.4)

$$\bar{J}_{31} = \cos\theta_{21}P_{31} - \frac{\cos\theta_{31}}{\sin\theta_{31}}\sin\theta_{21}P_{21} + \frac{\sin\theta_{21}}{\sin\theta_{31}}P_{32},$$

(4.5)

$$J_{31} = \cos\theta_{n-1\,n-2}P_{n-1\,n-3}$$

$$- \frac{\cos\theta_{n-1\,n-3}}{\sin\theta_{n-1\,n-3}}\sin\theta_{n-1\,n-2}P_{n-1\,n-2} + \frac{\sin\theta_{n-1\,n-2}}{\sin\theta_{n-1\,n-3}}P_{n-2\,n-3},$$

(4.6)

where the prime on $\bar{J}'_{k\,k-1}$ means the substitutions θ_{jk}, $p_{jk} \to \theta_{j+1\,k+1}$, $p_{j+1\,k+1}$ in $\bar{J}_{k\,k-1}$ and that on $J'_{k\,k-1}$ the substitutions θ_{jk}, $p_{jk} \to \theta_{j-1\,k-1}$, $p_{j-1\,k-1}$ in $J_{k\,k-1}$.

The operators $\bar{J}_{jk}$ and J_{jk} satisfy the commutation relations

$$[\bar{J}_{jk}, \bar{J}_{lm}] = -i(\delta_{jl}\bar{J}_{km} + \delta_{km}\bar{J}_{jl} - \delta_{jm}\bar{J}_{kl} - \delta_{kl}\bar{J}_{jm}), \quad (4.7)$$

$$[J_{jk}, J_{lm}] = i(\delta_{jl}J_{km} + \delta_{km}J_{jl} - \delta_{jm}J_{kl} - \delta_{kl}J_{jm}), \quad (4.8)$$

$$[\bar{J}_{jk}, J_{lm}] = 0. \quad (4.9)$$

The expressions for any $\bar{J}_{jk}$ and J_{jk} $(j, k \leqslant n-1)$ are obtained from (4.7) and (4.8) together with (4.3)−(4.6).

The infinitesimal operator of the representation corresponding to the boost $^b l_{n\,n-1}^{(n)}(\zeta)$ is obtained from (3.4) as follows:

$$J_{n\,n-1} = -\sin\theta_{n-1\,1}p_{n-1\,1} - i\rho'_n\cos\theta_{n-1\,1}, \quad (4.10)$$

where the symbol ρ'_n is used instead of ρ_n in (3.4). The operators J_{nj} $(1 \leqslant j < n-1)$ are obtained from the commutation relation

$$J_{nj} = i[J_{n\,n-1}, J_{n-1\,j}]. \quad (4.11)$$

Then, it is easy to see that the operators $J_{jk}(j, k \leqslant n)$ satisfy the commutation relations of SO$(n-1, 1)$.

It follows from the second relation of (4.2) that the bases (3.2) are specified by the numbers $\{\lambda_{n-1}\}$ which are classified by the group chain SO$(n-1) \supset$ SO$(n-2)$ $\supset \cdots \supset$ SO(2) and are considered to be the same as those in the infinitesimal method. The second-order Casimir operator is given in the present case as follows:

$$F^{(n-1,1)} = \sum_{j>k}^{n-1} J_{jk}^2 - \sum_{j}^{n-1} J_{nj}^2. \quad (4.12)$$

Making use of the relations (4.10), (4.11), (4.4), (4.6), and (4.8), we can express the right-hand side of (4.12) as follows:

$$F^{(n-1,1)} = \rho'_n(\rho'_n + n - 2) + \sum_{j>k}^{n-2} J_{jk}^2. \quad (4.13)$$

The second term on the right-hand side of (4.13) is the second-order Casimir operator of the first parameter group of SO$(n-2)$. It, therefore, follows from the first relation of (4.2) that the action of (4.13) on the bases (3.2) gives the eigenvalue

$$F^{(n-1,1)} = \rho'_n(\rho'_n + n - 2) + \sum_{j=2}^{[n/2]} m_{nj}(m_{nj} + n - 2j). \quad (4.14)$$

By comparing (4.1) and (4.14), we may take $\rho_n = \rho'_n$ without loss of generality. Thus the complex numbers ρ_n in Secs. 2 and 3 may be considered as the same one.

5. SIMPLE CASES

In this section, we give the boost matrix elements for the special cases $n=3$ and 4, though these are well-known[10,16].

(a) $n = 3$:

(i) Principal series, $-\infty < m < \infty$, integer or half-integer, (3.8) becomes

$$^b d_{m'm}^{(\nu)}(\zeta) = \frac{1}{2\pi}\left[\frac{\Gamma(m'+i\nu+\frac{1}{2})\Gamma(m-i\nu+\frac{1}{2})}{\Gamma(m'-i\nu+\frac{1}{2})\Gamma(m+i\nu+\frac{1}{2})}\right]^{1/2}$$

$$\times \int_0^{2\pi} d\theta \exp(-im'\theta)$$

$$\times (\cosh\zeta - \cos\theta\sinh\zeta)^{-1/2+i\nu}\exp(im\theta'). \quad (5.1)$$

This is integrated as follows:

$$^b d_{m'm}^{(\nu)}(\zeta) = (-1)^{2q}\left[\frac{\Gamma(m'+i\nu+\frac{1}{2})\Gamma(m-i\nu+\frac{1}{2})}{\Gamma(m'-i\nu+\frac{1}{2})\Gamma(m+i\nu+\frac{1}{2})}\right]^{1/2}$$

$$\times \frac{\Gamma(m+i\nu+\frac{1}{2})\Gamma(-m+i\nu+\frac{1}{2})}{\Gamma(p-q+i\nu+\frac{1}{2})\Gamma(-p-q+i\nu+\frac{1}{2})(2q)!}$$

$$\times (\cosh\tfrac{1}{2}\zeta)^{2p}(\sinh\tfrac{1}{2}\zeta)^{2q}$$

$$\times {}_2F_1(p+q+\tfrac{1}{2}-i\nu, p+q+\tfrac{1}{2}+i\nu; 2q+1;$$

$$-\sinh^2\tfrac{1}{2}\zeta), \quad (5.2)$$

where $p = (m+m')/2$ and $q = |m-m'|/2$.

(ii) $D^0(\sigma)$, $0 \leqslant \sigma < \frac{1}{2}$, $-\infty < m < \infty$, integer,

$$^b d_{m'm}^{(\sigma)}(\zeta) = [i\nu \to \sigma \text{ in } (5.2)]. \quad (5.3)$$

(iii) $D^+(\sigma)$, $-\sigma = 0$, $1/2$, $1, \cdots$, $m \geqslant -\sigma+\frac{1}{2}$,

$$^b d_{m'm}^{(\sigma)}(\zeta) = [i\nu \to \sigma \text{ in } (5.2)]. \quad (5.4)$$

(iv) $D^-(\sigma)$, $-\sigma = 0$, $1/2$, $1, \ldots$, $-m \geqslant -\sigma+\frac{1}{2}$,

$$^b d_{m'm}^{(\sigma)}(\zeta) = [i\nu \to \sigma \text{ in } (5.2)]. \quad (5.5)$$

It is noted that for $D^\pm(\sigma)$, we must get rid of the singularity of the gamma functions, when we write the normalized bases; i.e., the relations $\Gamma(-m)/\Gamma(-m')$ $= (-1)^{m-m'}\Gamma(m'+1)/\Gamma(m+1)$ for m and m' positive integers must be used.

(b) $n = 4$:

(i) Principal series, $|k_0| =$ integer or half-integer, $j = |k_0|$, $|k_0| + 1, \cdots$; (3.8) gives

$$^b d_{j'(m)j}^{(k_0, -1+i\nu)}(\zeta)$$

$$= \frac{1}{2}[(2j+1)(2j'+1)]^{1/2}\left[\frac{\Gamma(j-i\nu+1)(j'+i\nu+1)}{\Gamma(j+i\nu+1)\Gamma(j'-i\nu+1)}\right]^{1/2}$$

$$\times \int_0^\pi d\theta \sin\theta\, \overline{d_{k_0m}^{(j')}(\theta)}\,(\cosh\zeta - \cos\theta\sinh\zeta)^{-1+i\nu}d_{k_0m}^{(j)}(\theta'), \quad (5.6)$$

where $d_{mm'}^{(j)}(\theta)$ is the d matrix element of SO(3). This is easily integrated but their expressions are omitted. They satisfy the relations[16]

$$^b d_{j'(m)j}^{(k_0, -1+i\nu)}(\zeta) = (-1)^{j'-j}\,^b d_{j(m)j'}^{(k_0, -1+i\nu)}(\zeta),$$

$$^b d_{j'(m)j}^{(k_0, -1+i\nu)}(\zeta) = \,^b d_{j'(k_0)j}^{(m, -1+i\nu)}(\zeta),$$

$$^b d_{j'(m)j}^{(k_0, -1+i\nu)}(\zeta) = \,^b d_{j'(-m)j}^{(-k_0, -1+i\nu)}(\zeta), \quad (5.7)$$

$$^b d_{j'(m)j}^{(k_0, -1+i\nu)}(\zeta) = (-1)^{j'-jb} d_{j(m)j'}^{(k_0, -1-i\nu)}(\zeta),$$

$$^b d_{j'(m)j}^{(k_0, -1+i\nu)}(\zeta) = {}^b d_{j(m)j'}^{(k_0, -1-i\nu)}(-\zeta),$$

$$\overline{{}^b d_{j'(m)j}^{(k, -1+i\nu)}(\zeta)} = {}^b d_{j'(m)j}^{(k_0, -1-i\nu)}(\zeta).$$

(ii) $D^1(\sigma)$, $k_0 = 0$, $0 < -\sigma < 1$, $j \geq 0$,

$$^b d_{j'(m)j}^{(0,\sigma)}(\zeta) = [i\nu \to 0, \quad k_0 \to 0 \quad \text{in (5.6)}]. \qquad (5.8)$$

In this way, we can give any matrix elements step by step.[16]

APPENDIX

In this Appendix, it is shown that if we require the commutation relations for the infinitesimal operators of SO($n-1$, 1), (3.4) holds in general.

The action of the representation operator $R_{nn-1}^b(\zeta)$, which corresponds to the boost $^b t_{nn-1}^{(n)}(\zeta)$, on $f(\{\theta_{n-1}\})$ can be defined as follows[14]

$$R_{nn-1}^b(\zeta) f(\{\theta_{n-1}\})$$

$$= M(\theta_{n-1\,1}, \zeta) f(\{\theta_{n-2}\}, (\theta'_{n-1\,1}, \theta_{n-1\,2}, \ldots, \theta_{n-1\,n-2})),$$

(A1)

where the multiplier $M(\theta, \zeta)$ is

$$M(\theta_{n-1\,1}, \zeta) = \frac{\alpha(\theta_{n-1\,1})}{\alpha(\theta'_{n-1\,1})},$$

and $\theta'_{n-1\,1}$ is given by the relation (2.14). It is assumed that $\alpha(\theta)$ depends only on $\theta_{n-1\,1}$ because θ's except for $\theta'_{n-1\,1}$ do not change under the boost $^b t_{nn-1}(\zeta)$. The infinitesimal operator J_{nn-1} of $R_{nn-1}^b(\zeta)$ is obtained from (A1),

$$J_{nn-1} f(\{\theta_{n-1}\}) = i \frac{d}{d\zeta} [M(\theta_{n-1\,1}, \zeta) f(\{\theta_{n-2}\},$$

$$(\theta'_{n-1\,1}, \theta_{n-1\,2}, \ldots, \theta_{n-1\,n-2}))]_{\zeta=0}$$

$$= i \frac{\partial \theta'_{n-1\,1}}{\partial \zeta}\Big|_{\zeta=0} \left(\frac{\partial}{\partial \theta_{n-1\,1}} - \frac{1}{\alpha(\theta_{n-1\,1})} \frac{d\alpha(\theta_{n-1\,1})}{d\theta_{n-1\,1}} \right)$$

$$\times f(\{\theta_{n-1}\}). \qquad (A2)$$

Thus, $J_{n\,n-1}$ is given by using the relation (2.14) as follows,

$$J_{n\,n-1} = i \sin\theta_{n-1\,1} \left(\frac{\partial}{\partial \theta_{n-1\,1}} - \frac{1}{\alpha(\theta_{n-1\,1})} \frac{d\alpha(\theta_{n-1\,1})}{d\theta_{n-1\,1}} \right). \qquad (A3)$$

We calculate J_{nn-2} by making use of (A3) and (4.4) with $k \to n-2$ through

$$J_{nn-2} = i [J_{nn-1}, J_{n-1n-2}], \qquad (A4)$$

and require that the following relation holds,

$$J_{n-1\,n-2} = i [J_{n\,n-1}, J_{n\,n-2}]. \qquad (A5)$$

Then we obtain with $\theta_{n-1\,1} \to \theta$

$$\cos\theta \frac{d}{d\theta} \left(\frac{\sin}{\alpha(\theta)} \frac{d\alpha(\theta)}{d\theta} \right) = \sin\theta \frac{d^2}{d\theta^2} \left(\frac{\sin\theta}{\alpha(\theta)} \frac{d\alpha(\theta)}{d\theta} \right).$$

(A6)

This relation gives

$$\alpha(\theta) = c(\sin\theta)^{\rho_n} \left(\tan\frac{\theta}{2} \right)^{c_1},$$

$$J_{n\,n-1} = i \left(\sin\theta \frac{\partial}{\partial\theta} - \rho_n \cos\theta - c_1 \right),$$

(A7)

where c, c_1, and ρ_n are some constants. We can take $c_1 = 0$ without loss of generality, because the c_1 term gives a multiplicative factor in the representation matrix of $R_{n\,n-1}^b(\zeta)$ and we are dealing with the representation of the group SO($n-1$, 1). We, therefore, obtain the solution for $\alpha(\theta)$ and the expression for the generator and the multiplier as follows:

$$\alpha(\theta) = c(\sin\theta)^{\rho_n},$$

$$J_{n\,n-1} = i \left(\sin\theta \frac{\partial}{\partial\theta} - \rho_n \cos\theta \right),$$

(A8)

$$M(\theta, \zeta) = \left(\frac{\sin\theta}{\sin\theta'} \right)^{\rho_n} = (\cosh\zeta - \cos\theta \sinh\zeta)^{\rho_n},$$

where ρ_n may be any complex number up to now.

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Symmetries of the stationary Einstein–Maxwell equations. IV. Transformations which preserve asymptotic flatness[a]

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We give a series of transformations $\overset{k}{\beta}$, $k = 0,1,...$ which may be used to generate new stationary axially-symmetric vacuum solutions from ones already known. These transformations have the important property of *preserving asymptotic flatness*. As one example of their use, we show how to generate the Kerr metric from Schwarzschild. As a second example, we generate a new five-parameter vacuum solution which contains the $\delta = 2$ Tomimatsu–Sato solution as a special case.

1. INTRODUCTION

Except for the Schwarzschild and Kerr metrics and possibly one or two others, the exact solutions of Einstein's equations have played a very minor role thus far in the development of general relativity. This has been the case because the known exact solutions are mostly mathematical curiosities and do not satisfy the boundary conditions required by the rest of physics. Recently we have been working to change this situation, by trying to develop methods of exact solution which lead to practical and useful results.[1–5]

We have concentrated on the problem of the stationary, axially symmetric Einstein–Maxwell field. The field equations for this case are remarkably simple, and certainly possess a large class of physically realistic solutions. Yet only a handful are known today.

We have been attacking this problem via a systematic study of the symmetry transformations which leave the field equations invariant. Every such transformation provides us with a solution-generating technique, whereby any known solution may be used to generate a one-parameter family of new solutions. We have been able to write down the entire group $\mathbf{K}'$ of all symmetry transformations for the reduced Einstein–Maxwell equations. $\mathbf{K}'$ contains an infinite number of arbitrary parameters; and so does $\mathbf{K} \subset \mathbf{K}'$, the corresponding symmetry group for the vacuum field equations. Thus for the solution generation we have an infinite number of degrees of freedom. Geroch's conjecture is that this group freedom is sufficient to generate *all* stationary solutions.[6] We have succeeded in proving so for the static case.

$\mathbf{K}$ is given in terms of its action on an infinite hierarchy of fields and potentials, which we have shown must exist for a stationary metric (see II for details.) Before $\mathbf{K}$ can be applied to a given spacetime, we must have an effective procedure for calculating these quantities. This question is addressed in Sec. 2.

The individual transformations of $\mathbf{K}$ that we have previously dealt with do not preserve asymptotic flatness, and this runs contrary to our stated objectives. We have now identified an infinite-parameter subgroup $\mathbf{B} \subset \mathbf{K}$ which *does preserve asymptotic flatness* (see Sec. 3.) Applied to any stationary axially symmetric asymptotically flat metric, $\mathbf{B}$ produces new asymptotically flat metrics. In particular when Schwarzschild is used for the initial metric, one of the new metrics generated is Kerr. A rational way of generating Kerr from Schwarzschild had been sought for a long time.

Our hope is that with the aid of $\mathbf{B}$, it will be possible to produce a large variety of new and realistic solutions. One such solution is presented in this paper, in Sec. 4.

2. CALCULATION OF POTENTIALS

We turn first to the question of how to calculate the potentials for a given spacetime. In practical applications this is where most of the labor is involved, and it is important to have an efficient method.

The approach is to use generating functions $F_{AB}(t)$, $G_{AB}(s, t)$ defined by

$$F_{AB} = \sum_{n=0}^{\infty} t^n \overset{n}{H}_{AB},\tag{2.1}$$

$$G_{AB} = \sum_{n=0}^{\infty} s^m t^n \overset{mn}{N}_{AB}.\tag{2.2}$$

From the field equations and the recursion relation for $\overset{n}{H}_{AB}$, we have already shown that F_{AB} obeys

$$\nabla F_{AB} = - i\rho^{-1} f_A{}^X \widetilde{\nabla} F_{XB},\tag{2.3}$$

$$\nabla F_{AB} = it[(H_{AX} + H^*_{\bar{X}A}) \nabla F^X{}_B + F^X{}_B \nabla H_{AX}].\tag{2.4}$$

Recalling that

$$H_{AB} + H^*_{\bar{B}A} = 2f_{AB} + 2iz\,\epsilon_{AB},$$

we can write Eq. (2.4) as

$$(1 - 2tz)\,\nabla F_{AB} + 2t\rho\widetilde{\nabla} F_{AB} = itF^X{}_B\,\nabla H_{AX}\tag{2.5}$$

or

$$\nabla F_{AB} = itS^{-2}[(1 - 2tz)\,\nabla H_{AX} - 2t\rho\,\widetilde{\nabla} H_{AX}]F^X{}_B,\tag{2.6}$$

where

$$S^2 = (1 - 2tz)^2 + (2t\rho)^2.\tag{2.7}$$

To solve Eqs. (2.3), (2.5) we would like to decouple the variables, obtaining individual equations for each component of F_{AB}. This can be done most easily by temporary abandonment of the manifest covariance.

Write out the $A = 1$ components of Eq. (2.3):

$$\nabla F_{1B} = - i\rho^{-1} f_{12}\widetilde{\nabla} F_{1B} + i\rho^{-1} f_{11}\widetilde{\nabla} F_{2B}.$$

Divide by f_{11} and take the divergence to eliminate the F_{2B} term:

$$\nabla \cdot [f_{11}^{-1}(\nabla F_{1B} + i\rho^{-1}f_{12}\tilde{\nabla}F_{1B})] = 0.$$

Expand this, using the definition of ψ_{11} [Eq. (I.4.7)] to get

$$f_{11}\nabla^2 F_{1B} = \nabla H_{11} \cdot \nabla F_{1B} . \qquad (2.8)$$

This linear second-order equation for F_{1B} is a direct generalization of the Ernst equation.[7] The $A = 2$ components of Eq. (2.3) yield

$$f_{22}\nabla^2 F_{2B} = \nabla H_{22} \cdot \nabla F_{2B} \qquad (2.9)$$

Now write out the $A = 1$ components of Eq. (2.5):

$$(1 - 2tz)F_{1B,\rho} + 2t\rho F_{1B,z} = it(F_{2B}H_{11,\rho} - F_{1B}H_{12,\rho}),$$

$$(1 - 2tz)F_{1B,z} - 2tzF_{1B,\rho} = it(F_{2B}H_{11,z} - F_{1B}H_{12,z}). \qquad (2.10)$$

Here the elimination of F_{2B} can be done algebraically. The resulting first-order equation is

$$[(1 - 2tz)H_{11,z} + 2t\rho H_{11,\rho}]F_{1B,\rho} + [2t\rho H_{11,z}$$
$$- (1 - 2tz)H_{11,\rho}]F_{1B,z}$$
$$= it[H_{11,\rho}H_{12,z} - H_{11,z}H_{12,\rho}]F_{1B}$$

or

$$(1 - 2tz)\tilde{\nabla}H_{11} \cdot \nabla F_{1B} + 2t\rho \nabla H_{11} \cdot \nabla F_{1B}$$
$$= it(\nabla H_{22} \cdot \tilde{\nabla}H_{21})F_{1B} . \qquad (2.11)$$

Likewise for $A = 2$,

$$(1 - 2tz)\tilde{\nabla}H_{22} \cdot \nabla F_{2B} + 2t\rho \nabla H_{22} \cdot \nabla F_{2B}$$
$$= it(\nabla H_{22} \cdot \tilde{\nabla}H_{21})F_{2B} . \qquad (2.12)$$

Equation (2.11) can be solved by the method of characteristics.[8] Its solution will contain one arbitrary function, say g. When this is inserted in Eq. (2.8), we get a linear second-order ordinary differential equation for g. The two linearly independent solutions correspond to F_{11} and F_{12}. We do not need to solve Eqs. (2.9) and (2.12) for F_{2B}, since F_{2B} may be obtained algebraically from Eq. (2.10), once F_{1B} is known.

Now suppose that $F_{AB}(t)$ has been found. The next step is to construct $G_{AB}(s,t)$. It is remarkable that this can also be done algebraically, with no further integrations. Define two auxiliary generating functions,

$$G'_{AB} = \sum_{m=0}^{\infty} s^m \overset{m1}{N}_{AB}, \qquad (2.13)$$

$$'G_{AB} = \sum_{n=0}^{\infty} t^n \overset{1n}{N}_{AB} . \qquad (2.14)$$

The recursion relations, Eqs. (II.2.18) and (II.2.22), become

$$G_{AB}(s,t) - G^*_{BA}(t,s) = \epsilon_{AB} + F^*_{XA}(s) F^X_B(t), \qquad (2.15)$$

$$'G_{AB}(t) - G'^*_{BA}(t) = H^*_{XA}F^X_B(t), \qquad (2.16)$$

$$t^{-1}[G_{AB}(s,t) - \epsilon_{AB}] - s^{-1}[G_{AB}(s,t) + iF_{AB}(t)]$$
$$= iG'_{AX}(s) F^X_B(t), \qquad (2.17)$$

$$t^{-1}[F_{AB}(t) - i\epsilon_{AB}] = it['G_{AB}(t) + H_{AX}F^X_B(t)]. \qquad (2.18)$$

Solving Eqs. (2.18), (2.16), and (2.17) in that order,

we find

$$G_{AB}(t) = -t^{-1}\epsilon_{AB} - it^{-1}F_{AB}(t) - H_{AX}F^X_B(t), \qquad (2.19)$$

$$G'_{AB}(s) = s^{-1}\epsilon_{AB} + is^{-1}F^*_{BA}(s) - (H_{XB} + H^*_{BX}) F^{*X}_A(s)$$
$$= s^{-1}\epsilon_{AB} + is^{-1}F^*_{BA}(s) - 2izF^*_{BA}(s) - 2f_{XB}F^{*X}_A(s), \qquad (2.20)$$

$$G_{AB}(s,t) = (s-t)^{-1}[s\epsilon_{AB} - tF^*_{XA}(s) F^X_B(t)$$
$$+ 2stzF^*_{XA}(s) F^X_B(t)$$
$$- 2istf_{XY}F^{*X}_A(s) F^Y_B(t)]. \qquad (2.21)$$

The solution we have found satisfies Eq. (2.15) identically, and hence obeys all of the required conditions. However, it will not have a series expansion of the form Eq. (2.2) unless the pole at $s = t$ is absent. The necessary and sufficient condition for this is that when we set $s = t$, the remaining factors in $G_{AB}(s,t)$ must vanish. That is,

$$\epsilon_{AB} - (1 - 2tz) F^*_{XA}F^X_B - 2itf_{XY}F^{*X}_A F^Y_B = 0. \qquad (2.22)$$

Thus the existence of a $G_{AB}(s,t)$ which obeys the recursion relations imposes a set of four real (Hermitian) constraints on $F_{AB}(t)$.

In the (f, ω) parametrization of the metric (see I, Sec. 4) the constraints may be written out in more detail. They become

$$2iftF_{21} = (1 - 2tz - 2itf\omega) F_{11} + SF^*_{11}, \qquad (2.23)$$

$$2iftF_{22} = (1 - 2tz - 2itf\omega) F_{12} + SF^*_{12}, \qquad (2.24)$$

$$F^*_{11}F_{12} - F_{11}F^*_{12} = 2iftS^{-2}. \qquad (2.25)$$

Together they are consistent with the first integral found in III,

$$F_{11}F_{22} - F_{12}F_{21} = -S^{-1}. \qquad (2.26)$$

Zipoy--Voorhees metrics

As an example, we will calculate potentials for the Zipoy—Voorhees class of static metrics,[9,10] which includes the Schwarzschild metric and the static limits of the Tomimatsu—Sato metrics. We work in the usual prolate spheroidal coordinates x, y, related to cylindrical coordinates ρ, z by

$$\rho^2 = (x^2 - 1)(1 - y^2),$$

$$z = xy, \qquad (2.27)$$

$$\tilde{\nabla}x = -\rho^{-1}(x^2 - 1) \nabla y.$$

The Zipoy—Voorhees metric is given by[11]

$$H_{11} = f = \left(\frac{x-1}{x+1}\right)^{\delta}, \qquad H_{12} = 2iy(x - \delta). \qquad (2.28)$$

Equation (2.11) becomes

$$(1 - 2txy) F_{1B,y} - 2t(x^2 - 1) F_{1B,x} = 2t(x - \delta) F_{1B} . \qquad (2.29)$$

The corresponding characteristic equations are

$$\frac{dy}{1 - 2txy} = \frac{-dx}{2t(x^2 - 1)} = \frac{dF_{1B}}{2t(x - \delta) F_{1B}} \qquad (2.30)$$

with solutions

$$x - 2ty = a(x^2 - 1)^{1/2}, \qquad (2.31)$$

$$F_{1B} = \frac{b}{(x^2-1)^{1/2}} \left(\frac{x-1}{x+1}\right)^{\delta/2}. \qquad (2.32)$$

Equation (2.31) determines a family of characteristic curves in the x, y plane, and Eq. (2.32) tells how F_{1B} varies along the curves. Here a, b are constant along each curve, and we may now regard b as an arbitrary function of a.

To determine $b(a)$, we substitute Eqs. (2.31), (2.32) into Eq. (2.8). We find

$$(a^2 + 4t^2 - 1)\,b'' + 3ab' + (1 - \delta^2)\,b = 0 \qquad (2.33)$$

with solutions $R^{-1}(a \pm R)^\delta$, where

$$R = (a^2 + 4t^2 - 1)^{1/2}$$
$$= S(x^2-1)^{-1/2}. \qquad (2.34)$$

Thus, we obtain the generating functions

$$F_{11} = \frac{tc(t)}{(x^2-1)^{1/2}} \left(\frac{x-1}{x+1}\right)^\delta \frac{(a-R)^\delta}{R}$$
$$= \frac{tc(t)}{S} \left(\frac{x-2ty-S}{x+1}\right)^\delta, \qquad (2.35)$$

$$F_{12} = \frac{id(t)}{S} \left(\frac{x-2ty+S}{x+1}\right)^\delta,$$

where $c(t)$, $d(t)$ are yet to be determined.

The case $\delta = N$, an arbitrary integer, is the one of greatest interest, since this corresponds to the static limit of the Tomimatsu-Sato metrics. Suppose we make the simplest choice, $c = d = 1$. Then $F_{11} + itF_{12}$ will be a polynomial in t of degree N.[4] Hence

$$i\,\overset{n+1}{H_{11}} = \overset{n}{H_{12}}, \quad n \geqslant N. \qquad (2.36)$$

However, this choice violates Eq. (2.25). It would correspond to a more general gauge in which recursion relations (II.2.22) did not hold. In fact from Eq. (2.35) we calculate

$$F_{11}F_{12} = \frac{icd}{S^2} t \left(\frac{x-1}{x+1}\right)^\delta (1 - 4t^2)^\delta.$$

Therefore, a consistent choice is to take

$$c = d = (1 - 4t^2)^{-\delta/2}. \qquad (2.37)$$

The other potentials now follow from Eqs. (2.21), (2.23), and (2.24).

3. THE SUBGROUP B

In III, we saw what happened when the transformations of the vacuum symmetry group K were applied to flat space. For the infinitesimal transformations,

$$\overset{k}{\gamma_{11}} : H_{11} \to 1 + i\gamma(2r)^{k+1}P_{k+1}(\cos\theta),$$

$$\overset{k}{\gamma_{12}} : H_{11} \to 1 - 2\gamma(2r)^{k}P_{k}(\cos\theta), \qquad (3.1)$$

$$\overset{k}{\gamma_{22}} : H_{11} \to 1 - i\gamma(2r)^{k-1}P_{k-1}(\cos\theta).$$

Thus the $\overset{k}{\gamma_{AB}}$ generate weak gravitational fields of the electric multipole and magnetic multipole variety. All of the fields diverge at spatial infinity, and hence each transformation $\overset{k}{\gamma_{AB}}$ violates asymptotic flatness.

Is there some linear combination of the $\overset{k}{\gamma_{AB}}$ which is more well behaved? It would seem, for one thing, that we have "too many" transformations, since both $\overset{k}{\gamma_{11}}$ and $\overset{k+2}{\gamma_{22}}$ generate the same multipole. Logically, one should therefore devote particular attention to the combinations

$$\overset{k}{\beta} = \overset{k}{\gamma_{11}} + \overset{k+2}{\gamma_{22}}, \quad k = 0, 1, \cdots. \qquad (3.2)$$

They form an Abelian subgroup, which we call B. The infinitesimal transformations of B leave H_{11} for flat space invariant. Does that hold true also for the finite transformations? We shall show that it does.

Using the infinitesimal transformations as given in Eq. (II.3.1), we can calculate

$$\overset{k}{\beta} : \overset{mn}{N_{11}} \to \overset{mn}{N_{11}} + \beta[i(\overset{m+k+1,n}{N_{11}} - i\,\overset{m+k,n}{N_{21}})$$
$$- i(\overset{m,n+k+1}{N_{11}} + i\,\overset{m,n+k}{N_{12}}) + \overset{m1}{N_{11}}(\overset{k+1,n}{N_{11}} - i\,\overset{kn}{N_{21}})$$
$$+ (\overset{m,k+2}{N_{11}} + i\,\overset{m,k+1}{N_{12}})\overset{0n}{N_{11}} + \sum_s (\overset{m,s+1}{N_{11}} + i\overset{ms}{N_{12}})(\overset{k+1-s,n}{N_{11}} - i\,\overset{k-s,n}{N_{21}})$$

$$(3.3)$$

and also similar expressions for $\overset{mn}{N_{12}}$, $\overset{mn}{N_{21}}$, and $\overset{mn}{N_{22}}$. However, the form of Eq. (3.3) strongly suggests that a simplification would occur if we used instead certain linear combinations of the N_{AB}. We therefore define

$$P_{0n} = \overset{0n}{N_{11}} + i\,\overset{0,n-1}{N_{12}},$$
$$P_{mn} = \overset{mn}{N_{11}} - i\,\overset{m-1,n}{N_{21}} + i\,\overset{m,n-1}{N_{12}} + \overset{m-1,n-1}{N_{22}}, \quad m > 0. \qquad (3.4)$$

In particular, since $\overset{00}{N_{AB}} = \epsilon_{AB}$, we have

$$P_{01} = \overset{01}{N_{11}} + i = -i(H_{11} - 1). \qquad (3.5)$$

The transformation $\overset{k}{\beta}$ can now be written in terms of what it does to P_{mn}:

$$\overset{k}{\beta} : P_{0n} \to P_{0n} + \beta(-2iP_{0,n+k+1} + \sum_s P_{0s}P_{k+2-s,n})$$
$$P_{mn} \to P_{mn} + \beta(2iP_{m+k+1,n} - 2iP_{m,n+k+1}$$
$$+ \sum_s P_{ms}P_{k+2-s,n}), \qquad (3.6)$$

where the summations run from $s = 1$ to $s = k+2$.

For flat space the P_{mn} all vanish [see Eq. (II.4.19)]. We see from Eq. (3.6) that after a transformation of B (even a finite one) they must continue to vanish. In particular the relation $H_{11} = 1$ will be preserved, and flat space will remain flat space, in the same gauge, and in the same coordinates.

Now suppose we start with a space which is asymptotically flat. We can pick a gauge for the potentials such that $P_{mn} \to 0$ at spatial infinity. Since B preserves this condition, *it automatically preserves asymptotic flatness.*[12]

To adapt the generating function approach to our present needs, we may define

$$\overset{n}{J_A} = \overset{n}{H_{A1}} + i\,\overset{n-1}{H_{A2}}, \qquad (3.7)$$

$$R_A(t) = \sum t^n \overset{n}{J_A}$$

$$= F_{A1} + i/F_{A2}, \tag{3.8}$$

$$Q(s, t) = \sum s^m t^n P_{mn}$$

$$= G_{11} + i/G_{12} - isG_{21} + s/G_{22}. \tag{3.9}$$

Then R_A, Q bear much the same relation to each other as do F_{AB} and G_{AB}, and Q may be determined algebraically once R_A is known. In fact, from Eq. (2.21),

$$Q(s, t) = (s - t)^{-1}[2is/(1 - f_{XY}R^{*X}R^Y)$$

$$- t(1 - 2sz)R_X^*R^X]. \tag{3.10}$$

4. APPLICATIONS OF B

To illustrate how **B** may be used to generate asymptotically flat metrics, we will apply it to the Zipoy—Voorhees solution. We will show that the metrics produced include Kerr, NUT, and Tomimatsu—Sato, as well as others which are new.

A. Kerr-NUT from Schwarzschild

We start from Schwarzschild, which is the Zipoy—Voorhees metric for $\delta = 1$. From the results of Sec. 2A, we have

$$R_1 = - 2t(1 - 4t^2)^{-1/2}(x + 1)^{-1},$$

$$R_2 = - i(1 - 4t^2)^{-1/2}(1 + 2ty). \tag{4.1}$$

$Q(s, t)$ may now be calculated via Eq. (3.10) and expanded in powers of s and t to yield the P_{mn}'s. For example,

$$P_{01} = 2i(x + 1)^{-1}, \quad P_{11} = 4iy(x + 1)^{-1}, \tag{4.2}$$

$$P_{02} = 0, \quad P_{12} = 4i. \tag{4.3}$$

Actually, it is not even necessary to examine any others beyond these, since for $\delta = 1$, *all of the higher P_{mn}'s are fixed linear combinations of the above four.* This can be seen readily from the expansion of Eq. (4.1),

$$\overset{2k}{J_A} = c_k \overset{0}{J_A}, \quad \overset{2k+1}{J_A} = c_k \overset{1}{J_A}, \tag{4.4}$$

where c_k are the expansion coefficients of $(1 - 4t^2)^{-1/2}$. Thus

$$P_{03} = 2P_{01}, \quad P_{13} = P_{31} = 2P_{11},$$

$$P_{21} = 2P_{01} - 4i, \tag{4.5}$$

and so forth. Furthermore, one can prove that *these relationships*, Eqs. (4.3)—(4.5), *are preserved under* $\overset{k}{\beta}$.

The entire infinite set of transformation equations, Eq. (3.6), can therefore be reduced to repetitions, up to a factor, of a very small and manageable subset. For $\overset{0}{\beta}$ there are only two distinct equations,

$$P_{01} \rightarrow P_{01} + \overset{0}{\beta}P_{01}P_{11},$$

$$P_{11} \rightarrow P_{11} + \overset{0}{\beta}(P_{11}P_{11} + 8iP_{01} + 16), \tag{4.6}$$

while for $\overset{1}{\beta}$ there is only one:

$$P_{01} \rightarrow P_{01} + 4\overset{1}{\beta}(P_{01}P_{01} - 2iP_{01}). \tag{4.7}$$

The same recursion relations also cause the $\overset{k}{\beta}$ to repeat. We find that

$$\overset{2}{\beta} = 4\overset{0}{\beta}, \quad \text{etc.}$$

Therefore, when **B** is applied to Schwarzschild, we have effectively only two independent transformations, and will be able to generate a three-parameter $\delta = 1$ metric.

As discussed in Ref. 5, the equations may be treated as differential equations and integrated to find the transformations for finite β. Equation (4.2) supplies the initial conditions at $\beta = 0$. For $\overset{0}{\beta}$,

$$\frac{\partial P_{01}}{\partial \beta} = P_{01}P_{11},$$

$$\frac{\partial P_{11}}{\partial \beta} = P_{11}P_{11} + 8iP_{01} + 16, \tag{4.8}$$

with solutions

$$P_{01} = \frac{2i}{a\cos4\beta - ib\sin4\beta + 1},$$

$$P_{11} = 4i\left(\frac{b\cos4\beta - ia\sin4\beta}{a\cos4\beta - ib\sin4\beta + 1}\right), \tag{4.9}$$

where a, b are integration constants. From the initial conditions, $a = x$, $b = y$. To obtain agreement with conventional notation, let

$$P = \cos4\beta, \quad q = - \sin4\beta, \quad p^2 + q^2 = 1. \tag{4.10}$$

Then the Ernst potential for our solution is

$$\xi = \frac{1 - H_{11}}{1 + H_{11}} = \frac{- iP_{01}}{2 + iP_{01}} = \frac{1}{px + iqy}, \tag{4.11}$$

which is the known form for the Kerr metric.[7]

Solving Eq. (4.6) in a similar fashion, we find

$$P_{01} = 2i(1 + x\exp(8i\beta))^{-1},$$

$$\xi = x^{-1}\exp(- 8i\beta). \tag{4.12}$$

This corresponds to NUT space, with NUT parameter l given by

$$l/m = - \tan8\beta. \tag{4.13}$$

To obtain $\delta = 1$ metrics in which $\overset{0}{\beta}$, $\overset{1}{\beta}$ are both non-zero, we may simply perform $\overset{0}{\beta}$ and $\overset{1}{\beta}$ transformations in succession. This is permissible even when the parameters are finite, since **B** is Abelian. The three-parameter family of metrics which results is thus Kerr—NUT space.[13]

B. Generalization of Tomimatsu-Sato

For the Zipoy—Voorhees metric with $\delta = 2$,

$$R_1 = - 4t(1 - 4t^2)^{-1}(x + 1)^{-2}(x - 2ty),$$

$$R_2 = - i(1 - 4t^2)^{-1}[1 + 4ty + 4t^2(x - 1)^{-1}(x + 1 - 2y^2)]. \tag{4.14}$$

From this we calculate

$$P_{01} = 4ix(x + 1)^{-2}, \quad P_{02} = - 8iy(x + 1)^{-2},$$

$$P_{11} = 16ixy(x + 1)^{-2}, \quad P_{21} = - 16i(x + 1)^{-2}(x^2 + y^2),$$

$$P_{12} = 16i(x+1)^{-1}(x^2-1)^{-1}(x^3+x^2+y^2-3xy^2),$$

$$P_{22} = -64iy(x+1)^{-1}(x^2-1)^{-1}(x-y^2).$$

$$(4.15)$$

The recursion relations are

$$P_{0,2l+n} = 4^l P_{0n}, \quad n = 1, 2,$$

$$P_{2k+m,2l+n} = 4^{k+l} P_{mn}, \quad m, n = 1, 2. \tag{4.16}$$

Once more, one may prove that the transformations of $\overset{k}{B}$ preserve the recursion relations. For each $\overset{k}{\beta}$, six transformation equations will be necessary. The solution of six nonlinear coupled differential equations is not an easy matter. We therefore digress upon a general way we have found of linearizing them.

In order to handle all of the β's at once, we let $\overset{k}{\beta}$ denote the final constant values. The instantaneous values will be $\lambda\overset{k}{\beta}$, where $0 \leq \lambda \leq 1$ is the variable of integration. The original set of transformation equations may be written in terms of infinite matrices:

$$\frac{dP}{d\lambda} = PAP + BP - P\tilde{B}, \tag{4.17}$$

where

$$P = \begin{bmatrix} 0 & P_{01} & P_{02} & \cdots \\ 0 & P_{11} & P_{12} & \cdots \\ 0 & P_{21} & P_{22} & \cdots \\ \cdots & \cdots & \cdots & \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & \overset{0}{\beta} & \cdots \\ 0 & \overset{0}{\beta} & \overset{1}{\beta} & \cdots \\ \overset{0}{\beta} & \overset{1}{\beta} & \overset{2}{\beta} & \cdots \\ \cdots & \cdots & \cdots & \end{bmatrix}, \tag{4.18}$$

$$B = 2i \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \overset{0}{\beta} & \overset{1}{\beta} & \overset{2}{\beta} & \cdots \\ 0 & 0 & 0 & \overset{0}{\beta} & \overset{1}{\beta} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{bmatrix},$$

and the tilde denotes the transpose.

We try to solve Eq. (4.17) via an ansatz:

$$P = ND^{-1}, \tag{4.19}$$

where N, D are also infinite matrices. Inserting this in Eq. (4.17) gives

$$\frac{dN}{d\lambda} D^{-1} - ND^{-1} \frac{dD}{d\lambda} D^{-1} = ND^{-1}AND^{-1} + BND^{-1} - ND^{-1}\tilde{B}. \tag{4.20}$$

If we choose N, D to obey

$$\frac{dN}{d\lambda} = BN, \quad \frac{dD}{d\lambda} = \tilde{B}D - AN, \tag{4.21}$$

then Eq. (4.20) will be satisfied. We now have twice as many equations to solve, but they are linear. At $\lambda = 0$ we will impose the further initial conditions $N = P(0)$, $D = I$.

The solutions of Eq. (4.21) are

$$N(\lambda) = \exp(\lambda B) P(0),$$

$$D(\lambda) = \exp(\lambda \tilde{B})(I - C(\lambda) P(0)), \tag{4.22}$$

where

$$C(\lambda) = \int_0^\lambda \exp(-\lambda\tilde{B}) A \exp(\lambda B) \, d\lambda. \tag{4.23}$$

Hence the finite transformation is

$$P(1) = \exp(B) P(0)[I - C(1) P(0)]^{-1} \exp(-\tilde{B}). \tag{4.24}$$

Return now to the special case of the Zipoy—Voorhees metric for $\delta = 2$. The recursion relations enable us to replace P, A, B by 3×3 matrices. P is merely truncated, and now

$$A = \begin{bmatrix} 0 & 0 & \overset{0}{\beta} + 4\overset{2}{\beta} \\ 4\overset{1}{\beta} + 16\overset{3}{\beta} & \overset{0}{\beta} + 8\overset{2}{\beta} & \overset{1}{\beta} + 8\overset{3}{\beta} \\ \overset{0}{\beta} + 4\overset{2}{\beta} & \overset{1}{\beta} + 8\overset{3}{\beta} & \overset{2}{\beta} \end{bmatrix},$$

$$B = 2i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4\overset{1}{\beta} + 16\overset{3}{\beta} & \overset{0}{\beta} + 4\overset{2}{\beta} \\ 0 & 4\overset{0}{\beta} + 16\overset{2}{\beta} & 4\overset{1}{\beta} + 16\overset{3}{\beta} \end{bmatrix}. \tag{4.25}$$

The $\overset{k}{\beta}$'s repeat themselves for $\delta = 2$ also, starting this time with

$$\overset{4}{\beta} = 8\overset{2}{\beta} - 16\overset{0}{\beta}$$

We have four effective transformations and will be able to generate a five-parameter $\delta = 2$ metric.

The results of the calculation are

$$N = \begin{bmatrix} 0 & P_{01} & P_{02} \\ 0 & N_{11} & N_{12} \\ 0 & N_{21} & N_{22} \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & D_{11} & D_{12} \\ 0 & D_{21} & D_{22} \end{bmatrix}, \tag{4.26}$$

where

$$N_{11} = (P_{11} \cos 4\alpha_0 + \tfrac{1}{2}iP_{21} \sin 4\alpha_0) \exp(8i\alpha_3),$$

$$N_{12} = (P_{12} \cos 4\alpha_0 + \tfrac{1}{2}iP_{22} \sin 4\alpha_0) \exp(8i\alpha_3),$$

$$N_{21} = (2iP_{11} \sin 4\alpha_0 + P_{21} \cos 4\alpha_0) \exp(8i\alpha_3),$$

$$N_{22} = (2iP_{12} \sin 4\alpha_0 + P_{22} \cos 4\alpha_0) \exp(8i\alpha_3),$$

$$D_{11} = -\tfrac{1}{2}iP_{01} + (1 + \tfrac{1}{2}iP_{01} - \alpha_1 P_{21} - \tfrac{1}{2}\alpha_2 P_{11}) \cos 4\alpha_0 \exp(8i\alpha_3)$$
$$\quad - \tfrac{1}{8}(P_{11} + 16i\alpha_1 P_{11} + 2i\alpha_2 P_{21}) \sin 4\alpha_0 \exp(8i\alpha_3),$$

$$D_{12} = -\tfrac{1}{2}iP_{02} + (\tfrac{1}{2}iP_{02} - \alpha_1 P_{22} - \tfrac{1}{2}\alpha_2 P_{12}) \cos 4\alpha_0 \exp(8i\alpha_3)$$
$$\quad - \tfrac{1}{8}(-16i + P_{12} + 16i\alpha_1 P_{12} + 2i\alpha_2 P_{22}) \sin 4\alpha_0 \exp(8i\alpha_3),$$

$$D_{21} = -\tfrac{1}{8}(8\alpha_1 P_{11} + \alpha_2 P_{21}) \cos 4\alpha_0 \exp(8i\alpha_3)$$
$$\quad + \tfrac{1}{32}(16i - 8P_{01} + P_{21} - 16i\alpha_1 P_{21} - 8i\alpha_2 P_{11}) \sin 4\alpha_0$$

$$\quad \times \exp(8i\alpha_3),$$

$$D_{22} = \tfrac{1}{8}(8 - 8\alpha_1 P_{12} - \alpha_2 P_{22})\cos 4\alpha_0 \exp(8i\alpha_3)$$

$$+ \tfrac{1}{32}(-8P_{02} + P_{22} - 16i\alpha_2 P_{22} - 8i\alpha_2 P_{12})\sin 4\alpha_0$$

$$\times \exp(8i\alpha_3)$$

(4.27)

and where

$$\alpha_0 = \overset{0}{\beta} + 4\overset{2}{\beta}, \quad \alpha_1 = \overset{1}{\beta} + 8\overset{3}{\beta},$$

$$\alpha_2 = \overset{0}{\beta} + 12\overset{2}{\beta}, \quad \alpha_3 = \overset{1}{\beta} + 4\overset{3}{\beta}.$$

(4.28)

The transformation of P_{01} is

$$P_{01} \rightarrow \frac{P_{01}D_{22} - P_{02}D_{21}}{D_{11}D_{22} - D_{12}D_{21}}$$

(4.29)

When the initial values of P_{mn} are inserted, we obtain for the Ernst potential,

$$\xi = \exp(-i\gamma)\left(\frac{2px(x^2 - 1) - 2iqy(1 - y^2) - 2i(p\alpha + iq\beta)x(x^2 - y^2) + 2i(p\beta + iq\alpha)y(x^2 - y^2)}{p^2(x^4 - 1) - 2ipqxy(x^2 - y^2) + q^2(y^4 - 1) - 2i\alpha(x^2 + y^2 - 2x^2y^2) - 2i\beta xy(x^2 + y^2 - 2) + (\alpha^2 - \beta^2)(x^2 - y^2)^2} \right),$$

(4.30)

where we have put

$$p = \cos 4\alpha_0, \quad q = \sin 4\alpha_0, \quad p^2 + q^2 = 1, \quad \alpha = 16\alpha_1, \quad \beta = 4\alpha_2, \quad \gamma = 8\alpha_3.$$

(4.31)

As before, the fifth parameter is the mass m, contained in the dimensionless coordinate x.

Particular cases of this metric may be obtained by special choices of the parameters. For example, for $\alpha = \beta = \gamma = 0$ we recover the $\delta = 2$ Tomimatsu–Sato metric. For $p = 1$, $q = \alpha = \gamma = 0$ we recover the solution quoted in Ref. 5.

Similar considerations apply for higher values of δ. For example, for $\delta = 3$ a set of 4×4 matrix equations are involved. The independent transformations are $\overset{0}{\beta}, \ldots, \overset{5}{\beta}$, and they lead to a seven-parameter vacuum solution. Exclusion of the NUT parameter leaves a six-parameter asymptotically flat metric.

Note added in proof: We would like to thank John Wainwright, University of Waterloo, for verifying directly with an algebraic computer program that Eq. (4.30) does satisfy the Ernst equation.

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[11]The mass parameter is effectively concealed in the variable x. If asymptotically $x \sim r/m\delta$, where r is the radius, then m is the Schwarzschild mass.

[12]This statement refers to the local properties of the metric, and is intended in the simplest sense that the Riemann tensor tends to zero. Metrics with the same global anomaly as NUT space do arise, but these are always associated with a gravitational "magnetic monopole" parameter, and may be easily excluded afterwards. Due to the multipole character of the transformations, we strongly believe that no other problems such as angular singularities can occur, assuming that we start from a well-behaved metric.

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On invariant integration over SU(N)[a]

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We give a graphical algorithm for evaluation of invariant integrals of polynomials in SU(N) group elements. Such integrals occur in strongly coupled lattice gauge theory. The results are expressed in terms of totally antisymmetric tensors and Kronecker delta symbols.

For the strong coupling expansion of lattice gauge theory one requires invariant integrals over polynomials in elements of the fundamental gauge group.[1,2] To explicitly exhibit the invariant measure of a group is in principle straightforward but often in practice a rather tedious task. Beginning with some parametrization, i.e., a generalization of the Euler angles for the rotation group, one studies the group transformation properties of a small volume element in this parameter space. Fortunately, symmetry arguments can determine many integrals, thus sidestepping the explicit construction of the invariant measure. We will show how the symmetry properties of the groups SU(N) give rise to a set of rules for evaluation of the integrals arising in strongly coupled gauge theory. This generalizes to arbitrary N the rules of Ref. 2 for SU(3).

Given any compact Lie group, it is well known that there exists a unique normalized integration measure with the properties

$$\int dg f(g) = \int dg f(g_0 g g_1) = \int dg f(g^{-1}), \quad \int dg = 1, \quad (1)$$

where g is the group element being integrated over, $f(g)$ is an arbitrary function of g, and g_0 and g_1 are arbitrary fixed group elements. In this paper we are interested in the group SU(N); so, g represents an N by N unitary matrix of determinant one. We wish to evaluate integrals of the form

$$I = \int dg \, g_{i_1 j_1} \cdots g_{i_n j_n} g^{-1}_{k_1 l_1} \cdots g^{-1}_{k_m l_m}, \quad (2)$$

where matrix indices for the g's and g^{-1}'s are explicitly indicated. We introduce a generating function for such integrals

$$W(J,K) = \int dg \exp[\text{Tr}(Jg + Kg^{-1})], \quad (3)$$

where J and K are arbitrary N by N matrices. Integrals of the form of Eq. (2) are obtained from $W(J,K)$ by differentiating

$$I = \left(\frac{\partial}{\partial J_{j_1 i_1}} \cdots \frac{\partial}{\partial J_{j_n i_n}} \frac{\partial}{\partial K_{l_1 k_1}} \cdots \frac{\partial}{\partial K_{l_n k_n}} W(J,K) \right)\Bigg|_{J=K=0}. \quad (4)$$

We wish to express $W(J,K)$ in a convenient form that will permit a graphical evaluation of these derivatives.

We first eliminate the K dependence of W by expressing g^{-1} in terms of the cofactor of g. The cofactors of a matrix are easily extracted using the totally antisymmetric tensor $\epsilon_{i_1,\dots,i_N}$ which satisfies

$$\epsilon_{1,2,\dots,N} = 1. \quad (5)$$

Since g is of determinant one we obtain the simple expression

$$g^{-1}_{ij} = (\text{cof} g)_{ji}$$

$$= \frac{1}{(N-1)!} \epsilon_{j,i_1,\dots,i_{N-1}} \epsilon_{i,j_1,\dots,j_{N-1}} g_{i_1 j_1}, \cdots, g_{i_{N-1} j_{N-1}}. \quad (6)$$

Using this, multiple derivatives with respect to J can replace derivatives with respect to K; thus, we write

$$W(J,K) = \exp\left\{ \text{Tr}\left[K \left(\text{cof} \frac{\partial}{\partial J} \right) \right] \right\} W(J), \quad (7)$$

where

$$W(J) = \int dg \exp(\text{Tr} Jg). \quad (8)$$

To evaluate $W(J)$ we make use of the invariance of the integration measure, which immediately implies

$$W(J) = W(g_0 J g_1), \quad (9)$$

where g_0 and g_1 are arbitrary matrices in SU(N). In an appendix of Ref. 2 it is proven that any analytic function of J satisfying Eq. (9) is expressible as a power series in the determinant of J. Thus we write

$$W(J) = \sum_{i=0}^{\infty} a_i (\text{det} J)^i. \quad (10)$$

The fact that the integration measure is normalized implies

$$a_0 = 1. \quad (11)$$

We now derive a recursion relation to determine further a_n. Since elements of SU(N) have determinant one, $W(J)$ must satisfy the differential equation

$$\left(\text{det} \frac{\partial}{\partial J} \right) W(J) = W(J). \quad (12)$$

A combinatoric exercise in the Appendix shows

$$\left(\text{det} \frac{\partial}{\partial J} \right) (\text{det} J)^i = \frac{(i+N-1)!}{(i-1)!} (\text{det} J)^{i-1}. \quad (13)$$

From Eqs. (10), (12), and (13) we obtain

$$a_i = \frac{(i-1)!}{(i+N-1)!} a_{i-1}. \quad (14)$$

With Eq. (11) this is solved by

$$a_i = \frac{2! \, 3! \cdots (N-1)!}{i! \, (i+1)! \cdots (i+N-1)!}, \quad (15)$$

giving the expression

$$W(J) = \sum_{i=0}^{\infty} \frac{2! \cdots (N-1)!}{i! \cdots (i+N-1)!} (\text{det} J)^i. \quad (16)$$

g_{ij} =

g_{ij}^{-1} =

FIG. 1. Graphical represen-
tation of g and g^{-1}.

I =

FIG. 2. The generic integral
under consideration.

(a) δ_{ij} = i ——— j

(b) $\epsilon_{i_1 \cdots i_n}$ =

FIG. 3. (a) Representation of the Kronecker symbol;
(b) Representation of the antisymmetric tensor.

= N !

i —————— j = (N-1)! i ——— j

= (N-2)!

FIG. 4. Some combinatoric identities.

=

FIG. 5. Replacing g^{-1} with
the cofactors of g.

$\left(\right)^p$ = $\dfrac{2!\,3!\cdots(N-1)!}{(p+1)!\cdots(p+N-1)!}$ $\left(\right)^p$

+ PERMUTATIONS

FIG. 6. Evaluation of the integral. There are $(NP)!/[p!(N!)^p]$
distinct permutations to be summed.

a

b

FIG. 7. (a) Invariance of the
Kronecker symbol; (b) Invari-
ance of the antisymmetric
tensor.

= $\dfrac{1}{(N-1)!}$

= $\dfrac{1}{N!\,(N-1)!}$ = $\dfrac{1}{N}$

FIG. 8. Evaluation of the
integral $\int dg\, g_{ij}\, g^{-1}_{kl}$.

= $\left(\dfrac{1}{(N-1)!}\right)^2$

= a $\left(\right)$ + b $\left(\right)$

FIG. 9. The integral $\int dg\, g_{i_1 j_1}\, g^{-1}_{k_1 l_1}\, g_{i_2 j_2}\, g^{-1}_{k_2 l_2}$.

= $\dfrac{1}{N}$

= a $\left(\right)$ + b $\left(\right)$

= (Na + b) $\left(\right)$ + (Nb + a) $\left(\right)$

FIG. 10. Evaluation of the coefficients a and b. The closed
circles represent $\Sigma_i \delta_{ii} = N$.

Note that the determinant of a matrix is simply ex-
pressed in terms of the antisymmetric tensor ϵ

$$\det J = \frac{1}{N!}\, \epsilon_{i_1,\cdots,i_N}\, \epsilon_{j_i,\cdots,j_N}\, J_{i_1 j_1},\cdots,J_{i_N j_N}.\qquad (17)$$

A graphical notation is useful for carrying out the
derivatives Eq. (4). Directed vertical line segments
correspond to group elements. Upward directed lines
represent factors of g while downward directed lines
represented factors of g^{-1}, as illustrated in Fig. 1. The
ends of these line segments are labeled with the matrix
indices of the respective group elements. The line
direction runs from the first to the second index, as

shown in the figure. Figure 2 shows how the integral
of Eq. (2) appears in this notation.

We represent the Kronecker delta symbol δ_{ij} with an
undirected line segment connecting the indices i and j,
as shown in Fig. 3a. The antisymmetric symbol
$\epsilon_{i_1,\cdots,i_N}$ is represented by a vertex joining N lines from
the indices $i_1, \ldots, i_N$. These N lines are labeled with an
arrow representing the order of the associated indices
in the ϵ symbol, as shown in Fig. 3(b). Finally, when
two line segments are connected, a matrix product is
understood; i.e., the indices associated with the con-
nected ends are summed over. In the evaluation of group
integrals, products of ϵ symbols will often occur. Some

useful identities involving such products are:

$$\epsilon_{i_1,\ldots,i_N}\epsilon_{i_1,\ldots,i_N}=N!\,,$$

$$\epsilon_{i,i_1,\ldots,i_{N-1}}\epsilon_{j,i_1,\ldots,i_{N-1}}=(N-1)!\,\delta_{ij}, \tag{18}$$

$$\epsilon_{i,j,i_1,\ldots,i_{N-2}}\epsilon_{k,l,i_1,\ldots,i_{N-2}}=(N-2)!\,(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}).$$

These have the simple graphical representation shown in Fig. 4.

Evaluation of a group integral consists of a replacement of the directed lines in Fig. 2 with vertices and undirected lines, thus expressing the result in terms of antisymmetric ϵ and Kronecker δ symbols. The first step in this procedure is to convert all directed lines into a set of lines directed upward. This is accomplished using Eq. (6) which is shown graphically in Fig. 5. (If initially there are more downward lines than upward ones it would be equivalent and simpler to convert all lines to downward ones.) Once all lines have the same orientation, we can use Eqs. (16) and (17) to reduce these lines to a sum of terms involving ϵ symbols. Noting that the integral vanishes unless the number of group lines is a multiple of N, Eq. (16) becomes graphically Fig. 6. The indicated sum over permutations is over topologically distinct ways of connecting the group indices to pairs of ϵ vertices and does not include mere permutations of group indices coupled to the same vertex pair or permutations of the vertex pairs. The resulting sum for Np lines has $(Np)!/[p!(N!)^p]$ terms.

Certain identities on the group elements have a simple graphical representation. For example invariance of the Kronecker δ symbol

$$g_{ij}\delta_{jk}g_{kl}^{-1}=\delta_{il}\,, \tag{19}$$

is shown in Fig. 7(a). Invariance of the ϵ symbol

$$g_{i_1j_1}g_{i_2j_2}\cdots g_{i_Nj_N}\epsilon_{j_1,\ldots,j_N}=\epsilon_{i_1,\ldots,i_N}\,, \tag{20}$$

is shown in Fig. 7(b). Both of these identities must be true regardless of other lines present in the diagram.

We conclude this paper with some examples of simple integrals to illustrate the rules. First consider $p=1$ in Fig. 6. This immediately gives

$$\int dg\, g_{i_1j_1}\cdots g_{i_Nj_N}=\frac{1}{N!}\,\epsilon_{i_1,\ldots,i_N}\epsilon_{j_1,\ldots,j_N}\,. \tag{21}$$

Now consider the integral

$$I_{ijkl}=\int dg\, g_{ij}\, g_{kl}^{-1}\,, \tag{22}$$

shown graphically in Fig. 8. In this figure we use Fig. 5 to make all lines direct upwards, then we use Fig. 6 for $p=1$ to eliminate these lines, and we finally use an identity of Fig. 4 to reduce the result to

$$I_{ijkl}=\frac{1}{N}\,\delta_{jk}\delta_{il}\,. \tag{23}$$

As a final example consider the integral

$$I=\int dg\,(g_{i_1j_1}g_{k_1l_1}^{-1}g_{i_2j_2}g_{k_2l_2}^{-1})\,. \tag{24}$$

In Fig. 9 we use Fig. 5 to express I in terms of $2N$ upward lines. Use of Fig. 6 at this point would give an

expression with $(2N)!/(2!N!)$ terms; however, this evaluation can be simplified with some tricks. First note that the resulting terms will all have four, an even number, ϵ vertices both at the top and at the bottom of the diagram. These can be eliminated using identities similar to Eq. (18) to reduce the terms to sets of Kronecker δ symbols connecting separately indices at the top and at the bottom of the diagram. Furthermore note that a Kronecker δ cannot connect the indices i_1 and i_2 because they can be initially coupled only through an odd number of ϵ vertices. Using a similar conclusion on the indices j_1 and j_2, we see that the final answer for the integral must take the form

$$I=a(\delta_{i_1l_1}\delta_{i_2l_2}\delta_{j_2k_2}+\delta_{i_1l_2}\delta_{i_2l_1}\delta_{j_1k_2}\delta_{j_2k_1})$$
$$+b(\delta_{i_1l_1}\delta_{i_2l_2}\delta_{j_1k_2}\delta_{j_2k_1}+\delta_{i_1l_2}\delta_{i_2l_1}\delta_{j_1k_1}\delta_{j_2k_2}), \tag{25}$$

where only two independent coefficients are needed because of the $k_1l_1k_2l_2 \rightharpoonup k_2l_2k_1l_1$ symmetry of the integrand. The coefficients a and b can now be determined by multiplying by $\delta_{j_1k_1}$ and using Fig. 7(a) to reduce the integral to that in Fig. 8. This sequence of steps is illustrated in Fig. 10 and leads to the conclusion

$$a=\frac{1}{N^2-1}\,,\qquad b=\frac{-1}{N(N^2-1)}\,. \tag{26}$$

Inserting this in Eq. (25) gives the desired integral.

APPENDIX

Here we prove Eq. (13). Defining

$$f(J)=\left(\det\frac{\partial}{\partial J}\right)(\det J)^i\,, \tag{A1}$$

we first note that properties of the determinant imply

$$f(J)=f(g_0 J g_1), \tag{A2}$$

for arbitrary g_0 and g_1 in SU(N). By the theorem mentioned below Eq. (9), $f(J)$ must be a function only of $\det J$. By homogeneity we conclude

$$f(J)=C(N,i)(\det J)^{i-1}\,, \tag{A3}$$

where $C(N,i)$ will now be determined by a recursion relation. Setting $J_{ij}=\delta_{ij}$, we have

$$C(N,i)=\left(\det\frac{\partial}{\partial J}\right)(\det J)^i\bigg|_{J_{ij}=\delta_{ij}}\,. \tag{A4}$$

Writing $\det(\partial/\partial J)$ in terms of ϵ symbols and isolating the sum over minors of the last row gives

$$C(N,i)=\sum_{j=1}^{N}\frac{\partial}{\partial J_{N,j}}\left(\epsilon_{i_1,\ldots,i_{N-1},j}\frac{\partial}{\partial J_{1,i_1}}\cdots\frac{\partial}{\partial J_{N-1,i_{N-1}}}\right)$$

$$\times(\det J)^i\big|_{J_{ij}=\delta_{ij}}\,. \tag{A5}$$

When $j=N$ in this sum we obtain i times $C(N-1,i)$, while by symmetry all $(N-1)$ terms for $j\neq N$ are equal. Separating the sum over the next to the last row gives $C(N,i)$

$$=iC(N-1,i)+(N-1)\frac{\partial}{\partial J_{N,N-1}}\sum_{j=1}^{N}\frac{\partial}{\partial J_{N-1,j}}$$

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$$\times \left(\epsilon_{i_1, \dots, i_{N-2}, j, N-1} \frac{\partial}{\partial J_{1, i_1}} \cdots \frac{\partial}{\partial J_{N-2, i_{N-2}}} \right) (\det J)^i \Big|_{J_{ij} = \delta_{ij}}$$

(A6)

In this sum, when $j = N$ we obtain i times $C(N-2, i)$, when $j = N-1$ we get no contribution, and when $j \leqslant N-2$ we have $N-2$ equal terms. Repeating this process on further rows gives

$$C(N, i) = i\{C(N-1, i) + (N-1) C(N-2, i)$$
$$+ (N-1)(N-2) C(N-3, i) + \cdots (N-1)! C(1, i)\}.$$

(A7)

Combining (A7) for N and for $N-1$, we see

$$C(N, i) = (i + N - 1) C(N-1, i).$$

(A8)

Using the initial condition $C(1, i) = i$, we conclude

$$C(N, i) = \frac{(i + N - 1)!}{(i - 1)!},$$

(A9)

which gives Eq. (13).

[1]K. Wilson, Phys. Rev. D **10**, 2445 (1974); L. P. Kadanoff, Rev. Mod. Phys. **49**, 267.
[2]M. Creutz, "Feynman Rules for Lattice Gauge Theory" (to be published in Rev. Mod. Phys.).

Almost-structures and structures in Lorentzian manifolds. I. Almost-Hermite- and almost-product-(2×2)-structures

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We present a list of the most important almost-structures which have been found of interest in general relativity, in the null-bivectors formalism. We discusss some of the relevant properties of such almost-structures and various new or more or less known results. We also present theorems on the relations between almost-product-structures and almost-Hermitian-structures.

INTRODUCTION

Recently the complex differential geometry on Hermitian and Kählerian manifolds has been utilized in general relativity: the twistor theory and the heaven-space theory are the best examples (see the review by Flaherty[1] and his bibliography). In particular, the various almost-structures and structures on a differentiable (or analytic) manifold, an old and well-studied subject of the classical differential geometry, appear to be very useful when suitably generalized and modified for the application to the Lorentzian manifold of the general relativity. In this paper we present some considerations and facts concerning the almost-structures (structures will be considered in a forthcoming paper) in general relativity treated in the formalism of null bivectors.

Some of the results we present here are more or less known, but we think that collecting them in a concise and powerful formalism, well-adapted to general relativity, will be useful for further investigations.

In the context of this article a space—time will be a differentiable manifold M with a Lorentzian real valued metric g of signature $(+ - - -)$. The definition of the Riemann and the Weyl tensors are as in the classical article by Penrose.[2] We shall, moreover, suppose that our manifold should be parallelizable so that it admits global null tetrad fields (see Geroch[3]).

1. THE COMPLEX BIVECTOR FORMALISM

In this section we briefly recall the complex bivector formalism introduced by Debever[4] which is essentially a modification of the null tetrad formalism of Newmann and Penrose.[5] For a more detailed treatment see, e.g., Israel,[6] Pirani,[7] Bampi *et al.*[8]

At each point of space—time a set of complex-valued 2-forms is introduced:

$$Z^{(1)} = 2\sqrt{2}\, \theta^{(0)} \wedge \theta^{(1)},$$
$$Z^{(2)} = 2\sqrt{2}\, \theta^{(2)} \wedge \theta^{(3)}, \tag{1.1}$$
$$Z^{(3)} = 2(\theta^{(1)} \wedge \theta^{(2)} - \theta^{(0)} \wedge \theta^{(3)}),$$

where, in local coordinates,

$$\theta^{(0)} = n_\alpha dx^\alpha$$
$$\theta^{(1)} = -\bar{m}_\alpha dx^\alpha$$
$$\theta^{(2)} = -m_\alpha dx^\alpha \tag{1.2}$$
$$\theta^{(3)} = l_\alpha dx^\alpha, \qquad \alpha = 0,1,2,3$$

n and l are real null vector fields and m, $\bar{m}$ are null complex vector fields. The only nonvanishing scalar products are

$$g(l,n) = -g(m,\bar{m}) = 1, \tag{1.3}$$

and the metric tensor reads

$$g = \theta^{(0)} \overset{s}{\otimes} \theta^{(3)} - \theta^{(1)} \overset{s}{\otimes} \theta^{(2)}, \tag{1.4}$$

where $\overset{s}{\otimes}$ is the symmetrized tensorial product. Let Λ_2 be the vectorial space of complex-valued 2-forms (bivectors): we can introduce the following operations induced by its algebraic structure only:

"Duality": $* : \Lambda_2 \times \Lambda_2 \to \Lambda_2$,

$$F_{ij} \to F^*_{ij} = \tfrac{1}{2} i \epsilon_{ijkm} F^{km}, \tag{1.5}$$

where ϵ_{ijkm} is the usual Levi-Civita tensor; here, and in the following, Latin indexes $(i = 0,1,2,3)$ will denote the null-tetrad components of a tensor quantity.

"Scalar product" $(\ ,\ )$: $\Lambda_2 \times \Lambda_2 \to \mathbb{C}$,

$$(F,G) = \tfrac{1}{4} F_{ij} G^{ij}, \tag{1.6}$$

which is nonsingular, i.e., $(F,G) = 0 \; \forall \; G \Longrightarrow F = 0$.

"Commutator" $[\ ,\ ]$: $\Lambda_2 \times \Lambda_2 \to \Lambda_2$

$$[F,G] = F_{ij} G_k{}^j \theta^{(i)} \wedge \theta^{(k)}; \tag{1.7}$$

One can easily see that this operation induces a Lie algebra structure on the Λ_2 space.

"Bracket" $\{\ ,\ \}$: $\Lambda_2 \times \Lambda_2 \to S_2$,

$$\{F,G\} = -\tfrac{1}{2}(F_{ij} G_k{}^j + F^*_{ij} G^{*j}_k) \theta^{(i)} \otimes \theta^{(k)}, \tag{1.8}$$

where S_2 is the space of complex traceless symmetric tensors of rank 2.

We collect here for later convenience some useful relations:

$$\forall X \in T_p(M), \quad \forall\; F, G \in \Lambda_2:$$
$$g(F(X), G(X))$$
$$= [F,G](X,X) - \{F,G\}(X,X) + (F,G)\, g(X,X), \tag{1.9}$$

$$g(\{F,\bar{F}\}(X), \{F,\bar{F}\}(X)) = (F,F)(\bar{F},\bar{F})\, g(X,X), \tag{1.10}$$

$$F(F(X)) = -(F,F)X. \tag{1.11}$$

Let C_3 and $\bar{C}_3$, respectively, be the space of self-dual $(F^* = iF)$ and anti-self-dual $(F^* = -iF)$ 2-forms: one can easily show the following results:

(a) $C_3 \cap \overline{C}_3 = \phi$;

(b) $\Lambda_2 = C_3 \oplus \overline{C}_3$;

(c) if $\{\theta^{(i)}\}$ is a null tedrad as in (1.2), (1.3), (1.4), then $\{Z^{(\alpha)}\}$ as in (1.1) is a base for C_3 and $\{\overline{Z^{(\alpha)}}\}$ is a base for $\overline{C}_3$; we have, moreover,

$$Z^{(3)} = -[Z^{(1)}, Z^{(2)}] \tag{1.12}$$

$$(Z^{(\alpha)}, Z^{(\beta)}) \equiv \gamma^{\alpha\beta} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{1.13}$$

One can identify [due to the nonsingular nature of the scalar product (1.6)], with the usual isomorphism, C_3^* and C_3, and one can introduce the dual base

$$\{Z_{(\alpha)}\} \quad \text{and} \quad \{\overline{Z_{(\alpha)}}\}.$$

It turns out that the bases are related by

$$Z^{(\alpha)} = \gamma^{\alpha\beta} Z_{(\beta)},$$

$$Z_{(\alpha)} = \gamma_{\alpha\beta} Z^{(\beta)},$$

and

$$\gamma_{\alpha\beta} \equiv \gamma^{\alpha\beta}.$$

We collect here for later convenience some formulas:

$$(Z_{(\alpha)}, \overline{Z}_{(\beta)}) = 0, \tag{1.14}$$

$$[Z_{(\alpha)}, \overline{Z}_{(\beta)}] = 0, \tag{1.15}$$

$$\{Z_{(\alpha)}, Z_{(\beta)}\} = \{\overline{Z}_{(\alpha)}, \overline{Z}_{(\beta)}\} = 0. \tag{1.16}$$

We pass now to the bivectorial analysis. In this formalism the first Cartan structure equations are

$$dZ^{(\alpha)} = \epsilon^{\alpha\beta\gamma} \sigma_{(\beta)} \wedge Z_{(\gamma)}, \tag{1.17}$$

$\epsilon^{\alpha\beta\gamma}$ is the Levi-Civita three-dimensional tensor and the 1-forms $\sigma_{(\alpha)}$ are related to the usual Newman—Penrose[5] coefficients by

$$\sigma_{(1)} = -\sqrt{2}\,(\kappa\theta^{(0)} + \sigma\theta^{(1)} + \rho\theta^{(2)} + \tau\theta^{(3)}),$$

$$\sigma_{(2)} = -\sqrt{2}\,(\pi\theta^{(0)} + \mu\theta^{(1)} + \lambda\theta^{(2)} + \nu\theta^{(3)}), \tag{1.18}$$

$$\sigma_{(3)} = 2(\epsilon\theta^{(0)} + \beta\theta^{(1)} + \alpha\theta^{(2)} + \gamma\theta^{(3)}).$$

The second Cartan equations are

$$\Sigma_{(\alpha)} = d\sigma_{(\alpha)} - \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\sigma^{(\beta)} \wedge \sigma^{(\gamma)}, \tag{1.19}$$

where, in vacuum,

$$\Sigma_{(\alpha)} = \tfrac{1}{4}C_{\alpha\beta}Z^{(\beta)} \tag{1.20}$$

(the electrovacuum case will be treated in the following) and, in terms of Newman-Penrose quantities:

$$C_{\alpha\beta} = \begin{bmatrix} -2\psi_0 & -2\psi_2 & 2\sqrt{2}\psi_1 \\ -2\psi_2 & -2\psi_4 & -2\sqrt{2}\psi_3 \\ 2\sqrt{2}\psi_1 & -2\sqrt{2}\psi_3 & -4\psi_2 \end{bmatrix}. \tag{1.21}$$

(1.21) follows from the definition

$$C_{ijkm} - iC^*_{ijkm} = C_{\alpha\beta}Z^{(\alpha)}_{ij}Z^{(\beta)}_{km}. \tag{1.22}$$

Finally, the Bianchi identities read

$$d\Sigma^{(\alpha)} = \epsilon^{\alpha\beta\gamma}\sigma_{(\beta)} \wedge \Sigma_{(\gamma)}. \tag{1.23}$$

We derive now a relation that will be used later. From the first Cartan equation for $Z^{(3)}$ we have

$$dZ^{(3)} = \sigma_{(1)} \wedge Z^{(1)} - \sigma_{(2)} \wedge Z^{(2)} = \psi \wedge Z^{(3)} \tag{1.24}$$

where (see Ref. 9)

$$\psi = \sqrt{2}\,\{\sigma_{(1)2}\theta^{(0)} + \sigma_{(1)3}\theta^{(1)} - \sigma_{(2)0}\theta^{(2)} - \sigma_{(2)1}\theta^{(3)}\}. \tag{1.25}$$

We turn now to the expression for $\Sigma_{(\alpha)}$ in the electrovacuum case. Let F be the electromagnetic 2-form; it turns out (see Ref. 10) that if $F \in \Lambda_2$ and $F = \overline{F}$, $\tau = \{F, F\}$ is the electromagnetic energy—momentum tensor. This can easily be seen by noting that

$$\tau_{ij} = \tfrac{1}{2}(F_{ip}F_j{}^p + F^*_{ip}F^{*p}_j) = (-F_{ip}F_j{}^p + \tfrac{1}{4}g_{ij}F_{pq}F^{pq}). \tag{1.26}$$

Introducing the notation

$$A_{\alpha\bar\beta} \equiv \tfrac{1}{2}\{Z_{(\alpha)}, \overline{Z}_{(\beta)}\} \tag{1.27}$$

(and $A_\alpha{}^{\bar\beta} \equiv \tfrac{1}{2}\{Z_{(\alpha)}, \overline{Z}^{(\beta)}\}$ etc.), we have that $\forall T \in S_2$

$$T = T_{ij}\theta^{(i)} \otimes \theta^{(j)} = T^{\alpha\bar\beta}A_{\alpha\bar\beta};$$

moreover, $\{A_{\alpha\bar\beta}\}$ is a base for S_2 and we can introduce[8] the following operation:

$$\langle \, , \, \rangle : \; S_2 \times S_2 \to \mathbb{C}, \tag{1.28}$$

$$\langle \tau, \omega \rangle = \tau_{ij}\omega^{ij}.$$

With this notation, after some algebra (in which we have used the usual decomposition of Weyl tensor, see e.g. Refs. 8, 2), we obtain the required result:

$$\Sigma_{(\alpha)} = \tfrac{1}{4}C_{\alpha\beta}Z^{(\beta)} + \tau_{\alpha\bar\beta}\overline{Z}^{(\beta)}, \tag{1.29}$$

where τ is given by (1.26) and

$$\overline{\tau_{\alpha\bar\beta}} = \tau_{\beta\bar\alpha}. \tag{1.30}$$

In terms of Newman—Penrose quantities we have (see Refs. 6, 9)

$$\tau_{\alpha\bar\beta} = \begin{bmatrix} -2\phi_{00} & -2\phi_{02} & 2\sqrt{2}\phi_{01} \\ -2\phi_{20} & -2\phi_{22} & 2\sqrt{2}\phi_{21} \\ 2\sqrt{2}\phi_{10} & 2\sqrt{2}\phi_{12} & 4\phi_{11} \end{bmatrix}. \tag{1.31}$$

A digression on complex invariant of Weyl tensor is now appropriate. Following Penrose,[2] we can introduce the following complex quantities, tetrad- and coordinates-invariant:

$$\mathcal{G} = C_{\alpha\beta}C^{\alpha\beta}, \tag{1.32}$$

$$\mathcal{J} = C_\alpha{}^\beta C_\gamma{}^\alpha C_\beta{}^\gamma, \tag{1.33}$$

where $C_{\alpha\beta}$ is defined in (1.22). In our notation

$$\mathcal{G} = 2C_{11}C_{22} - 4C_{13}C_{23} + 6(C_{12})^2 \tag{1.34}$$

$$\mathcal{J} = 10(C_{12})^3 + 6C_{12}C_{11}C_{22} - 6C_{12}C_{23}C_{13}$$
$$- 3C_{11}(C_{23})^2 - 3C_{22}(C_{13})^2 + 12C_{12}C_{13}C_{23}. \tag{1.35}$$

We have the results (see Ref. 2): the space—time is algebraically special if and only if

$$\mathcal{G}^3 = \tfrac{54}{25}\mathcal{J}^2; \tag{1.36}$$

the space—time is harmonic type I if $\mathcal{J} = 0$, $\mathcal{G} \neq 0$ and equianharmonic type I if $\mathcal{J} \neq 0$, $\mathcal{G} = 0$.

2. ALMOST-HERMITE-CV-STRUCTURES

In this paragraph we shall adopt the definitions of Yano.[11] An *almost-Hermite-cv-structure* on (M, g) is a differentiable tensor field J of rank $(1, 1)$ with the properties:

$$J(J(X)) = -X, \qquad (2.1)$$

$$g(J(X), J(X)) = g(X, X), \qquad (2.2)$$

for any differentiable vector field on M. We have the following result:

Theorem 1: A tensor field J is an almost-Hermitian-cv-structure if and only if it is a 2-form such that

$$(J, J) = 1, \qquad (2.3)$$

$$\{J, J\} = 0 \qquad (2.4)$$

in fact, from (2.1) and (2.2) one can see that the tensor of rank $(0, 2)$ associated with the tensor J must be anti-symmetric, and from (1.11) and (1.9) one immediately has the results (2.3) and (2.4); vice versa, from (1.11) and (1.9) one can always see that (2.3) and (2.4) guarantee that (2.1) and (2.2) are satisfied.

From this theorem follows that in a space–time (i.e., a Lorentzian manifold) there cannot exist a real-valued almost-Hermitian-structure tensor (as pointed out by Flaherty) since $\{F, F\} = 0$ if and only if either $F \in C_3$ or $F \in \overline{C}_3$ so $F \notin \Lambda_2(\mathbb{R})$.

Theorem 2: $J = iZ^{(3)}$ is an almost-Hermite-cv-structure.

In fact, we can always choose locally a null tetrad as in (1.2), (1.3), (1.4) and a set of complex valued 2-forms as in (1.1). Moreover,

$$(Z^{(3)}, Z^{(3)}) = -1,$$

$$\{Z^{(3)}, Z^{(3)}\} = 0.$$

Theorem 3: There exist a null tetrad such that the most general almost-Hermite-cv-structure can be written

$$J = iZ^{(3)}.$$

In fact, $(J, J) = 1$, $\{J, J\} = 0$ so we can always choose $Z^{(3)} = -iJ$, $Z^{(1)}$, $Z^{(2)}$ as a base for C_3. $Z^{(1)}$ and $Z^{(2)}$ are determined, up to

$$Z^{(1)} \rightarrow e^{\alpha} Z^{(1)}$$

$$Z^{(2)} \rightarrow e^{-\alpha} Z^{(2)}, \qquad \alpha \in \mathbb{C},$$

by the relations

$$(Z^{(1)}, Z^{(1)}) = (Z^{(2)}, Z^{(2)}) = 0,$$

$$(Z^{(1)}, Z^{(2)}) = 1,$$

$$iJ = [Z^{(1)}, Z^{(2)}].$$

With an almost-Hermite-cv-structure we can construct another system of tetrad invariants: Let $A = A_\alpha Z^{(\alpha)}$ be an almost-Hermite-cv-structure in a general null-tetrad base. We can construct the following tetrad-invariant quantities:

$$P = A_\alpha A^\alpha = 1, \qquad (2.5)$$

$$Q = C_{\alpha\beta} A^\alpha A^\beta, \qquad (2.6)$$

$$R = C_\alpha{}^\beta C_\beta{}^\gamma A^\alpha A_\gamma, \qquad (2.7)$$

with $C_{\alpha\beta}$ as in (1.22). Let us consider the following invariant:

$$H = \mathcal{G} - 2R + \tfrac{1}{2} Q^2 \qquad (2.8)$$

with $\mathcal{G}$ as in (1.32).

Theorem 4: $H = 0$ if and only if a principal null tetrad for A coincides with a principal null tetrad for the Weyl tensor.

In fact, in the tetrad in which

$$A_\alpha = i\delta^3_\alpha$$

we have

$$H = 2C_{11} C_{22}.$$

The 2-form $J = J_{ab} \theta^{(a)} \wedge \theta^{(b)}$ is said to be the Kähler form of the almost-Hermite-cv-structure. J is called an *almost-Kähler-cv-structure* if the Kähler form is closed $(dJ = 0)$.

Theorem 5 (see Ref. 1): A space–time admits an almost-Kähler-cv-structure if and only if there exists a null tetrad in which $\rho = \mu = \tau = \pi = 0$.

In fact, let us work in the tetrad in which $J = iZ^{(3)}$ (see Theorem 3). From Cartan's first equation for $Z^{(3)}$ we have

$$dJ = idZ^{(3)} = i(\sigma_{(1)} \wedge Z^{(1)} - \sigma_{(2)} \wedge Z^{(2)}) = 0$$

if and only if $\sigma_{(1)} \wedge Z^{(1)} = \sigma_{(2)} \wedge Z^{(2)}$, that is,

$$\rho = \mu = \tau = \pi = 0.$$

An *almost-Tachibana-cv-structure* is an almost-Hermitian-cv-structure with the property:

$$J_{ab;c} + J_{ac;b} = 0. \qquad (2.9)$$

Due to the Lorentzian signature of the metric we have the following results which are no longer valid for Riemannian manifolds:

Theorem 6: There exist an almost-Tachibana-cv-structure if and only if $\sigma_{(1)} = \sigma_{(2)} = 0$. Every almost-Tachibana-cv-structure is an almost-Kähler-cv-structure.

In fact, in the tetrad in which $J = iZ^{(3)}$ we have that

$$Z^{(3)}_{ab;c} + Z^{(3)}_{ac;b} = 0 \text{ if and only is}$$

$$\sigma_{(1)} = \sigma_{(2)} = 0 \text{ so that } dZ^{(3)} = 0.$$

Theorem 7: In an almost-Tachibana space–time $H = 0$. Moreover, an almost-Tachibana space–time must be type D nonelectrovacuum.

In fact in the tetrad in which $J = iZ^{(3)}$ we have $H = 2C_{11} C_{22}$ and from Cartan's second equations and Theorem 6 we get

$$\Sigma_{(1)} = 0, \quad \Sigma_{(2)} = 0, \quad \Sigma_{(3)} = d\sigma_{(3)}.$$

We have now, from the general expression for $\Sigma_{(\alpha)}$

$$\Sigma_{(\alpha)} = \tfrac{1}{4} C_{\alpha\beta} Z^{(\beta)} - \tfrac{1}{48} R Z_{(\alpha)} + E_{\alpha\bar{\beta}} \overline{Z}^{(\beta)},$$

$$\Sigma_{(1)} \wedge Z^{(2)} = 0 \implies C_{11} = 0,$$

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$$\Sigma_{(1)} \wedge Z^{(3)} = 0 \implies C_{13} = 0,$$

$$\Sigma_{(2)} \wedge Z^{(3)} = 0 \implies C_{23} = 0,$$

$$\Sigma_{(2)} \wedge Z^{(1)} = 0 \implies C_{22} = 0,$$

and in electrovacuum space-time these relations imply $C_{12} = 0$. Under a conformal transformation of the metric

$$g \to \Omega^2 g$$

we have

$$J_{ab} \to \Omega^2 J_{ab},$$

$$J_a{}^b \to J_a{}^b,$$

so that one can search for conformal transformations that map a space—time with an almost-Hermite-cv-structure into another one with more restrictive properties.

Theorem 8: A space—time with an almost-Hermite-cv-structure is conformally related to an almost-Kähler-space—time if and only if $d\psi = 0$ where [see (1.25)]

$$\psi = -2\{\rho\theta^{(0)} + \tau\theta^{(1)} - \pi\theta^{(2)} - \mu\theta^{(3)}\}.$$

In fact, in the usual tetrad

$$d(\Omega^2 J) = i(2\Omega d\Omega \wedge Z^{(3)} + \Omega^2 dZ^{(3)});$$

moreover, by (1.24)

$$d(\Omega^2 J) = i(2\Omega d\Omega + \Omega^2 \psi) \wedge Z^{(3)}.$$

If $d\psi = 0$, we can make $d(\Omega^2 J) = 0$ by taking Ω such that

$$\psi = -d(\log\Omega^2),$$

and if $d(\Omega^2 J) = 0$, it follows that, in the tetrad chosen,

$$\psi = -d(\log\Omega^2).$$

Theorem 9: A vacuum space—time of type II or D is conformally related to an almost-Kählerian manifold (in general with complex metric). [See (1) for type D.]

In fact, first of all, there exists a tetrad in which

$$\Sigma_{(1)} = \tfrac{1}{4} C_{12} Z^{(2)},$$

$$\Sigma_{(2)} = \tfrac{1}{4} C_{12} Z^{(1)} + \tfrac{1}{4} C_{22} Z^{(2)},$$

$$\Sigma_{(3)} = \tfrac{1}{2} C_{12} Z^{(3)},$$

and $\sigma_{(1)} \wedge Z^{(2)} = 0$. From the Bianchi identity for $\Sigma^{(3)}$ and the first Cartan equation for $dZ^{(3)}$ we have immediately that

$$d(C_{12}^{2/3} Z^{(3)}) = 0$$

and, in this tetrad,

$$J = iZ^{(3)} \text{ is an almost-Hermite-cv-structure.}$$

Theorem 10: A space—time can be conformally related to an almost-Tachibana-manifold if and only if it is of type D.

In fact we have

$$(\Omega^2 Z_{ab}^{(3)})_{|c} + (\Omega^2 Z_{ac}^{(3)})_{|b} = 0$$

(where the $|$ means covariant derivation in the $\hat{g} = \Omega^2 g$ metric) if and only if

$$\sigma_{(1)} \wedge Z^{(2)} = 0, \quad \sigma_{(2)} \wedge Z^{(1)} = 0, \quad \text{and} \quad d\psi = 0.$$

These three conditions are met only in type D space—time.

Another interesting result can be obtained for vacuum space—time. Let K^a a Killing vector; from Killing equations $K_{a;b} + K_{b;a} = 0$ we have that $K_{a;b} \in \Lambda_2$. If we define $K = K_{a;b}\theta^{(a)} \wedge \theta^{(b)}$ and $\hat{K} = K - iK^*$ we have that $\hat{K} \in C_3$ and from the integrability conditions of Killing equations, $d\hat{K} = 0$.

In the case in which $(\hat{K}, \hat{K}) \neq 0$ we can always choose the tetrad in such a way that

$$\hat{K} = B_3 Z^{(3)},$$

and so we have the following Theorem:

Theorem 11: A vacuum space—time with a Killing vector such that $(\hat{K}, \hat{K}) \neq 0$ is conformally related to an almost-Kählerian-manifold.

3. ALMOST-PRODUCT-(2 × 2)-STRUCTURES

An *almost-product*-(2×2)-*structure* is a differentiable, real valued, *traceless* tensor field P of rank $(1,1)$ such that

$$P(P(X)) = X, \tag{3.1}$$

$$g(P(X), P(X)) = g(X, X). \tag{3.2}$$

We present now some algebraic facts concerning S_2, the space of symmetric, traceless 2-tensors:

$$A_{\alpha\bar{\beta}} \equiv \tfrac{1}{2}\{Z_{(\alpha)}, \bar{Z}_{(\beta)}\} \text{ is a base for } S_2:$$

$$S = S_{ij}\theta^{(i)} \otimes \theta^{(j)} = S^{\alpha\bar{\beta}} A_{\alpha\bar{\beta}}.$$

S is real if and only if

$$\overline{S^{\alpha\bar{\beta}}} = S^{\beta\bar{\alpha}}. \tag{3.3}$$

$S \in S_2$ is said to be reducible if there exist $F, G \in C_3$ such that

$$S = \{F, \bar{G}\}. \tag{3.4}$$

It can easily be shown [see, e.g., (8)] from the definitions of $\{\ ,\ \}$ and $(\ ,\ )$ that

$$S = \{F, \bar{G}\} \text{ if and only if}$$

$$g(S(X), S(X)) = \alpha g(X, X), \quad \alpha \in \mathbb{C}, \tag{3.5}$$

and

$$g(\{F, \bar{H}\}(X), \{F, \bar{H}\}(X)) = (F, F)(\bar{H}, \bar{H}) g(X, X) \tag{3.6}$$

Theorem 12: The most general almost product-(2×2)-structure can be written

$$P = \{Z^{(3)}, \bar{Z}^{(3)}\}.$$

In fact, from (3.2) it follows [see (3.5)] that there exist $F, G \in C_3$ such that

$$P = \{F, \bar{G}\}$$

From (3.6) and (3.2) we have, moreover, that

$$(F, F)(\bar{G}, \bar{G}) = 1. \tag{3.7}$$

So $(F, F) \neq 0$ and $(G, G) \neq 0$ and there exist a tetrad in which $F = F_3 Z^{(3)}$. In this tetrad the reality condition (3.3) gives

$$G_1 = G_2 = 0 \text{ and } G_3 = \alpha F_3, \quad \alpha \in \mathbb{R}, \quad \alpha \neq 0,$$

and (3.7) gives $\alpha = \pm 1/|F_3|^2$. Moreover, from the defination of $\{\ ,\ \}$ we have that

$$P = \{F_3 Z^{(3)}, (\pm \bar{F}_3/|F_3|^2)\bar{Z}^{(3)}\} = \pm \{Z^{(3)}, \bar{Z}^{(3)}\}.$$

Theorem 13: An almost-Hermite-cv-structure induces an almost-product-(2×2)-structure.

In fact let J be an almost-Hermite-cv-structure, from Theorem 1 we have $(J, J) = 1$, $J \in C_3$ so $P = \{J, \bar{J}\}$ is an almost-product-(2×2)-structure.

Theorem 14: An almost-product-(2×2)-structure induces an almost-Hermite-cv-structure.

In fact, it follows from Theorem 12 that there exists a tetrad in which

$$P = \{Z^{(3)}, \bar{Z}^{(3)}\}$$

and in this tetrad $J = iZ^{(3)}$ is an almost-Hermite-cv-structure.

We derive now some results on the relations between almost-product-(2×2)-structures and almost-Hermite-cv-structures.

Theorem 15: An almost-product-(2×2)-structure is a Killing tensor if and only if $\kappa = \sigma = \lambda = \nu = 0$ and $\psi + \bar{\psi} = 0$.

In fact, in the tetrad in which $P = \{Z^{(3)}, \bar{Z}^{(3)}\}$ from the tetrad component of $P_{ij} = -Z_{iP}^{(3)}\bar{Z}_j^{(3)P}$ we can see that

$$P_{(ik;j)} = 0 \Longleftrightarrow \begin{cases} \sigma_{(1)0} = \sigma_{(1)1} = \sigma_{(2)2} = \sigma_{(2)3} = 0, \\ \sigma_{(1)2} = -\overline{\sigma_{(1)2}}, \quad \sigma_{(1)3} = \overline{\sigma_{(2)0}}, \\ \sigma_{(2)1} = -\overline{\sigma_{(2)1}}. \end{cases}$$

Theorem 16: In an almost-Tachibana-space—time the induced almost-product-(2×2)-structure is a Killing tensor.

See Theorems 15 and 6.

An *almost-product-(2×2)-conservative-structure* is an almost-product-(2×2)-structure such that

$$\nabla \cdot P = P^{ij}_{;j} = 0. \tag{3.8}$$

By computing the tetrad components of $\nabla \cdot P$ we can easily show that

Theorem 17: $\nabla \cdot P = 0$ if and only if

$$\psi + \bar{\psi} = 0,$$

where ψ is as in (1.25).

Under a conformal transformation, we have

$$P_{ij} \to \Omega^2 P_{ij},$$

and we can easily show (this is essentially Lemma II of Debever's paper[9]):

Theorem 18: A space—time can be conformally related to an almost-product-(2×2)-conservative-manifold if and only if

$$d(\psi + \bar{\psi}) = 0.$$

In an interesting paper Debever[9] introduces the concept of pre-Maxwellian structure which turns out to be closely related to both almost Hermitian and product-(2×2)-structures. As we are here disregarding the integra-

bility conditions, we define, following Debever, an *almost-pre-Maxwellian-structure* as a *real* 2-form F such that

$$(F, F) \neq 0, \tag{4.1}$$

$$(F, F^*) \neq 0, \tag{4.2}$$

$$\nabla \cdot \{F, F\} = 0. \tag{4.3}$$

Theorem 19: There exists an almost-pre-Maxwellian-structure if and only if the space—time can be conformally related to an almost-product-(2×2)-conservative-manifold.

In fact, from (4.1) and (4.2) we have that there exists a tetrad in which

$$F = (1/\sqrt{2})(A_{(3)} Z^{(3)} + \bar{A}_{(3)} \bar{Z}^{(3)}),$$

so that

$$\{F, F\} = A_3 \bar{A}_3 P.$$

In an electrovacuum space—time we have a real 2-form F such that

$$d(F - iF^*) = 0$$

and

$$\nabla \cdot \{F, F\} = 0 \quad [\text{see } (1.26)].$$

Let us consider the electrovacuum space—times with a non-algebraically-special electromagnetic field:

$$(F, F) \neq 0, \quad (F, F^*) \neq 0.$$

Theorem 20: An electrovacuum space—time with a nonspecial electromagnetic field can be conformally related to an almost-product-(2×2)-conservative-manifold and to an almost-Kählerian-manifold.

In fact, in the tetrad in which

$$F - iF^* = F_3 Z^{(3)}$$

we have

$$\nabla \cdot \{F, F\} = 0 \quad \text{and} \quad d(F - iF^*) = 0$$

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An investigation of Dubourdieu's list of space-times which admit holonomy groups

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An investigation is made of fourteen space-times given by Dubourdieu which admit holonomy groups. It is shown that, although nine of these space-times admit only trivial vacuum gravitational fields, the remaining ones are of Bel–Petrov types III and N. Many of the latter metrics can be identified with known exact solutions of the source-free Einstein equations including the Kerr–Goldberg metric and certain type III metrics recently studied by the authors.

1. INTRODUCTION

In 1927 Dubourdieu published a pair of papers[1,2] in which he gave a list of metrics for four-dimensional Riemann spaces admitting various types of holonomy groups. The first paper considered metrics of signature $(- - - +)$, i.e., space—times, and exhibited seventeen line elements. The second paper was concerned with enumerating the corresponding positive definite metrics. Dubourdieu's work was evidently intended to complete the lists previously given by Cartan[3] for two- and three-dimensional Riemann and Weyl spaces. No details were given of the calculations except for the reference to Cartan's memoir which itself was rather sketchy.

Dubourdieu's approach is to assume that a Riemannian space V_4 (a) admits a line element of the form

$$ds^2 = - (\Theta^1)^2 - (\Theta^2)^2 - (\Theta^3)^2 + (\Theta^0)^2,$$

where the Θ^K are linearly independent real Pfaffian forms, and (b) remains invariant under the infinitesimal generators of the Lorentz group.

$$X_{12} \equiv x_1 \partial_2 - x_2 \partial_1, \quad X_{10} \equiv x_1 \partial_0 + x_0 \partial_1,$$

$$X_{13} \equiv x_1 \partial_3 - x_3 \partial_1, \quad X_{20} \equiv x_2 \partial_0 + x_0 \partial_2,$$

$$X_{23} \equiv x_2 \partial_3 - x_3 \partial_2, \quad X_{30} \equiv x_3 \partial_0 + x_0 \partial_3.$$

In these expressions, we have written $\partial_\lambda = \partial / \partial x^\lambda$ and for convenience, we write the coordinate indices as subscripts, i.e., $x^\mu \equiv (x_0, x_1, x_2, x_3)$. Dubourdieu then classified his results according to the structure of a pair of groups γ_r and G_s. The former is a subgroup of the Lorentz group since it describes how "les vecteurs sont transformés par le groupe." The latter, which Dubourdieu calls "le groupe d'holonomie," is known as the nonhomogeneous holonomy group in current terminology.[4]

Dubourdieu's papers appear to have attracted little attention and have been largely forgotten until we recently came across them. Examination of his work reveals that having performed the lengthy task of deriving his canonical metrics, Dubourdieu never bothered to determine the types of gravitational fields admited by his space—times—indeed, the Einstein field equations are never mentioned in his work. In the present paper, we use symbolic algebraic manipulation techniques on a digital computer to verify Dubourdieu's results and to derive the explicit forms of the Einstein field equations for each metric. We show that Dubourdieu obtained some metrics which are of interest in problems of gravitational and electromagnetic radiation, some of which have been independently rediscovered.[5–8]

2. DUBOURDIEU'S LIST

In this section, we consider 14 of the 17 metrics given by Dubourdieu in Ref. 1. We denote these by capital Latin numerals corresponding essentially to Dubourdieu's listing. However, Dubourdieu's line elements (I), (II), and (IV) have been omitted since they are too general to be of interest. We have rewritten the remaining metrics in a more systematic manner and have arranged them into four main cases denoted A, B, C, and D. The variables (t, u, v, w) of his work are written as (x_0, x_1, x_2, x_3) and we write $dx^\mu = (dx_0, dx_1, dx_2, dx_3)$ for simplicity. Unless specified to the contrary, all functions appearing in the Dubourdieu metrics are arbitrary functions of the indicated variables.

Case A

This is based on the line elements

$$ds^2 = dx_3(dx_0 + A\, dx_1 + B\, dx_2 + \Omega\, dx_3) - C^2(dx_1^2 + dx_2^2),$$

$$\text{(III)}$$

where A, B, C are functions of x_1, x_2, x_3 and $\Omega = \Omega(x_0, x_1, x_2, x_3)$. Case A involves the following eight subcases which are specialization of (III):

$$A,\ B \text{ as in (III)}, \quad \Omega = k/4(\partial_1 A - \partial_2 B)x_0 + P(x_1,\ x_2,\ x_3),$$

$$\text{(VI)}$$

where $k = \text{const}$ and $C = 1$;

$$A,\ B,\ C \text{ as in (III) and } \Omega = 0; \quad\quad\quad\quad \text{(VII)}$$

$$A = \alpha(x_1, x_2), \quad B = \Omega = 0, \quad C = \beta(x_1,\ x_2)(k^2 + x_3^2)^{1/2},$$

$$\text{(VIII)}$$

where $k = \text{const}$ and $\partial_2 \alpha = 4k\beta^2$;

$A = B = 0$, $C = 1$, Ω as in (III); $\qquad\qquad$ (X)

A, B, C as in (X) with $\Omega = Q(x_0,\ x_1,\ x_2)$; $\qquad$ (XI)

A, B, C as in (X) with $\Omega = R(x_1,\ x_2,\ x_3)$; $\qquad$ (XII)

A, B, C, Ω as in (XII) with $\qquad\qquad$ (XIII)

$$\Omega = x_1 \phi(x_3) + D(x_3, x_3(x_2 + ax_1) - bx_1)$$

where a, b are constants;

A, B, C, as in (X) with $\Omega = \gamma(x_1, x_3)$. $\qquad$ (XV)

Case B

This case contains a single metric, (V), which can be written in two forms:

$$ds^2 = x_2^2 \Phi^2(dx_0^2 - dx_1^2) - dx_2^2 - dx_3^2 \qquad\qquad (\text{V-1})$$

or

$$ds^2 = dx_0^2 - dx_1^2 - x_0^2 \Psi^2(dx_2^2 + dx_3^2), \qquad\qquad (\text{V-2})$$

where

$$\Phi = \Phi(x_0, x_1) \quad \text{and} \quad \Psi = \Psi(x_2, x_3).$$

Case C

This case consists of the metric

$$ds^2 = dx_0^2 - dx_1^2 - (x_1 + x_0)^2 \Psi^2(dx_2^2 + dx_3^2), \qquad\qquad (\text{IX})$$

where

$$\Psi = \Psi(x_2, x_3).$$

Case D

This case consists of the metric

$$ds^2 = \Phi^2(dx_0^2 - dx_1^2) - \Psi^2(dx_2^2 + dx_3^2), \qquad\qquad (\text{XIV})$$

where

$$\Phi = \Phi(x_0, x_1), \quad \Psi = \Psi(x_2, x_3),$$

and the two subcases

$$ds^2 = dx_0^2 - dx_1^2 - \Psi^2(dx_2^2 + dx_3^2), \qquad\qquad (\text{XVI})$$

$$ds^2 = \Phi^2(dx_0^2 - dx_1^2) - dx_2^2 - dx_3^2. \qquad\qquad (\text{XVII})$$

3. SUMMARY OF RESULTS

None of the 14 metrics listed in Section 2 is *ab initio* flat. However, when the vacuum field equations $R_{\alpha\beta} = 0$ are imposed, many of them become flat space—times. We will indicate this by saying that the curvature tensor, $R_{\alpha\beta\lambda\mu}$, of these metrics is Bel-Petrov type O. The results of our investigations of the Dubourdieu metrics are summarized in Table I.

The physically interesting Dubourdieu metrics are contained in Case A which is characterized by metric (III). This is apparently a mild generalization of the Kerr—Goldberg metric.[5]

$$ds^2 = dx_0(2dx_3 + 2U\ dx_1 + \{V - x_3\partial_1 U\}dx_0) - (dx_1^2 + dx_2^2),$$
$$\qquad\qquad (\text{KG})$$

where $U = U(x_0,\ x_1,\ x_2)$ and $V = V(x_0,\ x_1,\ x_2)$. To compare (III) with (KG), one must set $x_0 \to 2\hat{x}_3$ and $x_3 \to \hat{x}_0$ in (III). Then (III) reduces to (KG) upon identifying $A \to 2U$, $B \to 0$, $C^2 \to 1$, $\Omega \to V - x_0\partial_1 U$. The field equations for (III) are quite unmanageable, and we have yet to produce an exact solution more general than (KG). However, we have determined that (III) is of Bel-Petrov type III.

TABLE I. This table summarizes the results of our investigation of the Dubourdieu metrics. A double entry as for the metrics (I), (II) in the group structure column, indicates that both metrics have the same γ_6 but different inhomogeneous holonomy groups. Type O denotes a flat space—time.

Metric	Group structure		Case	Bel—Petrov type	Comments
(I)	γ_6	G_{10}	$\cdots$	$\cdots$	Too general to be of interest
(II)		G_6	$\cdots$	$\cdots$	Too general to be of interest
(III)	γ_4	G_8	A	Type II or N	Generalized Kerr—Goldberg metric
(IV)	γ_3	G_6	$\cdots$	$\cdots$	Too general to be of interest
(V)		G_3	B	Type O	
(VI)	γ_3	G_7	A_{sub}	Type III	Generalized Kerr—Goldberg metric (less general than III)
(VII)		G_6	sub	Type N	A specialization of metric (VI)
(VIII)	γ_3	G_4	A_{sub}	Type O	Flat subcase of (VII)
(IX)		G_4	C	Type O	
(X)	γ_3	G_7	A_{sub}	Type N	Harris—Zund metric
(XI)	γ_2	G_5	A_{sub}	Type O	Flat subcase of (X)
(XII)	γ_2	G_5	A_{sub}	Type N	Harris—Zund metric
(XIII)		G_4	A_{sub}	Type O	Flat subcase of (XI)
(XIV)	γ_2	G_6	D	Type O	Pokhariyal metric
(XV)	γ_2	G_3	A_{sub}	Type O	Flat subcase of (XI)
(XVI)	γ_1	G_3	D_{sub}	Type O	Flat subcase of (XIV)
(XVII)	γ_1	G_3	D_{sub}	Type O	Flat subcase of (XIV)

The metric (VI) is a subcase of (III) which also generalizes (KG). It seems likely that (VI) is the most general Case A metric which is reasonable to consider; however, we again have not yet been able to produce an exact solution of the field equations which is not (KG). Metric (VI) is of Bel-Petrov type III.

Metric (VIII) is a type N specialization of (III) which essentially corresponds to a plane wave metric in a noncanonical coordinate system. Metric (VIII) is a further specialization of (VII) which is of type O. In Sec. 4, our analyses are illustrated using metrics (VII) and (VIII).

Metrics (X)—(XIII) and (XV) are specializations of (VI) with (XI), (XIII), and (XV) of type O and (X) and (XII) of type N. The two type N metrics are special cases of a type III metric recently studied by the authors.[7] This metric, which is a subcase of (III) but not of (VI), is of the form

$$ds^2 = F(x_0, \ x_1, \ x_2, \ x_3)dx_0^2 - 2dx_0 dx_3$$

$$- \exp[x_2 R(x_0)]\{P(x_0)dx_1^2 + Q(x_0)dx_2^2\}, \qquad \text{(HZ)}$$

where F, P, Q, and R are arbitrary functions subject only to the restrictions imposed by the field equations. The metric (HZ) reduces to (X) upon setting $P=Q=1$, $R=0$, interchanging the variables x_0 and x_3, and identifying $F(-x_3/2, \ x_1, \ x_2, \ x_0)$ with $\Omega(x_0, \ x_1, \ x_2, \ x_3)$. Metric (HZ) also includes two types III metrics studied by the authors[6] as well as some special cases obtained by Petrov.[9]

The remaining metrics consititute cases B, C, and D, and all are of type O. Metric (XIV) has recently been discussed by Pokhariyal[8] and appears to be of interest geometrically since it is a class two product spacetime. However, Pokhariyal does not seem to require that it satisfy any particular field equations, and if $R_{\alpha\beta}=0$, then (XIV) is type O.

4. EXAMPLES

In this section, we utilize metric (VII) and its subcase (VIII) to illustrate our technique. The field equations for (VII) reduce to

$$\Delta \ln C = 0, \qquad (4.1)$$

$$- 2(\partial_2 A - \partial_1 B)\partial_2 \ln C + \partial_{22}A - \partial_{12}B - 4C^{-2}\partial_{13}\ln C = 0, \qquad (4.2)$$

$$2(\partial A - \partial_1 B)\partial_1 \ln C - \partial_{12}A + \partial_{11}B - 4C^{-2}\partial_{23}\ln C = 0, \qquad (4.3)$$

$$- 2\partial_{13}A - 2\partial_{23}B - 8C\partial_{33}C - \partial_1 B \partial_2 A/C^2 + \tfrac{1}{2}(\partial_2 A/C)^2$$

$$+ \tfrac{1}{2}(\partial_1 B/A)^2 = 0, \qquad (4.4)$$

where $\Delta = \partial_{11} + \partial_{22}$. A special solution may be obtained by setting $A = A(x_3)$ and $B = B(x_3)$ whereupon $R_{\alpha\beta}=0$ reduces to merely

$$\Delta \ln C = 0, \quad \partial_{13}\ln C = 0, \quad \partial_{23}\ln C = 0, \quad \partial_{33}C = 0.$$

It is not difficult to verify that these equations are satisfied by

$$C = (k_1 x_3 + k_2) \exp \omega (x_1, x_2),$$

where k_1, k_2 are constants and $\omega(x_1, x_2)$ is a solution of $\Delta \omega = 0$. The functions $A(x_3)$ and $B(x_3)$ are not restricted by the field equations. This solution is Bel-Petrov type N.

For the metric VIII, the nonvanishing components of the Riemann tensor are

$$R_{1212} = -\beta^2(k^2 + x_3^2)R_{11},$$

$$R_{1213} = -R_{23},$$

$$R_{1223} = R_{13},$$

$$R_{1313} = R_{2323} = -2R_{33}/\beta^2(k^2 + x_3^2).$$

Thus, if $R_{\alpha\beta}=0$, then $R_{\alpha\beta\mu}=0$ and the space—time is of type O. Note that when the restrictions of (VIII) are applied, (4.4) yields the condition $\partial_2 \alpha = 4k\beta^2$ stated by Dubourdieu.

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Continuum calculus. III. Skorohod's weak distributions in the evaluation of a class of Feynman path integrals

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Path integrals for functionals are studied from the point of view of the continuum calculus proposed earlier
[J. Math. Phys. **17**, 1988 (1976)]. The weak distributions of Skorohod in an infinite-dimensional Hilbert
space and the p-integral method of continuum calculus are employed to derive a formula for the
functional integral, which is in turn evaluated through a natural extension of the weak distribution
expression. Generalizations are made to measures with density functions in the function space. As a
demonstration, the formula is tested against the polynomial functionals studied by Friedrichs, and valid
results are obtained for the general case.

I. INTRODUCTION

Since Feynman[1] introduced his alternative formulation of quantum mechanics in terms of a propagator which was a functional integral, this mathematical device has found applications in many branches of physics. Quantum field theory proved to be a fertile ground for the application of the functional integral method.[2] Classical statistical mechanics[3,4] can also be analyzed in terms of functional integrals in the approach of molecular fields. Hopf's[5] turbulence theory is another classical example of the utility of this method.[6,7] Applications are also made in laser transmission,[8] polymer solution theory,[9] and thermodynamics of irreversible processes.[10] On the other hand the mathematical theory has not kept equal pace. Efforts[11-13] have been made to put the integral on a rigorous basis. One of the fundamental difficulties has been the nonexistence of a general translation-invariant Lebesgue measure, in the conventional measure theory, for an infinite dimensional space.[11] Practical calculation formulas have been proposed[14,15] for physical problems. However, compared to what is needed in the solution of realistic problems (e.g., turbulence), the available means remain far from adequate.

In a series of papers[16,17] (hereafter referred to as I and II) we put forward an operational calculus with the objective of treating this problem from a new perspective. We succeeded in characterizing a class of exponential functional integrals usually encountered in the Feynman[1] formulation. Applications were made to physics and probability theory, and valid results were obtained.[16]

In this paper, we intend to enlarge the scope of functionals that can be treated by the continuum calculus method. Let B be a Banach space over complex field which is also Hausdorff in the norm topology. C^B is the set of functions (forms), v, from B to C, C being the complex numbers. We construct a complex Banach algebra, A_B, out of C^B, by the usual method of completion, including all the ideal elements. A_B is Hausdorff. Let (B, S_B, μ) be a measure space on B, and (A_B, S_A, m) be a measure space on A_B, μ and m being the measures on the σ-algebras, S_B and S_A, of the spaces B and A_B, respectively. We propose to investigate the integral of a class of functionals, $\phi:A_B \to C$,

given by the general formula,

$$\phi[y] = \int_E \mu(dt)\alpha(t)g(y(t)), \quad E \subseteq B, \quad \alpha:B \to C \qquad (1.1)$$

for some complex-valued function, $g:C \to C$. A preliminary study has been made in Paper I, using a differential homotopy approach. There we found that a uniqueness theorem was needed. Here we shall proceed with a different method, i.e., the integration of weak distributions of Skorohod,[11] and obtain equivalent results without incurring the uniqueness problem.

In Sec. II we apply the weak distribution integration method to a sample functional, $\gamma[y]$. In Sec. III, we rework the problem using the p-integral method. A functional integral is obtained that is consistent with the weak distribution result. This enables us to generalize to a formal definition of a functional integral of the functionals of type (1.1). As further demonstration, in Sec. IV we recalculate the integrals for the polynomial functionals of Friedrichs[18] with a Gaussian measure. The outcome is also consistent with that from the differential homotopy approach.[16]

To make this paper reasonably self-contained, we summarize some of the major results obtained previously.[16,17] We have proposed an operational calculus with two operations, the r differentiation, denoted by R/Rt ($\cdot$), and the p integration, $Pdt \gg (\cdot)$. Heuristically speaking, the r (i.e., *rational*) differentiation when applied to a function $f(t)$, $f:B \to C$, studies the *instantaneous ratio* of $f(t)$ in the neighborhood of the point t, just as in ordinary differentiation, d/dt is a measure of the instantaneous *difference* of f. In formula form, the r derivative of $f(t)$, denoted by Rf/Rt, at the point t which is in the interior of the support of f, is a multiplicative function, $f_t^*(\cdot)$ [i.e., $f_t^*(\alpha s + \beta r) = f_t^*(s)^\alpha f_t^*(r)^\beta$, $\forall \alpha, \beta \in C$ and $r, s \in B$], such that for every $\epsilon > 0$ it gives

$$\left| f(t + b) - f(t)f_t^*(b) \right| < \epsilon \qquad (1.2)$$

whenever $\|b\| < \delta > 0$. It is related to the ordinary derivative through a correspondence theorem:

$$\frac{R}{Rt} f(t) = \exp\left[\frac{d}{dt} \ln f(t)\right]. \qquad (1.3)$$

This theorem immediately generalizes the r derivative to an equal level with differential derivatives. The p (i.e., *potential*) integral is then obtained as the *primi-*

tive of r differentiation, in the same sense that the oridinary integral is the primitive of differentiation. It is related to the ordinary integral by

$$P_E \mu(dt) \gg f(t) = \exp[\int_E \mu(dt) \ln f(t)], \quad E \subseteq B. \quad (1.4)$$

The functional integral is then characterized as the result of the interaction between the p integral and the ordinary N-fold (limit $N \to \infty$) integral. In operator form,

$$P_E \mu(dt) \gg \int_F m(dy(t)) (\cdot)$$
$$= \exp\{\int_E \mu(dt) \ln[\int_F m(dy(t)) (\cdot)]\}, \quad E \subseteq B, \quad F \subseteq A_B. \quad (1.5)$$

This formula was successfully applied to functionals of the exponential type[16]

$$\psi[y] = \exp[\int_E \mu(dt)g(y(t))], \quad E \subseteq B, \quad g: C \to C, \quad (1.6)$$

yielding the integral

$$P_E \mu(dt) \gg \int_F m(dy(t)) \cdot \psi[y]$$
$$= \exp \int_E \mu(dt) \ln[\int_F m(dy(t)) \exp g(y(t))]. \quad (1.7)$$

For details and applications, see Paper I.[16]

II. INTEGRATION WITH RESPECT TO A WEAK DISTRIBUTION

In this section, we adopt as the basis of development the concept of weak distribution as given by Skorohod.[11] The language is of Hilbert spaces, although some of the results can be extended to more general spaces. Let X be a complex separable (Hausdorff) Hilbert space and B^* the σ-algebra of measurable sets of X. Let L be a finite dimensional subspace of X under the action of the projection operator P_L. If m is some normalized measure on (X, B^*), we define the measure m_L on the σ-algebra B_L^* of L as,

$$m_L(A) \equiv m(\{x \in X: P_L x \in A\}), \quad A \in B_L^*. \quad (2.1)$$

The family of measures m_L, defined on all finite-dimensional subspaces L of X and satisfying a compatibility condition (see e.g., Skorohod[11]), is called a *weak distribution*, m_*. It has been shown that in order that m_* corresponds uniquely to a measure m on (X, B^*), the Minlos–Sazonov theorem will have to be satisfied.[11] We shall not investigate this point here. Rather we analyze the interaction of the continuum calculus with the use of a weak distribution in the following.

For the functional $\phi[y]$ defined in (1.1), we construct a cylinder functional, ϕ_N, under the projection P_N according to Friedrichs,[18]

$$\phi_N[P_N y] \equiv \phi[P_N y], \quad \forall y \in X. \quad (2.2)$$

In order to render the developments more transparent, we consider a simple case. Let U be the real unit interval, $[0,1]$, and R^U the collection of all L^2-integrable functions from U to R, R being the real number field. The Hilbert space X is then constructed as usual from R^U when all the ideal elements of R^U are included. The σ-algebra B_U^* on U consists of all the Borel sets of $[0,1]$; and the measure μ on B_U^* is the Borel measure. The inner product is defined as

$$(y, z) \equiv \int_U \mu(dt)y(t)z(t), \quad y, z \in X. \quad (2.3)$$

For illustration, we define a simple functional on X as

$$\gamma[y] \equiv \int_U \mu(dt)\alpha(t)y(t)^n, \quad (2.4)$$

where $\alpha(t)$ is some complex valued measurable weight on U. Under the projection operator P_N, $\gamma_N[y]$ is given by

$$\gamma_N[P_N y] = \sum_{i=1}^{N} \Delta_i \alpha_i y_i^n, \quad (2.5)$$

where $P_N y = (y_1, \ldots, y_N)$, and $y_i = y(\xi_i)$, $0 = t_0 < t_1 < \cdots < t_{i-1} \le \xi_i \le t_i < \cdots < t_N = 1$, and $\Delta_i = \mu([t_{i-1}, t_i])$. We can suppress Δ_i in later development by redefining the unit length for equal distance intervals. We are interested in the functional integral I_f of $\gamma[y]$:

$$I_f[\gamma] = \int_F m(dy) \cdot \gamma[y], \quad F \subseteq X. \quad (2.6)$$

Instead, we consider first the integration of the cylinder functional γ_N with respect to a weak distribution m_L, as defined in (2.1):

$$I_N[\gamma_N] = \int_F m_L(dy) \cdot \gamma_N[P_N y]. \quad (2.7)$$

When Fubini's theorem is applicable, we have

$$I_N[\gamma_N] = \int_{F_1} m_1(dy_1) \cdots \int_{F_N} m_N(dy_N) \cdot \gamma_N(y_1, \cdots, y_N). \quad (2.8)$$

As to the compatibility conditions, see Friedrichs.[18] The subspace measures m_i are induced by m_L, which are ordinarily taken to be Gaussian, e.g.,

$$m_i(dy_i) = \exp(-\tfrac{1}{2}y_i^2 \Delta_i)\sqrt{\Delta_i/2\pi}dy_i. \quad (2.9)$$

Here we simply choose the m_i's as Borel measures on R. (This choice is to be viewed from the perspective of later developments, since we are not dealing with an ordinary measure theory here.[19]) Let $F_i = [0, z_i], \forall i$,

$$I_N[\gamma_N] = \int_0^{z_1} dy_1 \cdots \int_0^{z_N} dy_N (\sum_{i=1}^{N} \alpha_i y_i^n)$$
$$= \sum_{i=1}^{N} \alpha_i \int_0^{z_1} dy_1 \cdots \int_0^{z_N} dy_N y_i^n$$
$$= \alpha_1 \left[\frac{z_1^{n+1}}{n+1} z_2 \cdots z_N \right] + \alpha_2 \left[z_1 \frac{z_2^{n+1}}{n+1} z_3 \cdots z_N \right] + \cdots$$
$$+ \alpha_N \left[z_1 \cdots z_{N-1} \frac{z_N^{n+1}}{n+1} \right]$$
$$= (\prod_{j=1}^{N} z_j) \left(\sum_i \alpha_i \frac{z_i^n}{n+1} \right). \quad (2.10)$$

A moment's reflection is needed here. It has been the hope that as $N \to \infty$ ($\Delta_i \to 0, \forall i$), the integral $I_N[\gamma_N]$ would converge under suitable conditions to the functional integral $I_f[\gamma]$ (Friedrichs[18]). Although many past works in physical applications have made[14,15] this assumption, in the present work we shall study aspects of this problem from a new starting point. Before we embark on a continuum approach, we rewrite (2.10) as:

$$I_N[\gamma_N] \overset{1}{=} \sum_{i=1}^{N} \alpha_i (\prod_{j=1}^{N} \int_0^{z_j} dy_j) \exp(\ln y_i^n)$$
$$\overset{2}{=} \sum_{i=1}^{N} \alpha_i (\prod_{j=1}^{N} \int_0^{z_j} dy_j) \exp(\sum_{k=1}^{N} \delta_{ik} \ln y_k^n)$$
$$\overset{3}{=} \sum_{i=1}^{N} \alpha_i (\prod_{j=1}^{N} \int_0^{z_j} dy_j \exp(\delta_{ij} \ln y_j^n))$$
$$\overset{4}{=} \sum_{i=1}^{N} \alpha_i \left[\prod_{j=1}^{N} z_j \exp \left(\delta_{ij} \ln \frac{z_j^n}{n+1} \right) \right] \quad (2.11)$$
$$\overset{5}{=} \sum_{i=1}^{N} \alpha_i (\prod_{j=1}^{N} z_j) \left(\prod_{j=1}^{N} \exp \delta_{ij} \ln \frac{z_j^n}{n+1} \right)$$

$$\overset{6}{=}\left(\prod_{j=1}^{N}z_j\right)\left(\sum_{i=1}^{N}\alpha_i\frac{z_i^n}{n+1}\right).$$

In the equality 2 of the above equation, we introduced an expression containing the Kronecker delta, δ_{ik}, with a purpose. Equality 4 will also prove to be important in future developments. As N goes to infinity, the summation term in (2.11) passes naturally to a Riemann integral, while the infinite product will pose a problem. It is precisely at this point that the continuum integral will make a contribution. We note also that since we admit complex numbers in the integrand, the introduction of the logarithmic function is justified as long as the resulting expression makes sense.

III. THE FUNCTIONAL INTEGRAL

In the previous section, we employed the integration with respect to a weak distribution to study the integral of a sample functional γ. The resulting expression involves an N product $(\prod_{j=1}^{N}y_j)$. The p-integral method developed in the continuum calculus is applicable to infinite products. Thus we apply the method in this section. As in Paper I,[16] the p-integral operator, $P\mu\,(dt)\gg(\cdot)$ when applied to an N-fold integral $(N\to\infty)$ can be written in operator form as:

$$P_E\,\mu(dt)\gg\int_F m\,(dy(t))(\cdot)$$

$$\equiv\lim_{\substack{N\to\infty\\\forall j,\mu\,(E_j)\to 0}}\prod_{j=1}^{N}\,(\int_{F_j}m_j(dy_j)(\cdot))^{\mu\,(E_j)},\qquad(3.1)$$

where E_i are subsets of E due to the partition P_N.[11,18] To prepare for the interaction, we write $\gamma[y]$ as

$$\gamma[y]=\int_U\mu(dr)\alpha\,(r)y(r)^n$$

$$=\int_U\mu(dr)\alpha\,(r)\exp\int\mu(dt)\delta(r,t)\ln y(t)^n.\qquad(3.2)$$

Application of (3.1) to (3.2) now takes the form,

$$P_U\mu(dt)\gg\int_F m\,(dy(t))\cdot\gamma[y]$$

$$=\int_U\mu(dr)\alpha\,(r)\cdot\exp\int_U\mu(dt)$$

$$\times\ln[\int_F m\,(dy(t))\exp\delta(r,t)\ln y(t)^n].\qquad(3.3)$$

This corresponds to equality 3 in the weak distribution development of (2.11). To evaluate the inner integral,

$$J[z]\equiv\int_0^{z(t)}m\,(dy(t))\exp\delta(r,t)\ln y(t)^n,\qquad(3.4)$$

for $F=[0,z(t)]$, m being the Borel measure, we refer to equality 4 of (2.11) and infer the corresponding results in the continuum case as

$$J[z]=z(t)\exp\left[\delta(r,t)\ln\frac{z(t)^n}{n+1}\right].\qquad(3.5)$$

Therefore, (3.3) becomes

$$\int_U\mu(dr)\alpha\,(r)\exp\int_U\mu(dt)\ln\left[z(t)\exp\delta(r,t)\ln\frac{z(t)^n}{n+1}\right]$$

$$=\int_U\mu(dr)\alpha\,(r)\exp\left[\left(\int_U\mu(dt)\ln z(t)\right)+\ln\frac{z(r)^n}{n+1}\right]$$

$$=[\exp\int_U\mu(dt)\ln z(t)]\left[\int_U\mu(dr)\alpha\,(r)\frac{z(r)^n}{n+1}\right].\qquad(3.6)$$

As expected, we obtain an ordinary integral for the original functional and a p integral representing the limit of the infinite product. For appropriate limits of

integration, both integrals in (3.6) exist, and we proceed to define the functional integral of γ to be

$$I_f[\,\gamma]\equiv[\,\exp\int_U\mu(dt)\ln z(t)]\left[\int_U\mu(dr)\alpha\,(r)\frac{z(r)^n}{n+1}\right].\qquad(3.7)$$

We take note of the integral (3.5) and generalize for arbitrary functions, $g(y)$, $g:R\to C$, that admit a Taylor series expansion (details omitted) to obtain

Theorem 3.1 Evaluation of the J integral: Let g be a function from R to C whose primitive exists and is denoted by G, so that $dG/dy=g$. If g is in class C^∞ (possessing continuous derivatives of all orders) and admits a Taylor expansion, then the J integral with appropriate limits, $F=[0,z(t)]$, and Borel measure m is

$$J=\int_F m\,(dy(t))\exp\delta(r,t)\ln g(y(t))$$

$$=\int_0^{z(t)}dy(t)\exp\delta(r,t)\ln g(y(t))$$

$$=z(t)\{\exp\delta(r,t)\ln[G(z(t))/z(t)]\}.\qquad(3.8)$$

It is plausible that relation (3.8) also holds for more general functions, $g(y)$. This will be left for a future study. Another direction of generalization is the calculation for more general measures. As in probability theory, for measures m, with a density function $\rho(y)$, $m(dy)=\rho(y)dy$, or $\rho(y)$ being the Radon–Nikodym derivative[20] of m with respect to the Borel measure, the above relation can be generalized by a simple transformation of variables.

Let $P(y)$ be the integral (or primitive) of $\rho(y)$, $dP(y)/dy=\rho(y)$, (if it exists), P corresponds to m in the Stieltjes sense. Suppose that the mapping $x=P(y)$ is bijective; there exists an inverse function, P^{-1}, such that $y=P^{-1}(x)$. When this is substituted into the function $g(y)$, we have

$$h(x)\equiv g(P^{-1}(x)).\qquad(3.9)$$

The J integral is then

$$J=\int_F m\,(dy(t))\exp\delta(r,t)\ln g(y(t))$$

$$=\int_0^{\bar{y}}dy(t)\rho(y(t))\exp\delta(r,t)\ln g(y(t))$$

$$=\int_0^{\bar{x}}dx(t)\exp\delta(r,t)\ln h(x(t)).\qquad(3.10)$$

This expression is of the form (3.8); therefore, we have

Theorem 3.2: J Integral for a general measure: For the measure m described above, the J integral in the expression

$$J=\int_F m\,(dy(t))\exp\delta(r,t)\ln g(y(t))\qquad(3.11)$$

is given by

$$J=\bar{x}(t)\exp\delta(r,t)\ln\frac{H(\bar{x}(t))}{\bar{x}(t)}$$

$$=P(\bar{y}(t))\exp\delta(r,t)\ln\frac{H(P(\bar{y}(t)))}{P(\bar{y}(t))}\qquad(3.12)$$

where

$$H(\bar{x}(t))\equiv\int_0^{\bar{x}}dx(t)h(x(t))$$

$$=\int_0^{\bar{y}}dy(t)\rho(y(t))g(y(t)).\qquad(3.13)$$

if the indicated integrals exist. In the above expression, t is fixed during integration.

We remark that in the above development we have restricted $y(t)$ to be a real valued function (i.e., A_B is a real Banach algebra) so that no integration in the complex plane was involved. To simplify the derivation, homogeneous lower limits were used. For the Banach algebra A_B of real functions, we then propose

Definition 3.1: The functional integral of $\phi[y]$: Let A_B be a real Banach algebra, and $\phi[y]$ the functional from A_B into C, as given by (1.1), its integral, $I_f[\phi]$, on a set $F \subseteq A_B$, is given by

$$I_f[\phi] \equiv P_E \mu(dt) \gg \int_F m(dy(t)) \cdot (\int_E \mu(dr)\alpha(r)g(y(r)))$$
$$= \int_E \mu(dr)\alpha(r)\{\exp \int_E \mu(dt)\ln \int_F m(dy(t))$$
$$\times \exp[\delta(r,t)\ln g(y(t))]\}. \qquad (3.14)$$

The treatment of the innermost integral

$$J = \int_F m(dy(t)) \exp[\delta(r,t)\ln g(y(t))] \qquad (3.15)$$

is according to Theorem 3.2, whenever the indicated integrals exist.

With this definition at hand, a host of interesting examples can be worked out, since the formula given is of quite general applicability. In order to demonstrate its validity, we test the definition against a class of polynomial functionals, whose integrals are known.[18]

IV. THE POLYNOMIAL FUNCTIONALS

To test the formulas given in Sec. III, we examine the polynomial functionals studied by Friedrichs.[18] Without loss of generality, we consider a completely diagonalized r-ics,

$$\xi[y] \equiv \int_0^1 ds\, l_r(s)y(s)^r \qquad (4.1)$$

for $y:[0,1] \to R$. Friedrichs[18] took, for m, the measure in the function space A_B to be Gaussian. For even numbers of r, $r = 2n$, n an integer, the problem is similar to the γ functional considered previously, except that here we integrate with respect to a Gaussian measure. According to Definition 3.1, the functional integral takes the form

$$I_f[\xi] = P_E \mu(dt) \gg \int_F m(dy(t)) \cdot \xi[y]$$
$$= \int_0^1 ds\, l_r(s) \exp \int_0^1 dt \ln \int_{-\infty}^{+\infty} dy(t) \exp[-\tfrac{1}{2}y(t)^2]$$
$$\times \exp[\delta(s,t)\ln y(t)^r], \qquad (4.2)$$

For the innermost integral

$$J = \int_{-\infty}^{\infty} dy(t) \exp[-\tfrac{1}{2}y(t)^2] \exp[\delta(s,t)\ln y(t)^r] \qquad (4.3)$$

we apply (3.12). Here $\rho(y) = \exp(-\tfrac{1}{2}y^2)$. Therefore, $P(y)$ is the error function

$$P(\bar{y}) = \int_0^{\bar{y}} dz \exp(-z^2/2) = \sqrt{\pi/2}\ \mathrm{erf}(\bar{y}/\sqrt{2}). \qquad (4.4)$$

$P(y)$ is then one-to-one, and the inverse exists

$$x = P(y) = \sqrt{\pi/2}\ \mathrm{erf}(y/\sqrt{2}), \quad y = P^{-1}(x). \qquad (4.5)$$

The J integral is then, by (3.12),

$$J = \lim_{\bar{x} \to \sqrt{\pi/2}} 2 \int_0^{\bar{x}} dx(t) \exp[\delta(r,t)\ln P^{-1}(x)^r]$$
$$= \lim_{\bar{y} \to \infty} 2 P(\bar{y}) \exp\left[\delta(r,t)\ln \frac{H(P(\bar{y}))}{P(\bar{y})}\right], \qquad (4.6)$$

where

$$\lim_{\bar{y} \to \infty} P(\bar{y}) = \int_0^{\infty} dz \exp(-\tfrac{1}{2}z^2) = \sqrt{\pi/2} \qquad (4.7)$$

and

$$\lim_{\bar{y} \to \infty} H(P(\bar{y})) = \int_0^{\infty} dz \exp(-\tfrac{1}{2}z^2)z^r$$
$$= [(2n!/n!\,2^{n+1}]\sqrt{2\pi}\ . \qquad (4.8)$$

Substitution of (4.7) and (4.8) into (4.6), and then (4.6) into (4.2) gives

$$I_f[\xi] = \int_0^1 ds\, l_r(s) \exp \int_0^1 dt \ln\left\{\sqrt{2\pi}\exp\left[\delta(r,t)\ln\frac{(2n)!}{n!\,2^n}\right]\right\}$$
$$= \left(\frac{\sqrt{2\pi}\,(2n)!}{n!\,2^n}\right)\int_0^1 ds\, l_r(s). \qquad (4.9)$$

Except for a normalization constant $\sqrt{2\pi}$, the result is identical to the solution given by Friedrichs.[18] For $r = 2$, the formula reduces to the case studied in Paper I[16] by the differential homotopy approach

$$\xi[y] = \int_0^1 ds\, l_2(s)y(s)^2,$$
$$I_f[\xi] = \sqrt{2\pi}\ \mathrm{Tr}\, l_2 = \sqrt{2\pi}\int_0^1 ds\, l_2(s), \qquad (4.10)$$

where Tr is the trace. Same results are obtained.

We remark that the formalism given by (3.14) is valid for general measures, i.e., it is independent of the Gaussian measure, and is also applicable to finite limits of integration in the function space A_B.

V. CONCLUDING REMARKS

In a series of studies,[16,17] we have applied the method of continuum calculus to the formulation of the functional integrals, in closed form, for two classes of representation of functionals:

$$\phi[y] = \int_E \mu(dt)\alpha(t)g(y(t)) \qquad (5.1)$$

and

$$\psi[y] = \exp \int_E \mu(dt)g(y(t)). \qquad (5.2)$$

These representations encompass a wide variety of functionals of interest in applications. Other types of functionals can, in principle, be similarly analyzed, although some recalcitrant cases may be expected. Type (5.2) functionals, which are relatively easy to treat in the present methodology, are, in fact, encountered most frequently in physical applications.[1] Type (5.1) functionals require more care and are analyzed here in view of Skorohod's weak distributions. The solution is given by (3.14). Due to the novelty of the subject, more mathematical elaboration and rigor will be required to assure the existence of the proposed formulas under suitable measure theoretical and topological conditions.[11-13] It is interesting to note that the idea of p integration, one branch of the continuum calculus, is independently arrived at in the product integral formulation of Dollard and Friedman.[21]

Another less explored but equally important area of investigation is the subject of functional differential equations. Donsker and Lions[22] have given an interesting study. Consider the turbulence theory of Hopf,[5] the characteristic functional of the probability distribution

of a flow field satisfies the equation,

$$\frac{\partial \Phi}{\partial t} = \int dk' \frac{\delta}{\delta z_j(k')} \left[\int dk'' \, \tilde{z}_i(k' + k'')k''_j \, \frac{\delta \Phi}{\delta z_i(k'')} \right]$$

$$- \mu \int dk \, |k|^2 z_i(k) \frac{\delta \Phi}{\delta z_i(k)} \, . \tag{5.3}$$

The solution of this equation for a general flow problem holds promise to the progress of one important branch of physics.[7] We shall have occasion to report a study on this in the future.

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Symmetries of the 3j coefficient

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An explicit form of the five Regge symmetries of the 3j coefficient is given. It is shown that a set of six $_3F_2(1)$ hypergeometric functions is necessary and sufficient to account for the 72 symmetries of the 3j coefficient, each accounting for 12 symmetries. The eight-element group, recently discussed by Lockwood, accounts for only eight symmetries of the 3j coefficient.

The 3j coefficient is given in Regge's[1] notation as follows:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{vmatrix} j_2+j_3-j_1 & j_3+j_1-j_2 & j_1+j_2-j_3 \\ j-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{vmatrix} = (3J). \tag{1}$$

The symmetries of the 3j coefficient have been discussed by Regge[1] in terms of the column and row permutations of the above square symbol.

The 12-element group of physical symmetries of the 3j coefficient consists of

(a) six permutations of three angular momenta and

(b) six permutations of the space reflection.

These correspond to the column permutations and the exchange of the lowest rows of the square symbol, respectively.

The symmetry operation on the square symbol that the rows can be exchanged with columns[1] implies the following symmetry operation on the 3j coefficient:

$$(3J) = R1 = \begin{pmatrix} j_1 & \dfrac{(j_2-m_2)+(j_3-m_3)}{2} & \dfrac{(j_2+m_2)+(j_3+m_3)}{2} \\ j_2-j_3 & \dfrac{(j_3-m_3)-(j_2-m_2)}{2} & \dfrac{(j_3+m_3)-(j_2+m_2)}{2} \end{pmatrix}. \tag{2}$$

The four possible ways of interchanging the rows of the square symbol[1] (left after the space reflection) imply the following symmetry operation on the 3j coefficient

$$(3J) = R2 = \begin{pmatrix} \dfrac{j_2+j_3-m_1}{2} & \dfrac{j_1+j_3-m_2}{2} & \dfrac{j_1+j_2-m_3}{2} \\ \dfrac{(j_1-m_1)+(j_1-j_2-j_3)}{2} & \dfrac{(j_2-m_2)+(j_2-j_1-j_3)}{2} & \dfrac{(j_3-m_3)+(j_3-j_1-j_2)}{2} \end{pmatrix}, \tag{3}$$

$$(3J) = R3 = \begin{pmatrix} \dfrac{j_2+j_3+m_1}{2} & \dfrac{j_1+j_3+m_2}{2} & \dfrac{j_1+j_2+m_3}{2} \\ \dfrac{(j_2+j_3-j_1)-(j_1+m_1)}{2} & \dfrac{(j_1+j_3-j_2)-(j_2+m_2)}{2} & \dfrac{(j_1+j_2-j_3)-(j_3+m_3)}{2} \end{pmatrix}, \tag{4}$$

$$(3J) = R4 = (-1)^J \begin{pmatrix} \dfrac{j_2+j_3-m_1}{2} & \dfrac{j_1+j_3-m_2}{2} & \dfrac{j_1+j_2-m_3}{2} \\ \dfrac{(j_2+j_3-j_1)-(j_1-m_1)}{2} & \dfrac{(j_1+j_3-j_2)-(j_2-m_2)}{2} & \dfrac{(j_1+j_2-j_3)-(j_3-m_3)}{2} \end{pmatrix}, \tag{5}$$

and

$$(3J) = R5 = (-1)^J \begin{pmatrix} \dfrac{j_2+j_3+m_1}{2} & \dfrac{j_1+j_3+m_2}{2} & \dfrac{j_1+j_2+m_3}{2} \\ \dfrac{(j_1+m_1)+(j_1-j_2-j_3)}{2} & \dfrac{(j_2+m_2)+(j_2-j_1-j_3)}{2} & \dfrac{(j_3+m_3)+(j_3-j_1-j_2)}{2} \end{pmatrix}, \tag{6}$$

where $j_1+j_2+j_3=J$. Equations (2)–(6) give an explicit form of the Regge symmetries in which the j's and m's of the

$3j$ coefficient will be replaced by algebraic expressions involving the original j's and m's. The five Regge symmetries are named as R1, R2, R3, R4, and R5, respectively.

The study of the Regge symmetries of the $3j$ coefficient in the form given above, makes the 72 symmetries more apparent. For example, the superposition of one of the symmetries of type (b), i.e., changing m_1, m_2, and m_3 to their negative values, on R1, will only interchange the second and third columns of R1. This symmetry is also obtained by the superposition of a symmetry of type (a) on R1. It is easy to see that the superposition of symmetries of type (b) on R2, R3, R4, and R5 will give rise to the set of symmetries which are also obtained by the superposition of the symmetries of type (a) on R2—R5. Thus, only 47 distinct symmetries can be obtained by the superposition of symmetries of type (a) and type (b) on the five Regge symmetries. But a careful study of the superposition of the symmetries of type (a) and type (b) on R1 and vice-versa yields 25 symmetries which are distinct from each other and distinct from the 47 obtained above. Exhausting all the possible combinations of symmetries of type (a), type (b), and the five Regge symmetries, one can verify that there are only 72 distinct symmetries of the $3j$ coefficient.

The series representation[2] for the $3j$ coefficient can be put in the form

$$(3J) = \delta(m_1 + m_2 + m_3)\frac{(-1)^{j_1 - j_2 - m_3}}{[(J+1)!]^{1/2}}\left[\prod_{i=1}^{3}\prod_{k=1}^{3}(\beta_i - \alpha_k)!\right]^{1/2}\sum_t(-1)^t\left[\prod_{i=1}^{3}\prod_{k=1}^{3}(\beta_i - t)!(t - \alpha_k)!\right]^{-1}, \qquad (7)$$

where $\beta_1 = j_1 + j_2 - j_3$, $\beta_2 = j_1 - m_1$, $\beta_3 = j_2 + m_2$, and $\alpha_1 = 0$, $\alpha_2 = j_2 - j_3 - m_1$, $\alpha_3 = j_1 - j_3 + m_2$. Let β_0 be the minimum of the β's and α_0 be the maximum of the α's. The number of terms in the above series representation, Eq. (7), is determined from $n = \beta_0 - \alpha_0$. It is easy to see that n takes nine different values as the $3j$ coefficient goes through its 72 symmetries.

By successively making the substitution $t - \alpha_k = s$, $k = 1,2,3$, and $\beta_i - t = s$, $i = 1,2,3$, in Eq. (7), the series representation for the $3j$ coefficient can be rearranged[3,4] into the generalized hypergeometric series with unit argument, viz., $_3F_2 (ABC;DE;1)$.

The summation over t in Eq. (7) will be replaced by

$$\frac{(-1)^P}{\Gamma(1-A)\Gamma(1-B)\Gamma(1-C)\Gamma(D)\Gamma(E)}\ _3F_2(ABC;DE;1). \qquad (8)$$

The parameters of the $_3F_2(1)$ series for the substitution $t - \alpha_k = s$, $k = 1,2,3$, are given below:

$$A = -(j_1 + j_2 - j_3), \quad B = -(j_1 - m_1), \quad C = -(j_2 + m_2), \quad D = j_3 - j_2 + m_1 + 1, \quad E = j_3 - j_1 - m_2 + 1, \quad P = \alpha_1 = \alpha_0, \qquad (9)$$

$$A = -(j_1 + m_1), \quad B = -(j_1 + j_3 - j_2), \quad C = -(j_3 - m_3), \quad D = j_2 - j_3 - m_1 + 1, \quad E = j_2 - j_1 + m_3 + 1, \quad P = \alpha_2 = \alpha_0, \qquad (10)$$

and

$$A = -(j_2 - m_2), \quad B = -(j_3 + m_3), \quad C = -(j_2 + j_3 - j_1), \quad D = j_1 - j_3 + m_2 + 1, \quad E = j_1 - j_2 - m_3 + 1, \quad P = \alpha_3 = \alpha_0. \qquad (11)$$

The parameters of the $_3F_2(1)$ series for the substitution $\beta_i - t = s$, $i = 1,2,3$, are given below:

$$A = -(j_1 + j_2 - j_3), \quad B = -(j_1 + m_1), \quad C = -(j_2 - m_2), \quad D = j_3 - j_2 - m_1 + 1, \quad E = j_3 - j_1 + m_2 + 1, \quad P = \beta_1 = \beta_0, \qquad (12)$$

$$A = -(j_1 - m_1), \quad B = -(j_1 + j_3 - j_2), \quad C = -(j_3 + m_3), \quad D = j_2 - j_3 + m_1 + 1, \quad E = j_2 - j_1 - m_3 + 1, \quad P = \beta_2 = \beta_0, \qquad (13)$$

and

$$A = -(j_2 + m_2), \quad B = -(j_3 - m_3), \quad C = -(j_2 + j_3 - j_1), \quad D = j_1 - j_3 - m_2 + 1, \quad E = j_1 - j_2 + m_3 + 1, \quad P = \beta_3 = \beta_0. \qquad (14)$$

The series representation, Eq. (1) of Ref. 2, corresponds to the $_3F_2(1)$ series given by Eq. (9). Since the $_3F_2(1)S$ correspond to the minimum of the β's or to the maximum of the α's, it is easy to see that the numerator parameters are all negative and the denominator parameters are positive. The upper limit of the summation index in the $_3F_2(1)S$, n is given by the minimum of the negative value of the numerator parameters. Choosing any one of the $_3F_2(1)$ series, say, that given by Eq. (9), and associating the $3j$ coefficient $\left(\begin{smallmatrix}j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{smallmatrix}\right)$ with the identity permutation of parameters, viz., $_3F_2(ABC;DE;1)$, it is shown that the 3! permutations of the three numerator parameters together with the two permutations of the denominator parameters will account for 12 symmetries of the $3j$ coefficient. Each of the six $_3F_2(1)S$ will account for 12 symmetries of the $3j$ coefficient. It is straightforward to list the symmetries accounted to by a single $_3F_2(1)$ series. As an illustration of the proof, the list of 12 symmetries accounted to by the permutations of parameters of the $_3F_2(1)$ series, given by Eq. (9), is given in Table I.

The advantage of describing the symmetries of the $3j$ coefficient in terms of permutations of parameters of $_3F_2(1)S$ is that the absolute symmetries can be clearly separated from those carrying the phase factors. The $_3F_2(1)S$ given by Eqs. (9), (10), and (11) account for the 36 absolute symmetries, each accounting for 12 of them. The $_3F_2(1)S$ given by Eqs. (12), (13), and (14) account for the 36 symmetries which carry the phase factor, each account-

ing for 12 of them. Thus the set of six $_3F_2(1)S$ is necessary and sufficient to account for the 72 symmetries of the $3j$ coefficient.

Since the number of terms in the $_3F_2(1)$ series representation for the $3j$ coefficient takes nine possible values, there should be nine sets of canonical parameters[2] of the $3j$ coefficient: $(n_i;\ a_i, b_i;\ c_i, d_i;\ i = 1, 9)$. It is elementary to calculate the nine sets of canonical parameters. The 72 symmetries can be partitioned into nine sets of eight each depending on the number of terms in the $_3F_2(1)S$ for the $3j$ coefficient. The list of eight symmetries accounted to by the $_3F_2(1)$ series when it has $j_1 + j_2 - j_3 + 1$ number of terms is given in Table II.

TABLE I. The list of 12 symmetries accounted to by the permutations of parameters of $_3F_2(1)$ series given by Eq. (9).

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad \begin{pmatrix} j_2 & j_1 & j_3 \\ -m_2 & -m_1 & -m_3 \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{j_1 + j_2 - m_3}{2} & \dfrac{j_2 + j_3 - m_1}{2} & \dfrac{j_1 + j_3 - m_2}{2} \\ \dfrac{(j_3 - m_3) + (j_3 - j_1 - j_2)}{2} & \dfrac{(j_1 - m_1) + (j_1 - j_2 - j_3)}{2} & \dfrac{(j_2 - m_2) + (j_2 - j_1 - j_3)}{2} \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{j_1 + j_2 + m_3}{2} & \dfrac{j_1 + j_3 + m_2}{2} & \\ \dfrac{(j_3 + m_3) + (j_3 - j_1 - j_2)}{2} & \dfrac{(j_2 + m_2) + (j_2 - j_3 - j_1)}{2} & \dfrac{j_2 + j_3 + m_1}{2} \\ & & \dfrac{(j_1 + m_1) + (j_1 - j_2 - j_3)}{2} \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{j_1 + j_3 + m_2}{2} & \dfrac{j_1 + j_2 + m_3}{2} & \dfrac{j_2 + j_3 + m_1}{2} \\ \dfrac{(j_3 + j_1 - j_2) - (j_2 + m_2)}{2} & \dfrac{(j_1 + j_2 - j_3) - (j_3 + m_3)}{2} & \dfrac{(j_2 + j_3 - j_1) - (j_1 + m_1)}{2} \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{j_2 + j_3 - m_1}{2} & \dfrac{j_1 + j_2 - m_3}{2} & \dfrac{j_1 + j_3 - m_2}{2} \\ \dfrac{(j_2 + j_3 - j_1) - (j_1 - m_1)}{2} & \dfrac{(j_1 + j_2 - j_3) - (j_3 - m_3)}{2} & \dfrac{(j_3 + j_1 - j_2) - (j_2 - m_2)}{2} \end{pmatrix}$$

$$\begin{pmatrix} j_1 & \dfrac{(j_2 + m_2) + (j_3 + m_3)}{2} & \dfrac{(j_2 - m_2) + (j_3 - m_3)}{2} \\ j_3 - j_2 & \dfrac{(j_2 + m_2) - (j_3 + m_3)}{2} & \dfrac{(j_2 - m_2) - (j_3 - m_3)}{2} \end{pmatrix}$$

$$\begin{pmatrix} j_2 & \dfrac{(j_3 - m_3) + (j_1 - m_1)}{2} & \dfrac{(j_3 + m_3) + (j_1 + m_1)}{2} \\ j_3 - j_1 & \dfrac{(j_1 - m_1) - (j_3 - m_3)}{2} & \dfrac{(j_1 + m_1) - (j_3 + m_3)}{2} \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{(j_2 + m_2) + (j_3 + m_3)}{2} & j_1 & \dfrac{(j_2 - m_2) + (j_3 - m_3)}{2} \\ \dfrac{(j_3 + m_3) - (j_2 + m_2)}{2} & j_2 - j_3 & \dfrac{(j_3 - m_3) - (j_2 - m_2)}{2} \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{(j_1 - m_1) + (j_3 - m_3)}{2} & j_2 & \dfrac{(j_1 + m_1) + (j_3 + m_3)}{2} \\ \dfrac{(j_3 - m_3) - (j_1 - m_1)}{2} & j_1 - j_3 & \dfrac{(j_3 + m_3) - (j_1 + m_1)}{2} \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{(j_2 + m_2) + (j_1 + m_1)}{2} & \dfrac{(j_2 - m_2) + (j_1 - m_1)}{2} & j_3 \\ \dfrac{(j_1 + m_1) - (j_2 + m_2)}{2} & \dfrac{(j_1 - m_1) - (j_2 - m_2)}{2} & j_2 - j_1 \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{(j_1 - m_1) + (j_2 - m_2)}{2} & \dfrac{(j_1 + m_1) + (j_2 + m_2)}{2} & j_3 \\ \dfrac{(j_2 - m_2) - (j_1 - m_1)}{2} & \dfrac{(j_2 + m_2) - (j_1 + m_1)}{2} & j_1 - j_2 \end{pmatrix}$$

TABLE II. The list of eight symmetries accounted to by the $_3F_2(1)$ series when it has $j_1 + j_2 - j_3 + 1$ number of terms.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \qquad \begin{pmatrix} j_2 & j_1 & j_3 \\ -m_2 & -m_1 & -m_3 \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{(j_2+m_2)+(j_1+m_1)}{2} & \dfrac{(j_2-m_2)+(j_1-m_1)}{2} & j_3 \\[2mm] \dfrac{(j_1+m_1)-(j_2+m_2)}{2} & \dfrac{(j_1-m_1)-(j_2-m_2)}{2} & j_2-j_1 \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{(j_2-m_2)+(j_1-m_1)}{2} & \dfrac{(j_2+m_2)+(j_1+m_1)}{2} & j_3 \\[2mm] \dfrac{(j_2-m_2)-(j_1-m_1)}{2} & \dfrac{(j_2+m_2)-(j_1+m_1)}{2} & j_1-j_2 \end{pmatrix}$$

$$(-1)^J \begin{pmatrix} \dfrac{(j_2-m_2)+(j_1-m_1)}{2} & \dfrac{(j_2+m_2)+(j_1+m_1)}{2} & j_3 \\[2mm] \dfrac{(j_1-m_1)-(j_2-m_2)}{2} & \dfrac{(j_1+m_1)-(j_2+m_2)}{2} & j_2-j_1 \end{pmatrix}$$

$$(-1)^J \begin{pmatrix} \dfrac{(j_2+m_2)+(j_1+m_1)}{2} & \dfrac{(j_2-m_2)+(j_1-m_1)}{2} & j_3 \\[2mm] \dfrac{(j_2+m_2)-(j_1+m_1)}{2} & \dfrac{(j_2-m_2)-(j_1-m_1)}{2} & j_1-j_2 \end{pmatrix}$$

$$(-1)^J \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \qquad (-1)^J \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$$

The expression for the $3j$ coefficient in terms of canonical parameters, Eq. (4) of Ref. 2, is invariant for three different values of n. Since the canonical parameters c and d change their values as n takes different values, only eight symmetries are accounted to by the eight-element group, discussed by Lockwood. Thus, the set of eight symmetries corresponding to an n_i ($i = 1, 9$) will form a group by itself in the canonical parametrization. Table II clearly indicates the separating of the absolute symmetries from those carrying the phase factor. The absolute symmetries are explained by the $_3F_2(1)$ series given by Eq. (9), and the symmetries carrying the phase factor are explained by the $_3F_2(1)$ series given by Eq. (12).

ACKNOWLEDGMENTS

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A new form of the Mayer expansion in classical statistical mechanics[a]

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New expressions are given for the expansion coefficients in the Mayer expansion (and thus the virial expansion). These promise to be useful in applications, as well as provide a simple rigorous proof of the convergence of the Mayer series and some of its properties.

We will present a new form for the Mayer series coefficients, that is formally patterned on cluster expansions recently introduced into quantum statistical mechanics.[1] The new form has already been useful in a theoretical application (to the study of Coulomb shielding),[2] and may have other practical applications. This form leads to a proof of convergence we believe to be much simpler than the standard proofs: see Ref. 3 for one of the standard proofs, and further references. This paper may also serve as an introduction to some of the techniques used in Ref. 1, and suggest lines of development for more complicated systems.

To set the stage for the general situation, we first consider a system with a single species of particles interacting via a two-body potential $v(x)$. In the Mayer series,

$$\beta p = \sum_1^{\infty} b_n z^n, \tag{1}$$

we give our form of the first three coefficients explicitly (the b_n term for a general system will be given later):

$$b_1 = 1, \tag{2}$$

$$b_2 = -\tfrac{1}{2}\beta \int_0^1 ds \int d^3x\, v(x) \exp(-\beta s v(x)), \tag{3}$$

$$
\begin{aligned}
b_3 = \tfrac{1}{3}\beta^2 \int_0^1 ds_1 \int_0^1 ds_2 \int d^3x \int d^3y\, [v(x)v(y) \\
+ s_1 v(x)v(x+y)] \cdot \exp\{-\beta[s_1 v(x) + s_2 v(y) \\
+ s_1 s_2 v(x+y)]\}.
\end{aligned}
\tag{4}
$$

The usual expressions for the b_n may be found in most textbooks on statistical mechanics, such as Refs. 3 and 4. We now assume stability of the potential v,

$$\sum_{1 \leqslant i < j \leqslant n} v(x_i - x_j) \geqslant -Bn \tag{5}$$

for all n, x_i and some $B \geqslant 0$; and likewise its integrability

$$\|v\|_1 = \int |v(x)|\, d^3x < \infty. \tag{6}$$

An important aspect of our expressions for the b_n is that the arguments in the exponentials, as in (3) and (4), may be bounded as though all the s_i equal one. Thus

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$$s_1 v(x) + s_2 v(y) + s_1 s_2 v(x+y) \geqslant -3B \tag{7}$$

for $0 \leqslant s_1, s_2 \leqslant 1$. One sees easily the bounds

$$|b_2| \leqslant \tfrac{1}{2}\beta \|v\|_1 \exp(2\beta B), \tag{8}$$

$$|b_3| \leqslant \tfrac{1}{3}\beta^2 \|v\|_1^2 \exp(3\beta B). \tfrac{3}{2}. \tag{9}$$

Later we will see a bound for the nth term of this system

$$|b_n| \leqslant \beta^{n-1} \|v\|_1^{n-1} \exp(n\beta B)(\exp(n-1)/n). \tag{10}$$

Thus we get as a sufficient condition for the convergence of the Mayer series (1)

$$z \exp(\beta B) e \|v\|_1 \beta < 1. \tag{11}$$

We may compare the sufficient condition obtained in Ref. 3:

$$z \exp(2\beta B) e \int d^3x \,|\exp(-\beta v(x)) - 1| < 1. \tag{12}$$

If $v = v_0 + v_r$, where v_0 satisfies (5) and v_r is nonnegative, we will give an improvement (in the Appendix) that allows the condition (11) to be weakened, to include the case of hard cores for example, becoming on a par with condition (12). If $v(x)$ is repulsive, i.e., nonnegative, the alternating sign property of the b_n's, as is easily seen for the first three terms, is also immediate for the general term. This is not so clear in the standard expressions for b_n.

Whereas one may perform the integrals over the s parameters in (2) and (3) to recover the usual expressions, we will instead manipulate the three-body partition function $Z^{(3)}$ to extract these forms. This will yield a telling insight into the general problem. We calculate $Z_\Lambda^{(3)}$, the partition function in a volume Λ:

$$
Z_\Lambda^{(3)} = \frac{1}{3!} \int_{\Lambda^3} d^3x_1 d^3x_2 d^3x_3
$$

$$
\times \exp\{-\beta[v(1,2) + v(1,3) + v(2,3)]\}, \tag{13}
$$

where we use the obvious notation $v(i,j)$ for $v(x_i - x_j)$. The basic trick is use of the identity

$$\exp(f(1)) = \exp(f(0)) + \int_0^1 ds\, f'(s) \exp(f(s)). \tag{14}$$

We first use the parameter s_1 to decouple the interaction of particle 1 from the other particles (that is, to separate the uncoupled contribution)

$$\exp\{-\beta[v(1,2)+v(1,3)+v(2,3)]\}$$

$$= \exp[-\beta v(2,3)] + \int_0^1 ds_1 (-\beta)(v(1,2)+v(1,3))$$

$$\times \exp\{-\beta[s_1(v(1,2)+v(1,3))+v(2,3)]\}. \qquad (15)$$

The second term on the right side of (15) may be replaced by

$$2\int_0^1 ds_1(-\beta)v(1,2)\exp\{-\beta[s_1(v(1,2)+v(1,3))+v(2,3)]\}, \qquad (16)$$

since the integral in (13) is symmetric in the three variables. We now use the parameter s_2 to decouple the system of particles 1 and 2 from the remaining particle, replacing (16) by

$$2\int_0^1 ds_1(-\beta)v(1,2)\exp(-\beta s_1 v(1,2))$$

$$+ 2\int_0^1 ds_1 \int_0^1 ds_2(-\beta)^2 v(1,2)$$

$$\times [v(2,3)+s_1v(1,3)]\exp(-\beta W(s_1,s_2)), \qquad (17)$$

with

$$W(s_1, s_2) = [s_1 v(1,2) + s_1 s_2 v(1,3) + s_2 v(2,3)].$$

The first term on the right side of (15) may be replaced by

$$1 + (-\beta)\int_0^1 ds_1 v(1,2)\exp(-\beta s_1 v(1,2)). \qquad (18)$$

We define

$$K_\Lambda^{(1)} \equiv \int_\Lambda d^3x = |\Lambda| \, b_1 = |\Lambda|, \qquad (19)$$

where $|\Lambda|$ is the volume of Λ.

$$K_\Lambda^{(2)} \equiv -\tfrac{1}{2}\beta\int_0^1 ds \int_\Lambda d^3x_1 \int_\Lambda d^3x_2 \, v(x_1 - x_2) \qquad (20)$$

$$\times \exp[-\beta s v(x_1 - x_2)] \underset{\Lambda \to \infty}{\longrightarrow} |\Lambda| \, b_2. \qquad (21)$$

Limits as $\Lambda \to \infty$ are always through boxes whose minimum width approaches infinity.

$$K_\Lambda^{(3)} \equiv k_\Lambda^{(3)}(a) + k_\Lambda^{(3)}(b) \qquad (22)$$

$$\underset{\Lambda \to \infty}{\longrightarrow} |\Lambda| \, b_3, \qquad (23)$$

where

$$k_\Lambda^{(3)}(a) \equiv \tfrac{1}{3}\beta^2 \int_0^1 ds_1 \int_0^1 ds_2 \int_\Lambda d^3x_1 \int_\Lambda d^3x_2 \int_\Lambda d^3x_3$$

$$\times v(1,2)v(2,3)\exp(-\beta W(s_1, s_2)), \qquad (24)$$

$$k_\Lambda^{(3)}(b) \equiv \tfrac{1}{3}\beta^2 \int_0^1 ds_1 \int_0^1 ds_2 \int_\Lambda d^3x_1 \int_\Lambda d^3x_2 \int_\Lambda d^3x_3$$

$$\times v(1,2)s_1v(1,3)\exp(-\beta W(s_1, s_2)). \qquad (25)$$

Collecting, we have

$$Z_\Lambda^{(3)} = (1/3!)(K_\Lambda^{(1)})^3 + K_\Lambda^{(1)}K_\Lambda^{(2)} + K_\Lambda^{(3)}. \qquad (26)$$

This is the coefficient of z^3 in the expansion

$$Z_\Lambda = \exp\left(\sum_1^\infty K_\Lambda^{(n)} z^n\right) \qquad (27)$$

(whose convergence in some region we will soon prove) and as is familiar

$$\frac{1}{|\Lambda|}\sum_1^\infty K_\Lambda^{(n)} z^n \underset{\Lambda \to \infty}{\longrightarrow} \beta p = \sum_1^\infty b_n z^n. \qquad (28)$$

The identifications we have made are clear if one is familiar with the usual treatment of the Mayer expansion, since the $K_\Lambda^{(n)}$ we have defined for $n = 1,2,3$ are connected. However this paper is self-contained, and one may merely accept the definitions above, and

postpone the identifications in (27) and (28). We proceed to a mathematical treatment of the general problem.

We consider the grand canonical partition function

$$Z_\Lambda = \sum_0 z^n Z_\Lambda^{(n)} \qquad (29)$$

with

$$Z_\Lambda^{(n)} = \frac{1}{n!}\int_\Lambda dx^{(n)}\exp(-\beta V^{(n)}). \qquad (30)$$

If X is a subset of integers

$$\int_\Lambda dx^X = \prod_{i \in X}\left(\int_\Lambda d^3x_i \int d\epsilon_i\right) \qquad (31)$$

(n) is identified with the set of the first n integers. The $\int d\epsilon_i$ is a sum over internal freedoms of the particle, spins, charges, etc.

$$V^X = \sum_{\substack{y \subset X \\ |y| \leqslant 2}} v(y)$$

$$= \sum_{\alpha \in X} v_1(x_\alpha, \epsilon_\alpha) + \sum_{(\alpha,\beta) \in X} v_2(x_\alpha, \epsilon_\alpha, x_\beta, \epsilon_\beta)$$

$$= \sum_{\alpha \in X} v_1(\alpha) + \tfrac{1}{2}\sum_{\substack{\alpha,\beta \in X \\ \alpha \neq \beta}} v_2(\alpha, \beta). \qquad (32)$$

$|X|$ is the number of elements in X. We have assumed one- and two-body potentials, not necessarily translational invariant. The present scheme does not accomodate many-body potentials without modification. A more literal mimicking of Ref. 1 allows this possibility. We assume of the potentials:

[A0] *Measurability*: v_1 and v_2 are measurable functions.

[A1] *Stability*: There is a $B \geqslant 0$ such that

$$V^X \geqslant -|X|B. \qquad (33)$$

[A2] *Integrability*:

$$\|v_2\| = \sup_{x_1, \epsilon_1}\int d^3x_2 \int d\epsilon_2 \, |v_2(x_1, \epsilon_1, x_2, \epsilon_2)| < \infty. \qquad (34)$$

(In the Appendix we will give an improvement allowing the weakening of condition [A2].) We define the degeneracy s by

$$\int_\Lambda d^3x \int d\epsilon = |\Lambda|s. \qquad (35)$$

Proposition 0: Z_Λ is an entire function of z.

In fact the sum in (29) is dominated term by term by the sum

$$\sum_{n=0}^\infty \frac{1}{n!}(z\exp(\beta B)|\Lambda|s)^n. \qquad (36)$$

We now generalize the use of s_1 in (15) and s_2 in (17), introducing a sequence of parameters, s_i, where s_i decouples the system of the first i particles from the remaining particles. If W^X is of the form

$$\sum_{y \subset X} w(y)$$

then for $X' \subset X$ we define

$$W^{X,X'} = \sum_{y \subset X'} w(y) + \sum_{y \subset (X-X')} w(y) = W^{X'} + W^{X-X'}. \qquad (37)$$

One has removed from the W^X all interactions between particles in X' and particles in $(X - X')$. To the set (t) of t particles, $t > 1$, we associate σ_{t-1}, a sequence of parameters $s_1, s_2, \ldots, s_{t-1}$ and a potential $W^{(t)}(\sigma_{t-1})$ defined by an inductive process

$$W_0^{(t)} = V^{(t)}, \tag{38}$$

$$W_i^{(t)} = (1 - s_i)W_{i-1}^{(t),(i)} + s_i W_{i-1}^{(t)}, \quad i = 1, \cdots, t - 1, \tag{39}$$

$$W_{t-1}^{(t)} = W^{(t)}(\sigma_{t-1}). \tag{40}$$

Thus $W(s_1, s_2)$ in (17) is our $W^{(3)}(\sigma_2)$. Referring to (32) we may also write

$$W^{(t)}(\sigma_{t-1})$$
$$= \sum_{i=1}^{t} v_1(i) + \sum_{1 \leq i < j \leq t} s_i \cdot s_{i+1} \cdots s_{j-1} v_2(i,j). \tag{41}$$

Proposition 1: Stability of interpolated potentials: If the s_i satisfy $0 \leq s_i \leq 1$ then

$$W^{(t)}(\sigma_{t-1}) \geq -tB \tag{42}$$

We observe that from $W^{(t)} \geq -Bt$ all t, there follows $W^{(t),(i)} \geq -Bt$. The proposition follows by noting that the induction step (39) in the construction of $W^{(t)}(\sigma_{t-1})$ implies $W_i^{(t)}$ satisfies the inequality for all i, as the convex sum of terms that do so.

When s_1 was employed in (15), the next particle, 2 in (16), was coupled by $v(1,2)$ differentiated from the exponent. When s_2 was employed in (17), the next particle, in (17), was coupled by either $v(1,3)$ to particle 1, or by $v(2,3)$ to particle 2, in the potentials differentiated from the exponent. These two possibilities are separated, respectively, into the two terms (25) and (24). In general we introduce a function $\eta(i)$, $i = 1, 2, \ldots, t - 1$ with $\eta(i)$ a positive integer satisfying

$$\eta(i) \leq i \tag{43}$$

to specify a term, wherein the potential differentiated from the exponent by s_i is $v(i + 1, \eta(i))$. Thus (24) has η^a specifying it with

$$\eta^a(1) = 1, \quad \eta^a(2) = 2 \tag{44}$$

and (25) has η^b specifying it with

$$\eta^b(1) = 1, \quad \eta^b(2) = 1. \tag{45}$$

It is natural to associate a tree graph to each function η. The vertices in the graph are numbered from 1 to t, the vertices $i + 1$ and $\eta(i)$, $i = 1, 2, \cdots, t - 1$, are connected by lines. In particular, η^a and η^b have respectively the following graphs associated to themselves.
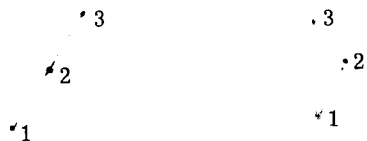
A somewhat similar use of a tree graph analysis to study the Mayer series may be found in Ref. 5.

We define

$$\int d\sigma_{t-1} = \int_0^1 ds_1 \cdots \int_0^1 ds_{t-1} \tag{46}$$

and introduce a function $f(\eta, \sigma_{t-1})$ by

$$f(\eta, \sigma_{t-1}) = \prod_{i=2}^{t-1} s_{i-1} s_{i-2} \cdots s_{\eta(i)}, \, t > 2$$

$$f(\eta, \sigma_1) = 1, \tag{47}$$

where $s_{i-1} s_{i-2} \cdots s_{\eta(i)}$ is understood to be 1 if $\eta(i) = i$. $f(\eta, \sigma_{t-1})$ is the product of s_i's differentiated from the exponent by the sequence of decouplings. We now can define $K_{\Lambda}^{(t)}$ in the general situation

$$K_{\Lambda}^{(t)} = \sum_{\eta} k_{\Lambda}^{(t)}(\eta). \tag{48}$$

This is the general form of (22), and the sum is over all η satisfying (43). As a general form of (20), (24), or (25) we have

$$k^{(t)}(\eta) = \frac{(-\beta)^{t-1}}{t} \int d\sigma_{t-1} \int_{\Lambda} dx^{(t)} f(\eta, \sigma_{t-1})$$
$$\times \prod_{i=1}^{t-1} v(i + 1, \eta(i)) \exp[-\beta W^{(t)}(\sigma_{t-1})]. \tag{49}$$

We can finally specify the general form of the Mayer expansion coefficients, if the interaction is translation invariant

$$b_n = \frac{(-\beta)^{n-1}}{n} \sum_{\eta} \int d\sigma_{n-1} \int_{R^3} dx^{(n-1)} \int d\epsilon_n f(\eta, \sigma_{n-1})$$
$$\times \prod_{i=1}^{n-1} v(i + 1, \eta(i)) \exp[-\beta W^{(n)}(\sigma_{n-1})]. \tag{50}$$

The following proposition is merely an observation in our formulation.

Proposition 2: If the two-body potentials are nonnegative, then $(-1)^n b_n \geq 0$.

We will have need for the following purely numerical estimate from Ref. 6.

Proposition 3:

$$\sum_{\eta} \int d\sigma_{t-1} f(\eta, \sigma_{t-1}) \leq \exp(t - 1). \tag{51}$$

To prove this we first note that the left side of (51) is less than or equal

$$\int_0^1 ds_1 \cdots \int_0^1 ds_{t-1}$$
$$\times \sum_{\eta} f(\eta, \sigma_{t-1}) \exp\left(\sum_{i=1}^{t-1} s_{t-1} s_{t-2} \cdots s_i\right) \tag{52}$$

If one performs the s_i integrals in the indicated order, using the estimate

$$\int_0^1 ds \, v \exp(sv) \leq \exp(v) \tag{53}$$

at each stage, the proposition follows. Proposition 3 is the most complicated estimate we need, we detail its steps for $t = 4$

$$\sum_{\eta} f(\eta, \sigma_3) = (1 + s_1)(1 + s_2 + s_2 s_1) \tag{54}$$

$$\int d\sigma_{t-1} \sum_{\eta} f(\eta, \sigma_3)$$
$$= \int_0^1 ds_1 \int_0^1 ds_2 \int_0^1 ds_3 (1 + s_1)(1 + s_2 + s_2 s_1)$$
$$\leq \int_0^1 ds_1 \int_0^1 ds_2 \int_0^1 ds_3 (1 + s_1)(1 + s_2 + s_2 s_1) \tag{55}$$
$$\times \exp(s_3 s_2 s_1 + s_3 s_2 + s_3)$$
$$\leq \int_0^1 ds_1 \int_0^1 ds_2 (1 + s_1) \exp(s_2 s_1 + s_2) e$$
$$\leq \int_0^1 ds_1 \exp(s_1) \exp(2) \leq \exp(3).$$

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Proposition 4:

$$|K_\Lambda^{(t)}| \leqslant \beta^{t-1}(\|v_2\|)^{t-1} \exp(t\beta B) \frac{\exp(t-1)}{t} s \, |\Lambda|. \qquad (56)$$

This follows easily from the last proposition and the definitions.

Proposition 5: If the interaction is translation invariant

$$\lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} K_\Lambda^{(n)} = b_n. \qquad (57)$$

Proposition 6:

$$Z_\Lambda^{(n)} = \sum_{1 \leqslant i \leqslant n} \frac{i}{n} K_\Lambda^{(i)} Z_\Lambda^{(n-i)} \qquad (58)$$

This expression is related to the Kirkwood—Salsburg equations and similar equations.[3] Equation (58) is directly derived by introducing a sequence of decoupling parameters to isolate connected configurations containing particle 1. It is an easy combinatoric exercise.

Proposition 7: For z sufficiently small

$$Z_\Lambda = \exp[\sum_n z^n K_\Lambda^{(n)}]. \qquad (59)$$

By Proposition 0 and Proposition 4, both sides of the equation are analytic functions of z for z sufficiently small. Using Proposition 6 we then have

$$z \frac{\partial}{\partial z} \sum_0^\infty z^n Z_\Lambda^{(n)} = \sum_{n,i} i z^i K_\Lambda^{(i)} Z_\Lambda^{(n-i)} z^{n-i} \qquad (60)$$

from which follows

$$z \frac{\partial}{\partial z} Z_\Lambda = \sum_{i=1}^\infty i z^i K_\Lambda^{(i)} Z_\Lambda \qquad (61)$$

yielding (59) and proving Proposition 7.

In fact, the equality in (59), and the convergence of the series in the exponent are ensured for

$$z \exp(\beta B) e \|v_2\| \beta < 1. \qquad (62)$$

Proposition 8: If the interaction is translation invariant, then for z sufficiently small [satisying (62)]

$$\lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \ln(Z_\Lambda) = \sum_{n=1}^\infty b_n z^n. \qquad (63)$$

This last proposition follows from (56), (57), and (59) by the easy mathematical theorem:

Let $|a_{ik}| \leqslant r_k$ with $\sum r_k < \infty$. Assume

$$\lim_{i \to \infty} a_{ik} = c_k$$

then

$$\lim_{i \to \infty} \sum_k a_{ik} = \sum_k c_k.$$

This is a very special case of the dominated convergence theorem.

Thus we have the basic properties of the Mayer series rigorously developed in the regime given by (62) by an entirely elementary argument. In the Appendix our

procedure will be modified to give an improvement extending the regime of validity, weakening condition (62).

APPENDIX

We split the interaction into two parts

$$V^X = V_0^X + V_r^X \qquad (A1)$$

each constructed as in (32), V_0^X from v_{01} and v_{02} satisfying assumptions [A0], [A1], and [A2], and V_r^X from $v_{r1} = 0$ and v_{r2}, with v_{r2} measurable and nonnegative. We define $u(i,j)$

$$u(i,j) = \exp[-\beta v_{r2}(i,j)] - 1. \qquad (A2)$$

We will indicate a modification of the expansion of the paper that will replace (62) as the limiting inequality by

$$z \exp(\beta B) e \beta [\|v_{02}\| + \beta^{-1}\|u\|] < 1. \qquad (A3)$$

Different split ups (A1) may be considered to find a best result; this may be virtually equivalent to equation (12) as a limiting condition.

The modification of the procedure consists in interpolating $\exp[-\beta v_{r2}(i,j)]$ as $1 + su(i,j)$. This replaces $W^{(t)}(\sigma_{t-1})$ of (41) by

$$W_0^{(t)}(\sigma_{t-1}) - \frac{1}{\beta}$$
$$\times \sum_{1 \leqslant i < j \leqslant t} \log[1 + s_i \cdot s_{i+1} \cdots s_{j-1} u(i,j)], \qquad (A4)$$

where $W_0^{(t)}(\sigma_{t-1})$ is $W^{(t)}(\sigma_{t-1})$ of the paper calculated for V_0^X. The $v(i,j)$ of (49) and (50) then are replaced by

$$v_{02}(i,j) - \frac{1}{\beta} \frac{u(i,j)}{1 + s_i \cdots s_{j-1} u(i,j)}. \qquad (A5)$$

The substitutions of (A4) and (A5) into (49) and (50) easily yield (A3) as a sufficient condition for convergence.

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O(3) shift operators : The general analysis

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O(3) shift operators are constructed in terms of tensor operators $T(j,\mu)$ and the O(3) generators. These are of type B_l^k, $k = -j,...,j$, where B_l^k raises l by k, $l(l+1)$ being the eigenvalue of the O(3) Casimir operator. Various convenient normalizations of these operators are constructed, and their properties and uses considered.

1. INTRODUCTION

In some previous papers,[1-10] shift operators were constructed out of the enveloping algebra of the generators l_i of O(3) [or SU(2)] and the components $T(j,\mu)$, $\mu = -j,\ldots,j$, of a tensor representation of O(3). These operators were essentially of type O_l^k, $k = -j,\ldots, j$, which raise the value of l, where $l(l+1)$ is the eigenvalue of the O(3) Casimir operator L^2, by k, when acting on eigenstates of L^2 corresponding to arbitrary values of m, the eigenvalue of l_0. Operators have also been given[11-14] which raise the value of l by ± 1 but which are valid only when acting on states for which $m = l$.

A general analysis of the operators O^k for arbitrary j values has not been given, their construction and properties having been considered only for the special cases where $j = 1$,[3,7,8,9] $j = 2$,[1,2,4] and $j = \frac{1}{2}$.[5,6] For these particular j values, the O_l^k were found to be extremely useful for the classification and analysis of irreducible unitary representations (IUR) of various low-dimensional groups possessing an O(3) subgroup. For $j = 1$, Stone[7] appears to have been the first person to explicitly construct such operators and apply them to the group O(4). Miller[9] also showed that the l-stepping operators arising in the type E and F factorizations of second order differential equations given by Infeld and Hull[8] can be constructed from the enveloping algebra of the three-dimensional Euclidean group, and so these are essentially also O_l^k-type shift operators for the $j = 1$ case. Hughes[3] showed how these operators could be used to give a unified analysis of the IUR of O(4), the homogeneous Lorentz group O(3,1), and the Euclidean group. The $j = 2$ case arose for Sl(3,R),[4] and for SU(3) in the O(3) basis,[1,2] where the shift operators were used to obtain an algorithm for the calculation of eigenvalues of the O(3) scalar operators used to resolve the state labelling problem. The $j = \frac{1}{2}$ case was applicable to the group O(3)$\wedge (T_2 \times \overline{T}_2)$,[5] and to the analysis of IUR of SU(3) in an SU(2) basis.[6]

In this paper the case of general j is treated. The cases where j is integral or half-integral have to be dealt with in a slightly different manner, in that if l is raised by a half-integral amount, so must m, since l and m must be integral or half-integral together. Thus, for integral j, the shift operators may be chosen to commute with l_0, whereas when j is half-integral they

are chosen to raise its eigenvalue by $\pm \frac{1}{2}$. Also, the hermiticity properties of the shift operators in the two cases are quite different, due to the fact that the operators $T(j,\mu)$ form a closed set with respect to taking hermitian conjugate for j integral, but not for half-integral j.[15]

The commutation relations of the $T(j,\mu)$ among themselves are not needed, and therefore not considered in this paper, so it is not assumed that the l_i and $T(j,\mu)$ generate a group. However, once mutual commutation relations are given for the $T(j,\mu)$, a Lie algebraic structure may ensue. For instance, SU(3) and Sl(3,R) are formed from the l_i and $\{T(2,\mu), \mu = -2,\ldots,2\}$; SU(3) in an SU(2) basis and SU(2.1) are generated by the set consisting of the l_i, $T(\frac{1}{2}, \pm \frac{1}{2})$, $\overline{T}(\frac{1}{2}, \pm \frac{1}{2})$, and $T(0,0)$, and the group O(3)$\wedge (T_2 \times \overline{T}_2)$ is generated by the i_i, $T(\frac{1}{2}, \pm \frac{1}{2})$, and $\overline{T}(\frac{1}{2}, \pm \frac{1}{2})$.

We start off in Sec. 2 by constructing the shift operators B_l^k for the integral j case, which commute with l_0 and shift l by k, and $B_{l,m}^{k,\pm 1/2}$ for j half-integral, which shift m by $\pm \frac{1}{2}$ and l by k. These operators depend explicitly on the l values of the states upon which they act, a fact noted by Joseph[10] for the particular case of the step operators of Infeld and Hull.[8] If they depended on l only through the combination $l(l+1)$, this could be replaced by the operator L^2; however, their dependence on l is more complicated, and so we introduce the operator R whose eigenvalue is l. The B operators contain l and m in the denominator and to half-integral powers, and so are not the most convenient to use in practical applications since when l and m are replaced by the operators R and l_0, it is highly desirable that these operators occur only to positive integral powers. Therefore, in Sec. 3 we obtain more convenient normalized operators A_l^k and O_l^k for integral j, together with their half-integral j counterparts. The expressions for O_l^k in terms of the $T(j,\mu)$ and $l_{\pm}$ are such that l and m occur only in the denominator and only to integral power and so can be replaced conveniently by, respectively, R and l_0. The resulting expression for O^k is a linear combination of products of positive integral powers of the l_i and R with the $T(j,\mu)$, and is valid even when not explicitly acting on states of definite l and m values.

The O_l^k do have the disadvantage that their matrix

elements contain m-dependent terms, these being such as to guarantee that the lowering operators O_l^{-k}, $k > 0$, never lower l below $|m|$, which would be inconsistent with properties of IUR of O(3). The operators A_l^k do not contain m-dependent terms in their matrix elements and are the operators to use when determining minimum l values occurring in IUR of groups containing an O(3) subgroup as in the case, for example of Sl(3,R).[4]

In Sec. 4 we consider the hermiticity properties of the shift operators for the integral j case. The set $\{T(j, \mu)\}$ is, in this case, closed under Hermitian conjugation, and so the shift operators satisfy hermiticity properties of the type

$$\langle \gamma, l, m \,|(O_l^{+k})^\dagger |\, \gamma', \, l + k, m \rangle = \alpha_{k,l} \langle \gamma, l, m \,|O_{l+k}^{-k}|\, \gamma', l + k, m \rangle.$$

$\alpha_{k,l}$ turns out to be $(2l + 1)/(2l + 2k + 1)$, which shows in particular that the O(3) scalar operator O_l^0 is Hermitian. If one replaced l by the operator R, then $(O_l^{+k})^\dagger$ can be obtained by taking the Hermitian conjugate of each term in its expression. Before the resulting operator can be compared with O_l^{-k} however, R must be brought to the right of all $T(j, \mu)$, and this necessitates knowledge of the commutator $[R, T(j, \mu)]$. Although these can be worked out, this method is more laborious than the one using matrix elements and so is not employed here; in operator form the above hermiticity relation becomes in fact

$$(O^k)^\dagger(2R + 1) = O^{-k}(2R - 2k + 1).$$

For half-integral j it is not possible to close the set $\{T(j, \mu)\}$ with respect to Hermitian conjugation in a self-consistent manner, so that a new set of tensor operators $\{\bar{T}(j, \mu)\}$ have to be introduced, together with corresponding shift operators; this is done in Sec. 5. The counterpart of the above hermiticity relation is found to be

$$(\bar{O}^{k, \pm 1/2})^\dagger(2R + 1) = (-1)^{-j \pm 1/2} O^{-k, \pm 1/2}(2R - 2k + 1).$$

In Sec. 6 it is shown how the properties of the shift operators may be used to calculate the above mentioned commutators $[R, T(j, \mu)]$ for the integral j case, and Sec. 7 treats the same problem for half-integral values of j.

Finally, in Sec. 8, we discuss the relation of the shift operators obtained in this paper with those derived by other authors, and ways in which the work of this paper may be extended and generalized.

2. CONSTRUCTION OF THE SHIFT OPERATORS B_l^k AND $B_{l,m}^{k, \pm 1/2}$

Let $T(j, \mu)$, $\mu = -j, \ldots, j$ and j integral or half-integral, be a $(2j + 1)$-dimensional tensor representation[15] of O(3) [or SU(2)], whose commutators with the generators l_0, $l_\pm$ of O(3) are

$$[l_\pm, T(j, \mu)] = [(j \mp \mu)(j \pm \mu + 1)]^{1/2} T(j, \mu \pm 1), \quad (2.1)$$

$$[l_0, T(j, \mu)] = \mu T(j, \mu), \quad (2.2)$$

where here, as throughout the rest of the paper, the positive real value of the square root is taken.

We shall suppose the $T(j, \mu)$ act upon states $|\gamma, l, m\rangle$,

where $l(l + 1)$ and m are the eigenvalues of, respectively, the O(3) Casimer L^2 and l_0, and γ denotes an additional collection of labels needed to completely specify the states. Now[15]

$$\langle \gamma', l', m' |\, T(j, \mu) |\, \gamma, l, m \rangle$$

$$= (-1)^{l'-m'} \begin{pmatrix} l' & j & l \\ -m' & \mu & m \end{pmatrix} \langle \gamma', l' \| T(j) \| \gamma, l \rangle \quad (2.3)$$

where

$$\begin{pmatrix} l' & j & l \\ -m' & \mu & m \end{pmatrix}$$

is a Wigner 3-j symbol and the $\langle \gamma', l' \| T(j) \| \gamma, l \rangle$ are reduced matrix elements of $T(j, \mu)$.

We treat the cases of integral and half-integral j separately, considering first of all the case where j, and therefore also μ, is integral. The shift operators in this case are constructed from the operators

$$Q_{\pm \mu} = T(j, \mp \mu) l_\pm^\mu, \quad \mu \geq 0, \quad (2.4)$$

which, since they commute with l_0, leave unchanged the m-values of states upon which they act.

Using (2.3) and (2.4), we obtain

$$\langle \gamma', l', m |\, Q_{\pm \mu} |\, \gamma, l, m \rangle$$

$$= (-1)^{l'-m} \begin{pmatrix} l' & j & l \\ -m & \mp \mu & m \pm \mu \end{pmatrix}$$

$$\times \left[\frac{(l \pm m + \mu)!\,(l \mp m)!}{(l \pm m)!\,(l \mp m - \mu)!} \right]^{1/2} \langle \gamma', l' \| T(j) \| \gamma, l \rangle.$$

Let

$$P_{\pm \mu} = \left[\frac{(l \pm m)!\,(l \mp m - \mu)!}{(l \pm m + \mu)!\,(l \mp m)!} \right]^{1/2} Q_{\pm \mu} \quad (2.5)$$

so that, for $\mu = j, \ldots, j$,

$$\langle \gamma', l', m |\, P_\mu |\, \gamma, l, m \rangle$$

$$= (-1)^{l'-m} \begin{pmatrix} l' & j & l \\ -m & -\mu & m + \mu \end{pmatrix} \langle \gamma', l' \| T(j) \| \gamma, l \rangle. \quad (2.6)$$

This then implies that

$$P_\mu |\, \gamma, l, m \rangle = \sum_{k'=-j}^{j} (-1)^{l + k' - m} \begin{pmatrix} j & l & l + k' \\ -\mu & m + \mu & -m \end{pmatrix}$$

$$\times \sum_{\gamma'} \langle \gamma', l + k' \| T(j) \| \gamma, l \rangle \,|\gamma', l + k', m \rangle, \quad (2.7)$$

where the symmetry of the 3-j symbols under even transpositions of the columns has been used.

Now the 3-j symbols satisfy the orthogonality relation[15]

$$\sum_{m_1, m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix}$$

$$= (2j_3 + 1)^{-1} \delta_{j_3 j_3'} \delta_{m_3 m_3'} \delta(j_1 j_2 j_3), \quad (2.8)$$

where $\delta(j_1 j_2 j_3) = 1$ if j_3 lies between $|j_1 - j_2|$ and $(j_1 + j_2)$, and vanishes otherwise. Hence

$$\left(\sum_{\mu=-j}^{j}\begin{pmatrix}j & l & l+k\\ -\mu & m+\mu & -m\end{pmatrix}P_\mu\right)|\gamma,l,m\rangle$$

$$=(-1)^{l+k-m}[2(l+k)+1]^{-1}\delta(j,l,l+k)$$

$$\times\sum_{\gamma'}\langle\gamma',l+k||T(j)||\gamma,l\rangle\,|\gamma',l+k,m\rangle. \tag{2.9}$$

Hence a shift operator changing l by k, $k=-j,\ldots,j$, is

$$B_l^k=\sum_{\mu=-j}^{j}\begin{pmatrix}j & l & l+k\\ -\mu & m+\mu & -m\end{pmatrix}P_\mu$$

$$=a_0^k(l,m)Q_0+\sum_{\mu=1}^{j}[a_\mu^k(l,m)Q_{+\mu}+(-1)^{2l+j+k}a_\mu^k(l,-m)Q_{-\mu}] \tag{2.10}$$

where, for $\mu=0,\ldots,j$,

$$a_\mu^k(l,m)=\left[\frac{(l-m-\mu)!\,(l+m)!}{(l+m+\mu)!\,(l-m)!}\right]^{1/2}$$

$$\times\begin{pmatrix}j & l & l+k\\ -\mu & m+\mu & -m\end{pmatrix}. \tag{2.11}$$

The action of B_l^k on $|\gamma,l,m\rangle$ is given by

$$B_l^k|\gamma,l,m\rangle=(-1)^{l+k-m}[2(l+k)+1]^{-1}\delta(j,l,l+k).$$

$$\sum_{\gamma'}\langle\gamma',l+k||T(j)||\gamma,l\rangle\,|\gamma',l+k,m\rangle. \tag{2.12}$$

Above we have used the well-known property

$$\begin{pmatrix}j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}=(-1)^{j_1+j_2+j_3}\begin{pmatrix}j_1 & j_2 & j_3\\ -m_1 & -m_2 & -m_3\end{pmatrix}, \tag{2.13}$$

which, as it stands, is valid only for j_1,j_2,j_3 nonnegative. When any of the j's happen to be negative one must, in the phase factor, replace j by $-(j+1)$. Thus, if j_1 is negative one has

$$\begin{pmatrix}j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}$$

$$=(-1)^{-(j_1+1)+j_2+j_3}\begin{pmatrix}j_1 & j_2 & j_3\\ -m_1 & -m_2 & -m_3\end{pmatrix}.$$

A similar argument applies to the corresponding relation for the vector coupling coefficients.

An alternative method of calculating B_l^k would be by the requirement that

$$[L^2,B_l^k]=k(k+2l+1)B_l^k. \tag{2.14}$$

It is not difficult to check, using recursion relations for the 3-j symbols, that the B_l^k given by (2.10) do indeed satisfy (2.14).

For the case of half-integral j, the shift operators must change not only l, but also m, by half-integral amounts. We consider only those which shift m by $\pm\frac{1}{2}$, denoting them by $B_{l,m}^{k,\pm1/2}$. They are constructed from the operators

$$Q_{\pm\mu}^+=T(j,\mp\mu)l_\pm^{(\mu\pm1/2)},\quad Q_{\pm\mu}^-=T(j,\mp\mu)l_\pm^{(\mu\mp1/2)}, \tag{2.15}$$

where $\mu=\frac{1}{2},\frac{3}{2},\ldots,j$. One can check that

$$[l_0,Q_{\pm\mu}^+]=\tfrac{1}{2}Q_{\pm\mu}^+,\quad [l_0,Q_{\pm\mu}^-]=-\tfrac{1}{2}Q_{\pm\mu}^-, \tag{2.16}$$

from which it follows that $Q_{\pm\mu}^+$ and $Q_{\pm\mu}^-$ change m by $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Letting

$$P_{\pm\mu}^\pm=\left[\frac{(l\pm m)!\,(l\mp m-\mu-\tfrac{1}{2})!}{(l\pm m+\mu+\tfrac{1}{2})!\,(l\mp m)!}\right]^{1/2}Q_{\pm\mu}^\pm, \tag{2.17}$$

$$P_{\mp\mu}^\pm=\left[\frac{(l\mp m)!\,(l\pm m-\mu+\tfrac{1}{2})!}{(l\mp m+\mu-\tfrac{1}{2})!\,(l\pm m)!}\right]^{1/2}Q_{\mp\mu}^\pm, \tag{2.18}$$

one finds that the following shift operators change l by k and m by $\pm\frac{1}{2}$:

$$B_{l,m}^{k,\pm1/2}=\sum_{\mu=-j}^{j}\begin{pmatrix}j & l & l+k\\ -\mu & m+\mu\pm\frac{1}{2} & -m\mp\frac{1}{2}\end{pmatrix}P_\mu^\pm$$

$$=\sum_{\mu=1/2}^{j}[a_\mu^k(l,m,\pm\tfrac{1}{2})Q_\mu^\pm+(-1)^{2l+j+2k}a_\mu^k(l,-m,\mp\tfrac{1}{2})Q_{-\mu}^\pm], \tag{2.19}$$

where, for $\mu=\frac{1}{2},\ldots,j$,

$$a_\mu^k(l,m,\pm\tfrac{1}{2})=\begin{pmatrix}j & l & l+k\\ -\mu & m+\mu\pm\frac{1}{2} & -m\mp\frac{1}{2}\end{pmatrix}$$

$$\times\left[\frac{(l\pm m)!\,(l\mp m-\mu-\tfrac{1}{2})!}{(l\pm m+\mu+\tfrac{1}{2})!\,(l\mp m)!}\right]^{1/2}. \tag{2.20}$$

The action of $B_{l,m}^{k,\pm1/2}$ on $|\gamma,l,m\rangle$ is given by

$$B_{l,m}^{k,\pm1/2}|\gamma,l,m\rangle=(-1)^{l+k-m\pm1/2}[2(l+k)+1]^{-1}\delta(j,l,l+k).$$

$$\sum_{\gamma'}\langle\gamma',l+k||T(j)||\gamma,l\rangle\,|\gamma',l+k,m\pm\tfrac{1}{2}\rangle. \tag{2.21}$$

3. NORMALIZATION OF B_l^k AND $B_{l,m}^{k,\pm1/2}$

The aim in this section is to obtain more convenient normalized forms of B_l^k and $B_{l,m}^{k,\pm1/2}$ in which the coefficients of $Q_{\pm\mu}$ or $Q_{\pm\mu}^\mp$ contain l and m only in the numerator and only to integral powers. This will enable us later on to replace m and l by the operators l_0 and R (whose eigenvalue is l), so that the resulting forms of the shift operators will be independent of the m and l values of the states upon which they act.

In order to do this, we need to know the manner in which the 3-j symbols appearing in (2.11) and (2.20) depend on l and m. Specifically, we shall be interested only in those parts of the 3-j symbols which contain l and m either to half-integral powers or in the denominator, and we shall refer to them as SR-D (short for square root-denominator) parts.

We consider the 3-j symbol $\begin{pmatrix}j & l & l+k\\ \mu & -\mu-m-\lambda & m+\lambda\end{pmatrix}$, where λ is either 0 or $\pm\frac{1}{2}$, and where for the time being we restrict ourselves to the case where k and μ are both $\geq|\lambda|$. We shall need the formula

$$\begin{pmatrix}j & l & l+k\\ j & -j-m-\lambda & m+\lambda\end{pmatrix}=(-1)^{-j+l+m+\lambda}\left[\frac{(2j)!\,(2l-j+k)!\,(l+j+m+\lambda)!\,(l+k-m-\lambda)!}{(j+k)!\,(j-k)!\,(2l+j+k+1)!\,(l-j-m-\lambda)!\,(l+k+m+\lambda)!}\right]^{1/2} \tag{3.1}$$

and the recursion relation

$$\begin{pmatrix} j & l & l+k \\ \mu-1 & -m-\lambda-\mu+1 & m+\lambda \end{pmatrix}$$

$$= -\left[\frac{(l-m-\lambda+k+1)(l+m+\lambda+k)}{(j+\mu)(j-\mu+1)}\right]^{1/2}$$

$$\times \begin{pmatrix} j & l & l+k \\ \mu & -m-\mu-\lambda+1 & m+\lambda-1 \end{pmatrix}$$

$$-\left[\frac{(l-m-\lambda-\mu+1)(l+m+\lambda+\mu)}{(j+\mu)(j-\mu+1)}\right]^{1/2}$$

$$\times \begin{pmatrix} j & l & l+k \\ \mu & -m-\mu-\lambda & m+\lambda \end{pmatrix}. \qquad (3.2)$$

In the expansion of $\begin{pmatrix} j & l & l+k \\ \mu & -m-\lambda-\mu & m+\lambda \end{pmatrix}$, first consider terms (i.e., linear factors) involving l but not m; it is clear from (3.2) that these do not depend on the value of μ. This therefore implies that the l-, but not m-, dependent SR-D part of $\begin{pmatrix} j & l & l+k \\ \mu & -\mu-m-\lambda & m+\lambda \end{pmatrix}$ and $\begin{pmatrix} j & l & l+k \\ j & -j-m-\lambda & m+\lambda \end{pmatrix}$ are the same, and therefore from (3.1) equal to $[(2l-j+k)!/(2l+j+k+1)!]^{1/2}$. The recursion relation (3.2) also shows that they contain no terms depending on m but not on l.

We next show by induction that the l- and m-dependent SR-D part of $\begin{pmatrix} j & l & l+k \\ \mu & -\mu-m-\lambda & m+\lambda \end{pmatrix}$ is

$$\begin{cases} \left[\dfrac{(l+m+\mu+\lambda)!\,(l-m+k-\lambda)!}{(l-m-\mu-\lambda)!\,(l+m+k+\lambda)!}\right]^{1/2}, & |\lambda| \leq k \leq \mu, \\[4mm] \left[\dfrac{(l+m+k+\lambda)!\,(l-m+k-\lambda)!}{(l+m+\mu+\lambda)!\,(l-m-\mu-\lambda)!}\right]^{1/2}, & |\lambda| \leq \mu \leq k. \end{cases} \qquad (3.3)$$

From (3.1) we see that (3.3) is true for $\mu = j$. Suppose it is true for some μ with $k+1 \leq \mu \leq j$; one can then by means of a straightforward calculation show that it is also true for $k \leq \mu - 1 \leq j - 1$.

We must next consider the case where μ and k are both $\leq -|\lambda|$. First of all, from (2.13) it is easy to see that the l- and m-dependent SR-D part of $\begin{pmatrix} j & l & l+k \\ \mu & -\mu-m-\lambda & m+\lambda \end{pmatrix}$ is obtained from that of $\begin{pmatrix} j & l & l+k \\ \mu & -\mu-m-\lambda & m+\lambda \end{pmatrix}$ by simply replacing m by $-(m+2\lambda)$. In order to consider the case where $k \leq -|\lambda|$, observe that[15]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}(2j_3+1)^{-1/2}$$

$$\times \langle j_1, m_1, j_2, m_2 | j_1, j_2, j_3, -m_3\rangle, \qquad (3.4)$$

where $\langle j_1, m_1, j_2, m_2 | j_1, j_2, j_3, -m_3\rangle$ is a vector-coupling coefficient. Note that (3.4) can be regarded as the definition of 3-j symbols in terms of vector-coupling coefficients. We take this definition to be valid for negative, as well as positive, values of $j_1, j_2,$ and j_3.

From (3.4) it then follows that

$$\begin{pmatrix} j & -l-1 & -l-1+k \\ \mu & -\mu-m-\lambda & m+\lambda \end{pmatrix} \Big/ \begin{pmatrix} j & l & l-k \\ \mu & -\mu-m-\lambda & m+\lambda \end{pmatrix}$$

$$= (-1)^{2l+1/2}$$

$$\times \frac{\langle j,\mu,-l-1,-\mu-m-\lambda | j,-l-1,-l-1+k,-m-\lambda\rangle}{\langle j,\mu,l,-\mu-m-\lambda | j,l,l-k,-m-\lambda\rangle}.$$

Now from the way in which vector coupling coefficients are defined, namely, as coefficients in the expansion

of one set of basis elements for the carrier space of the direct product representation $D^{j_1} \times D^{j_2}$ in terms of another, where the basis elements depend on j_1 and j_2 only through $j_1(j_1+1)$ and $j_2(j_2+1)$ and are therefore unchanged if j_1 or j_2 are replaced by $-(j_1+1)$ or $-(j_2+1)$ respectively, it is clear that the vector-coupling coefficients themselves are unchanged by replacing j_1 or j_2 by $-(j_1+1)$ or $-(j_2+1)$, respectively. Hence

$$\langle j,\mu,-l-1,-\mu-m-\lambda | j,-l-1,-l-1+k,-m-\lambda\rangle$$

$$= \langle j,\mu,l,-\mu-m-\lambda | j,l,l-k,-m-\lambda\rangle,$$

so

$$\begin{pmatrix} j & -l-1 & l-1+k \\ \mu & -\mu-m-\lambda & m+\lambda \end{pmatrix} = (-1)^{2l+1/2}$$

$$\times \begin{pmatrix} j & l & l-k \\ \mu & -\mu-m-\lambda & m+\lambda \end{pmatrix}. \qquad (3.5)$$

Note that in deriving (3.5), one could first apply (2.13) to both of the 3-j symbols; as a result one would instead get $(-1)^{2l+2k+1/2}$ as the phase factor. This apparent discrepancy is immediately resolved if one follows the procedure given directly after (2.13). It is then found that the phase factor is just that appearing in (3.5).

We now consider the normalization of the B_l^k. The coefficient given by (2.11) can be rewritten:

$$a_\mu^k(l,m) = (-1)^{j+k+2l}\left[\frac{(l-m-\mu)!\,(l+m)!}{(l+m+\mu)!\,(l-m)!}\right]^{1/2}$$

$$\times \begin{pmatrix} j & l & l+k \\ \mu & -m-\mu & m \end{pmatrix}. \qquad (3.6)$$

One can show by induction that the factor

$$[(2j)!\,(2l-j+k)!/(j+k)!\,(j-k)!\,(2l+j+k+1)!]^{1/2}$$

occurs in $\begin{pmatrix} j & l & l+k \\ \mu & -m-\mu & m \end{pmatrix}$ for all $\mu \geq 0$. Thus we first normalize away this factor, which is common to all the $a_\mu^k(l,m)$, and define

$$A_l^k = (-1)^{-k+l-m}\left[\frac{(j+k)!\,(j-k)!\,(2l+j+k+1)!}{(2j)!\,(2l-j+k)!}\right]^{1/2} B_l^k. \qquad (3.7)$$

Then, by a straightforward calculation one gets

$$A_l^k = \beta_0^k(l,m)Q_0 + \sum_{\mu=1}^{j}[\beta_\mu^k(l,m)Q_{+\mu} + (-1)^{j+k}\beta_\mu^k(l,-m)Q_{-\mu}], \qquad (3.8)$$

where, for $\mu = 0, \ldots, j$,

$$\beta_\mu^k(l,m) = (-1)^{j+3l-m}$$

$$\times \left[\frac{(j+k)!\,(j-k)!\,(2l+j+k+1)!\,(l-m-\mu)!\,(l+m)!}{(2j)!\,(2l-j+k)!\,(l+m+\mu)!\,(l-m)!}\right]^{1/2}$$

$$\times \begin{pmatrix} j & l & l+k \\ \mu & -\mu-m & m \end{pmatrix}, \qquad (3.9)$$

and where in obtaining the coefficients of $Q_{-\mu}$ we have used $(-1)^{2(l+m)} = 1$, since l and m are integral or half-integral together.

The action of A_l^k on $|\gamma, l, m\rangle$ is given by

$$A_l^k |\gamma, l, m\rangle = \left[\frac{(j+k)!(j-k)!(2l+j+k+1)!}{(2j)!(2l-j+k)!(2l+2k+1)^2} \right]^{1/2} \delta(j, l, l+k)$$

$$\times \sum_{\gamma'} \langle \gamma', l+k \| T(j) \| \gamma, l \rangle | \gamma', l+k, m \rangle. \quad (3.10)$$

For the normalization of the $B_{l,m}^{k,\pm1/2}$ we proceed in an exactly analogous manner. Thus define

$$A_{l,m}^{k,\pm1/2} = (-1)^{-k+l-m\pm1/2}$$

$$\times \left[\frac{(j+k)!(j-k)!(2l+j+k+1)!}{(2j)!(2l-j+k)!} \right]^{1/2} B_{l,m}^{k,\pm1/2}. \quad (3.11)$$

Then

$$A_{l,m}^{k,\pm1/2} = \sum_{\mu=1/2}^{j} [\beta_\mu^k(l, m, \pm\tfrac{1}{2}) Q_{+\mu}^\pm + (-1)^{k+j\pm1} \beta_\mu^k(l, -m, \mp\tfrac{1}{2}) Q_{-\mu}^\pm], \quad (3.12)$$

where, for $\mu = \tfrac{1}{2}, \ldots, j$,

$$\beta_\mu^k(l, m, \tfrac{1}{2}) = (-1)^{3l+j-m+1/2}$$

$$\times \left[\frac{(j+k)!(j-k)!(2l+j+k+1)!(l-m-\mu-\tfrac{1}{2})!(l+m)!}{(2j)!(2l-j+k)!(l+m+\mu+\tfrac{1}{2})!(l-m)!} \right]^{1/2}$$

$$\times \begin{pmatrix} j & l & l+k \\ \mu & -m-\mu-\tfrac{1}{2} & m+\tfrac{1}{2} \end{pmatrix}. \quad (3.13)$$

The action of $A_{l,m}^{k,\pm1/2}$ on $|\gamma, l, m\rangle$ is given by

$$A_{l,m}^{k,\pm1/2} |\gamma, l, m\rangle$$

$$= \left[\frac{(j+k)!(j-k)!(2l+j+k+1)!}{(2j)!(2l-j+k)!(2l+2k+1)^2} \right]^{1/2} \delta(j, l, l+k)$$

$$\times \sum_{\gamma'} \langle \gamma', l+k \| T(j) \| \gamma, l \rangle | \gamma', l+k, m \pm \tfrac{1}{2} \rangle. \quad (3.14)$$

We see from (3.10) and (3.11) that the matrix elements of A_l^k and $A_{l,m}^{k,\pm1/2}$ are independent of m.

Going back to the case of integral j, we consider the SR-D part of the coefficient $\beta_\mu^k(l, m)$, noting first that the part which depends on l but not on m has been normalized away. It is easy to show that the l- and m-dependent SR-D part of $\beta_\mu^k(l, m)$ for $\mu, k \geq 0$ is $[(l-m+k)!(l+m)!/(l+m+k)!(l-m)!]^{1/2}$, which precisely equals $\beta_j^k(l, m)$. From (3.8) we see that the l- and m-dependent SR-D part of the coefficient of $Q_{-\mu}$ in A_l^k is $[(l+m+k)!(l-m)!/(l-m+k)!(l+m)!]^{1/2}$, which is just $(-1)^{j+k}$ times the coefficient of Q_{-j}. Define, for $k \geq 0$

$$O_l^k = \left[\frac{(l+m+k)!(l-m+k)!}{(l+m)!(l-m)!} \right]^{1/2} A_l^k. \quad (3.15)$$

Then

$$O_l^k = \gamma_0^k(l, m) Q_0 + \sum_{\mu=1}^{j} [\gamma_\mu^k(l, m) Q_{+\mu} + (-1)^{j+k} \gamma_\mu^k(l, -m) Q_{-\mu}], \quad (3.16)$$

where, for $\mu = 0, \ldots, j$

$$\gamma_\mu^k(l, m) = (-1)^{j+3l-m} \left[\frac{(j+k)!(j-k)!(2l+j+k+1)!(l-m-\mu)!(l+m+k)!(l-m+k)!}{(2j)!(2l-j+k)!(l+m+\mu)!((l-m)!)^2} \right]^{1/2}$$

$$\times \begin{pmatrix} j & l & l+k \\ \mu & -\mu-m & m \end{pmatrix}. \quad (3.17)$$

From the preceding discussion it is clear that $\gamma_\mu^k(l, m)$ contains no l- and m-dependent SR-D part since the normalization in (3.15) cancels out the term $[(l+m+k)!/(l+m)!]^{-1/2}$ that occurs in $\beta_\mu^k(l, m)$ and yields $(l-m+k)!/(l-m)!$, which is not an SR-D part since $k \geq 0$. In particular the coefficient of Q_{+j} is

$$\gamma_j^k(l, m) = (l-m+k)!/(l-m)!, \quad (3.18)$$

and the coefficient of Q_{-j} is $(-1)^{j+k} \gamma_j^k(l, -m)$.

The action of O_l^k is given by

$$O_l^k |\gamma, l, m\rangle$$

$$= \left[\frac{(j+k)!(j-k)!(2l+j+k+1)!(l+m+k)!(l-m+k)!}{(2j)!(2l-j+k)!(l+m)!(l-m)!(2l+2k+1)^2} \right]^{1/2}$$

$$\times \delta(j, l, l+k) \sum_{\gamma'} \langle \gamma', l+k \| T(j) \| \gamma, l \rangle | \gamma', l+k, m \rangle. \quad (3.19)$$

O_l^k therefore has the desired property that both l and m occur only to integral powers and only in the numerator. l and m can therefore be replaced by the operators R and l_0, respectively, remembering that all terms involving R must be placed to the right of Q_μ in order that, when O_l^k acts upon $|\gamma, l, m\rangle$, R be replaceable by its eigenvalue l. O_l^k is then written entirely in terms of the operators l_i, $T(j, \mu)$, and R, and is thus in a form which is independent of the particular state upon which it acts.

Before defining O_l^{-k}, observe that

$$\frac{\beta_\mu^k(-(l+1), m)}{\beta_\mu^{-k}(l, m)} = (-1)^{-2l-1/2}$$

$$\times \begin{pmatrix} j & -l-1 & -l-1+k \\ \mu & -\mu-m & m \end{pmatrix} / \begin{pmatrix} j & l & l-k \\ \mu & -\mu-m & m \end{pmatrix},$$

which together with (3.5) shows that $\beta_\mu^{-k}(l, m) = \beta_\mu^k(-(l+1), m)$, so

$$A_l^{-k} = A_{-(l+1)}^k. \quad (3.20)$$

In order to preserve this relationship for O_l^{-k} we accordingly define for $k \geq 0$

$$O_l^{-k} = (-1)^k \left[\frac{(l+m)!(l-m)!}{(l+m-k)!(l-m-k)!} \right]^{1/2} A_l^{-k}. \quad (3.21)$$

Then

$$O_l^{-k} = O_{-(l+1)}^k$$

$$= \delta_0^{-k}(l, m) Q_0 + \sum_{\mu=1}^{j} [\delta_\mu^{-k}(l, m) Q_{+\mu} + (-1)^{j+k} \delta_\mu^{-k}(l, -m) Q_{-\mu}], \quad (3.22)$$

where

$$\delta_\mu^{-k}(l,m) = (-1)^{j+3l-k-m}$$

$$\times \left[\frac{(2l+j-k+1)!\,(j+k)!\,(j-k)!\,(l-m-\mu)!\,[(l+m)!\,]^2}{(2j)!\,(2l-j-k)!\,(l+m+\mu)!\,(l-m-k)!\,(l+m-k)!} \right]^{1/2}$$

$$\times \begin{pmatrix} j & l & l-k \\ \mu & -\mu-m & m \end{pmatrix}. \tag{3.23}$$

Following the steps of the above argument, we now consider the normalized shift operators $O_{l,m}^{k,\pm1/2}$. Define

$$d_\mu^k(l,m,\tfrac12) = (-1)^{j+3l-m+1/2} \left[\frac{(j+k)!\,(j-k)!\,(2l+j+k+1)!\,(l-m-\mu-\frac12)!\,(l+m+k+\frac12)!\,(l-m+k-\frac12)!}{(2j)!\,(2l-j+k)!\,(l+m+\mu+\frac12)!\,[(l-m)!\,]^2} \right]^{1/2}$$

$$\times \begin{pmatrix} j & l & l+k \\ \mu & -m-\mu-\frac12 & m+\frac12 \end{pmatrix}. \tag{3.26}$$

It is easy to show that $O_{l,m}^{k,\pm1/2}$ and $d_\mu^k(l,m,\tfrac12)$ contain no l- and m-dependent SR-D part.

The action of $O_{l,m}^{k,\pm1/2}$ on $|\gamma,l,m\rangle$ is given by

$$O_{l,m}^{k,\pm1/2}|\gamma,l,m\rangle = \left[\frac{(j+k)!\,(j-k)!\,(2l+j+k+1)!\,(l+m+k\pm\frac12)!\,(l-m+k\mp\frac12)!}{(2j)!\,(2l-j+k)!\,(l-m)!\,(l+m)!\,(2l+2k+1)^2} \right]^{1/2}$$

$$\times \delta(j,l,l+k) \sum_{\gamma'} \langle\gamma',l+k\|T(j)\|\gamma,l\rangle\,|\gamma',l+k,m+\tfrac12\rangle. \tag{3.27}$$

To define $O_{l,m}^{-k,\pm1/2}$, note that $b_\mu^k(-(l+1),m,\tfrac12) = (-1)^{-1/2}b_\mu^k(l,m,-\tfrac12)$, so

$$A_{-(l+1),m}^{k,\pm1/2} = (-1)^{-1/2} A_{l,m}^{-k,\pm1/2}. \tag{3.28}$$

Thus for $k \geqslant \tfrac12$, define

$$O_{l,m}^{-k,\pm1/2} = (-1)^{k-1/2} \left[\frac{(l+m)!\,(l-m)!}{(l+m-k\pm\frac12)!\,(l-m-k\mp\frac12)!} \right]^{1/2} A_{l,m}^{-k,\pm1/2}, \tag{3.29}$$

so that

$$O_{l,m}^{-k,\pm1/2} = O_{-(l+1),m}^{k,\pm1/2} = \sum_{u=1/2}^{j} [t_\mu^{-k}(l,m,\pm\tfrac12)Q_\mu^\pm + (-1)^{k+j+1}t_\mu^{-k}(l,-m,\mp\tfrac12)Q_{-\mu}^\pm], \tag{3.30}$$

where

$$t_\mu^{-k}(l,m,\tfrac12) = (-1)^{3l+j-m+k} \left[\frac{(j+k)!\,(j-k)!\,(2l+j-k+1)!\,(l-m-\mu-\frac12)!\,[(l+m)!\,]^2}{(2j)!\,(2l-j-k)!\,(l+m-k+\frac12)!\,(l-m-k-\frac12)!\,(l+m+\mu+\frac12)!} \right]^{1/2}$$

$$\times \begin{pmatrix} j & l & l-k \\ \mu & -m-\mu-\frac12 & m+\frac12 \end{pmatrix}. \tag{3.31}$$

for $k \geqslant \tfrac12$

$$O_{l,m}^{k,\pm1/2} = \left[\frac{(l+m+k\pm\frac12)!\,(l-m+k\mp\frac12)!}{(l-m)!\,(l+m)!} \right]^{1/2} A_{l,m}^{k,\pm1/2}. \tag{3.24}$$

Then

$$O_{l,m}^{k,\pm1/2} = \sum_{\mu=1/2}^{j} [d_\mu^k(l,m,\pm\tfrac12)Q_\mu^\pm + (-1)^{k+j+1}d_\mu^k(l,-m,\mp\tfrac12)Q_{-\mu}^\pm], \tag{3.25}$$

where, for $\mu = \tfrac12,\ldots,j$,

The fact that we can choose our shift operators to satisfy the symmetry relations (3.20) and (3.22) [or (3.28) and (3.30)] is due to the fact that the IUR of O(3) are labelled by $l(l+1)$ rather than by l itself, so one may describe them as corresponding to either $l=0$, $\tfrac12$, 1, $\tfrac32,\cdots$ or $l=-1$, $-\tfrac32$, -2, $-\tfrac52,\cdots$. In passing from the positive l to the negative l description, the roles of O_l^k and O_l^{-k} (or of $O_{l,m}^{k,\pm1/2}$ and $O_{l,m}^{-k,\pm1/2}$) are interchanged,

since in the negative l description, $O_{-(l+1)}^k$ and $O_{-(l+1),m}^{k,\pm1/2}$, by raising l, lower $l(l+1)$ in precisely the same way as O_l^{-k} and $O_{l,m}^{-k,\pm1/2}$ do in the positive l description.

For the case when j is integral, as a result of (3.20) and (3.22) we note that $O_l^0 = A_l^0$ is invariant under the replacement of l by $-(l+1)$, and therefore contains l always in the combination $l(l+1)$. This means that R

occurs only in the combination $R(R+1) = L^2 = l_+l_- + l_0(l_0 - 1)$; O_l^0 is therefore an O(3) scalar operator which can be written entirely in terms of l_i and $T(j,\mu)$.

Regarding the actions of $O_l^{\pm k}$, $O_{l,m}^{\pm k, 1/2}$, and $O_{l,m}^{\pm k, -1/2}$ on $|\gamma, l, m\rangle$, since their matrix elements are m-dependent, they are not as convenient to use for the classification and analysis of IUR of groups with respect to the IUR of O(3) subgroups as are the corresponding A operators. The additional term $[(l+m)\cdots(l+m-k+1)(l-m)\cdots \times (l-m-k+1)]^{1/2}$ in the definition of O_l^{-k} over A_l^{-k} guarantees that O_l^{-k} cannot shift l down below $|m|$ since one of the above factors must vanish whenever $l < |m|+k$. Similarly for the case of $O_{l,m}^{-k,\pm 1/2}$. If one uses the A operators instead of the O operators in such an analysis, one removes all considerations of the internal structure of IUR of O(3).

We end this section by giving the precise forms of $O_l^{\pm k}$ for the case $j=1$, and of $O_{l,m}^{\pm k, 1/2}$ and $O_{l,m}^{\pm k, -1/2}$ for the case $j=\frac{1}{2}$. These are

$$O_l^0 = T(1,-1)l_+ + \sqrt{2}T(1,0)l_0 - T(1,1)l_-, \qquad (3.32)$$

$$O_l^1 = T(1,-1)l_+(R-l_0+1) + \sqrt{2}T(1,0)(R-l_0+1)(R+l_0+1)$$
$$+ T(1,1)l_-(R+l_0+1), \qquad (3.33)$$

$$O_l^{-1} = -T(1,-1)l_+(R+l_0) + \sqrt{2}T(1,0)(R+l_0)(R-l_0)$$
$$- T(1,1)l_-(R-l_0). \qquad (3.34)$$

These are precisely the operators obtained for the groups O(4), O(3,1), and E(3) [3] when one identifies $T(1,1) = -q_{+1}$, $T(1,0) = q_0$, $T(1,-1) = q_{-1}$ and divides by two. Similarly, when $j=2$, one obtains the operators for SU(3) in an O(3) basis[1,2] and Sl(3,R).[4]

For the case when $j=\frac{1}{2}$, one obtains

$$O_{l,m}^{+1/2,\pm 1/2} = -T(\tfrac{1}{2},\mp\tfrac{1}{2})l_\pm - T(\tfrac{1}{2},\pm\tfrac{1}{2})(R\pm l_0+1), \qquad (3.35)$$

$$O_{l,m}^{-1/2,\pm 1/2} = -T(\tfrac{1}{2},\mp\tfrac{1}{2})l_\pm + T(\tfrac{1}{2},\pm\tfrac{1}{2})(R\mp l_0), \qquad (3.36)$$

and these are precisely the operators obtained for the case of O(3)$\wedge(T_2 \times \overline{T}_2)$ [5] and SU(3) in an SU(2) basis[6] when one denotes $T(\tfrac{1}{2},\pm\tfrac{1}{2}) = -q_{\pm 1/2}$.

In the half-integral j case one can in general construct $(2j+1)^2$ shift operators $O_{l,m}^{k,r}$, $k,r = j,\ldots,j$, which when applied to $|\gamma, l, m\rangle$ change l and m by any amounts within the above range. The preceding analysis, however, provides only the expressions for $O_{l,m}^{k,\pm 1/2}$. However, by the action of $O_{l,m}^{k,\pm 1/2}$ followed by the repeated application of $l_\pm$ to $|\gamma, l, m\rangle$, it is easily seen that one is able to shift l and m by any desired amount within the above range. Thus our restriction to the construction of only $O_{l,m}^{k,\pm 1/2}$ causes no serious loss of generality.

4. HERMITICITY PROPERTIES OF THE O_l^k

Just as $l_\pm$ are related by the hermiticity conditions $l_\pm^\dagger = l_\mp$, the $O_l^{\pm k}$ are also related by hermiticity. These may be investigated in two ways, depending on whether or not we replace l and m by operators R and l_0, respectively. If we do, then we may take the Hermitian conjugate of O^{+k} directly and compare it with O^{-k}, noting that when R is used the subscript "l" is removed from $O_l^{\pm k}$ since it is superfluous; this, however, involves knowledge of the commutators of R and the Q_μ, which will be considered in Sec. 6, and so in this section we

use the alternative approach which, since $O_l^{\pm k}$ must act to the right upon $|\gamma, l, m\rangle$, involves comparing matrix elements of the shift operators. First, note that O_l^k acts to the right upon $|\gamma, l, m\rangle$ as a raising operator, so $(O_l^k)^\dagger$ acts to the left upon $\langle \gamma, l, m|$ as a raising operator and, unlike O_{l+k}^{-k}, does not act to the right upon $|\gamma, l+k, m\rangle$ as a lowering operator. The advantage of replacing l by R is that then, like O^{-k}, $(O^k)^\dagger$ does in fact act to the right as a lowering operator. This advantage is, however, small compared to the disadvantages caused by the increase in computational complexity. We therefore find $\alpha_{k,l}$ in the equation

$$\langle \gamma, l, m |(O_l^k)^\dagger |\gamma', l+k, m\rangle = \alpha_{k,l}\langle \gamma, l, m|$$
$$\times O_{l+k}^{-k}|\gamma', l+k, m\rangle. \qquad (4.1)$$

Before calculating $\alpha_{k,l}$ explicitly we note that the relation $O_{-(l+1)}^{\pm k} = O_l^{\mp k}$ immediately imposes a condition on $\alpha_{k,l}$. To see this, suppress γ and m from the states and label them by $l(l+1)$ rather than l; then, taking the complex conjugate of (4.1) and replacing l by $-(l+1)$, we have

$$\langle (l-k)(l-k+1) |O_l^{-k} |l(l+1)\rangle$$
$$= \alpha_{k,-(l+1)}^*\langle (l-k)(l-k+1) |(O_{l-k}^{+k})^\dagger |l(l+1)\rangle$$

so that

$$\langle l(l+1) |O_{l+k}^{-k} |(l+k)(l+k+1)\rangle$$
$$= \alpha_{k,-(l+k+1)}^*\langle l(l+1) |(O_l^{+k})^\dagger |(l+k)(l+k+1)\rangle$$

from which follows

$$\alpha_{k,l}\alpha_{k,-(l+k+1)}^* = 1. \qquad (4.2)$$

To calculate $\alpha_{k,l}$ explicitly, first observe that

$$\langle \gamma, l, m |(A_l^{+k})^\dagger |\gamma', l+k, m\rangle$$
$$= \frac{f_{l,k}^* \alpha_{k,l}}{f_{-(l+k+1),k}} \langle \gamma, l, m |A_{l+k}^{-k} |\gamma', l+k, m\rangle$$

where

$$f_{l,k} = \left[\frac{(l+m)!(l-m)!}{(l+m+k)!(l-m+k)!}\right]^{1/2}.$$

But

$$f_{-(l+k+1),k} = (-1)^{-k}f_{l,k} = (-1)^{-k}f_{l,k}^*,$$

so that

$$\langle \gamma, l, m |(A_l^{+k})^\dagger |\gamma', l+k, m\rangle$$
$$= (-1)^k\alpha_{k,l}\langle \gamma, l, m |A_{l+k}^{-k} |\gamma', l+k, m\rangle. \qquad (4.3)$$

Since the matrix elements of A_l^k are independent of m, it follows that $\alpha_{k,l}$ is independent of m.

Using Eq. (3.10) and the fact that $\delta(j,l,l+k) = \delta(j,l+k,l)$, we obtain

$$\frac{\langle \gamma, l, m |(A_l^{+k})^\dagger |\gamma', l+k, m\rangle}{\langle \gamma, l, m |A_{l+k}^{-k} |\gamma', l+k, m\rangle}$$
$$= \frac{(2l+1)\langle \gamma', l+k||T(j)||\gamma, l\rangle^*}{(2l+2k+1)\langle \gamma, l||T(j)||\gamma', l+k\rangle}. \qquad (4.4)$$

We now impose the following hermiticity condition on $T(j,\mu)$:

$$T(j,\mu)^\dagger = (-1)^\mu T(j,-\mu). \qquad (4.5)$$

Here we note that (4.5) is defined in a self-consistent manner, for if one forms the Hermitian conjugate of it and lets μ be replaced by $-\mu$, recalling that in this case μ is an integer, one arrives at the condition (4.5) once again.

Using (2.3) and (4.5) with $\mu = 0$, we have

$$\langle \gamma', l+k \| T(j) \| \gamma, l \rangle$$
$$= (-1)^{-l-k+m} \langle \gamma', l+k, m \mid T(j,0) \mid \gamma, l, m \rangle / \begin{pmatrix} l+k & j & l \\ -m & 0 & m \end{pmatrix}$$
$$= (-1)^{-l-k+m} \langle \gamma, l, m \mid T(j,0) \mid \gamma', l+k, m \rangle^* / \begin{pmatrix} l+k & j & l \\ -m & 0 & m \end{pmatrix}$$
$$= (-1)^{-l-k+m} \left[(-1)^{l-m} \begin{pmatrix} l & j & l+k \\ -m & 0 & m \end{pmatrix} \times \langle \gamma, l \| T(j) \| \gamma', l+k \rangle \right]^* / \begin{pmatrix} l+k & j & l \\ -m & 0 & m \end{pmatrix}.$$

Using property (2.13) of the 3-j symbol, together with the property that an odd transposition of the column introduces a factor of $(-1)^{j_1+j_2+j_3}$, we obtain

$$\langle \gamma', l+k \| T(j) \| \gamma, l \rangle$$
$$= (-1)^{-k} \langle \gamma, l \| T(j) \| \gamma', l+k \rangle^*, \qquad (4.6)$$

so that

$$\langle \gamma', l, m \mid (A^{+k})^{\dagger} \mid \gamma', l+k, m \rangle$$
$$= \frac{(-1)^{-k}(2l+1)}{(2l+2k+1)} \langle \gamma, l, m \mid A^{-k}_{l+k} \mid \gamma', l+k, m \rangle. \qquad (4.7)$$

Hence

$$\alpha_{k,l} = (2l+1)/(2l+2k+1). \qquad (4.8)$$

Using the definition of A^k_l in terms of B^k_l, one easily finds

$$\langle \gamma, l, m \mid (B^{+k}_l)^{\dagger} \mid \gamma', l+k, m \rangle$$
$$= \alpha_{k,l} \langle \gamma, l, m \mid B^{-k}_{l+k} \mid \gamma', l+k, m \rangle. \qquad (4.9)$$

Equations (4.1) and (4.8) imply that if one replaces l by R in the expressions for $O^{\pm k}_l$, the resulting operators satisfy the hermiticity relation

$$(O^k)^{\dagger}(2R+1) = O^{-k}(2R-2k+1). \qquad (4.10)$$

This can in principle be verified directly once one knows the form of $[Q_{\mu}, R]$, and will be done for the particular case of $J = 1$ in Sec. 6.

It is clear from (4.1) and (4.8) [or from (4.10)] that the O(3) scalar operator O^0_l is Hermitian and therefore has real eigenvalues which may be used in specific applications to provide a partial label for the states $|\gamma, l, m\rangle$ as, for instance, was done in the case of IUR of SU(3) analyzed with respect to an O(3) basis. [1,2]

Finally, observe that, if one wishes, one may define shift operators G^k satisfying $(G^{+k})^{\dagger} = G^{+k}$ by

$$G^k = O^k(2R+2k+1)(2R+k+1)^{-1}. \qquad (4.11)$$

This, however, reintroduces R into the denominator and so may not be particularly advantageous.

5. HERMITICITY PROPERTIES OF THE $O^{k,\pm 1/2}_{l,m}$

Unlike the case of integral j considered in Sec. 4, one cannot adopt definition (4.5) for half-integral j, since such a definition would not be self-consistent. If one takes the Hermitian conjugate of (4.5) and replaces μ by $-\mu$, one obtains

$$T(j,\mu)^{\dagger} = (-1)^{-\mu} T(j,-\mu) \neq (-1)^{\mu} T(j,-\mu)$$

if μ is half-integral. For this reason one is therefore compelled, if one wishes to bring hermiticity into the analysis, to consider, for half-integral j, a new set of operators $\overline{T}(j,\mu), \mu = -j, \ldots, j$, which form a $(2j+1)$-dimensional tensor representation of O(3) and whose commutators with l_0 and $l_{\pm}$ are exactly analogous to those of the $T(j,\mu)$.

It is trivial to check that the following hermiticity condition is self-consistent:

$$T(j,\mu)^{\dagger} = (-1)^{-j+\mu} \overline{T}(j,-\mu). \qquad (5.1)$$

All the computations conducted in previous sections in deriving shift operators $B^{k,\pm 1/2}_{l,m}$, $A^{k,\pm 1/2}_{l,m}$, and $O^{k,\pm 1/2}_{l,m}$ can now be taken over when using $\overline{T}(j,\mu)$. The operators $\overline{B}^{k,\pm 1/2}_{l,m}$, $\overline{A}^{k,\pm 1/2}_{l,m}$, and $\overline{O}^{k,\pm 1/2}_{l,m}$ thus formed will have exactly the same properties and mathematical forms in terms of the $\overline{T}(j,\mu)$ as do their counterparts in terms of the $T(j,\mu)$.

Following the manner in which the constant of proportionality $\alpha_{k,l}$ of the previous section was found, one obtains the following relations:

$$\langle \gamma', l \pm k, m + \tfrac{1}{2} \mid O^{\pm k,+1/2}_{l,m} \mid \gamma, l, m \rangle$$
$$= \frac{(-1)^{j+1/2}(2l+1)}{(2l \pm 2k+1)} \langle \gamma', l \pm k, m + \tfrac{1}{2} \mid (\overline{O}^{\mp k,-1/2}_{l \pm k, m+1/2})^{\dagger} \mid \gamma, l, m \rangle \qquad (5.2)$$

and

$$\langle \gamma', l \pm k, m - \tfrac{1}{2} \mid O^{\pm k,-1/2}_{l,m} \mid \gamma, l, m \rangle$$
$$= \frac{(-1)^{j-1/2}(2l+1)}{(2l \pm 2k+1)} \langle \gamma', l \pm k, m - \tfrac{1}{2} \mid (\overline{O}^{\mp k,-1/2}_{l \pm k, m-1/2})^{\dagger} \mid \gamma, l, m \rangle. \qquad (5.3)$$

Note that care has to be taken when deriving (5.2) and (5.3) since they involve terms like

$$\begin{pmatrix} l & j & l+k \\ -m & -\tfrac{1}{2} & m+\tfrac{1}{2} \end{pmatrix}^*,$$

which have imaginary phase factors.

In operator form, (5.2) and (5.3) become

$$O^{\pm k,+1/2}(2R \pm 2k+1) = (-1)^{j+1/2}(\overline{O}^{\mp k,-1/2})^{\dagger}(2R+1) \qquad (5.4)$$

and

$$O^{\pm k,-1/2}(2R \pm 2k+1) = (-1)^{j-1/2}(\overline{O}^{\mp k,+1/2})^{\dagger}(2R+1). \qquad (5.5)$$

(5.4) and (5.5) can in principle be derived once one knows the commutators $[Q^+_{\pm\mu}, R]$ and $[Q^-_{\pm\mu}, R]$; this will be illustrated for the case of $j = \tfrac{1}{2}$ in Sec. 7.

6. EVALUATION OF $[R, Q_{\mu}]$

In this section we obtain expressions for $[R, Q_{\mu}]$ and show that $[R, Q_{\mu}](2R+2j)\cdots(2R-2j+l)$ involves only terms of the form Q_{μ} multiplied by positive integral powers of R and l_0.

We make use of the fact that, for $k = -j, \ldots, j$,

$$[R, B_l^k] = k B_l^k \tag{6.1}$$

providing, of course, the above operators act to the right on $|\gamma, l, m\rangle$. In terms of the P_μ of (2.5), this becomes

$$[R, P_\mu] = \sum_{\mu', k=-j}^{j} k(2l+2k+1) \begin{pmatrix} j & l & l+k \\ -\mu & m+\mu & -m \end{pmatrix} \times \begin{pmatrix} j & l & l+k \\ -\mu' & m+\mu' & -m \end{pmatrix} P_{\mu'},$$

where we have made use of the orthogonality relation (2.8). Employing (2.5), we get the following expression for $[R, Q_{+\mu}]$, where $\mu = -j, \ldots, j$:

$$[R, Q_\mu]$$
$$= \sum_{\mu'=-j}^{j} \left[\frac{(l+m+\mu)!\,(l-m)!}{(l+m)!\,(l-m-\mu)!} \right]^{\tau(\mu)/2}$$
$$\times \left[\frac{(l+m)!\,(l-m-\mu')!}{(l+m+\mu')!\,(l-m)!} \right]^{\tau(\mu')/2}$$
$$\times \left\{ \sum_{k=-j}^{j} k(2l+2k+1) \begin{pmatrix} j & l & l+k \\ -\mu & m+\mu & -m \end{pmatrix} \right.$$
$$\left. \times \begin{pmatrix} j & l & l+k \\ -\mu' & m+\mu' & -m \end{pmatrix} \right\} Q_{\mu'}, \tag{6.2}$$

where $\tau(\mu) = \mu/|\mu|$.

Using arguments similar to those employed in Sec. 3, it can be shown that the coefficients of Q_μ, in (6.2) contain no l- or m-dependent SR-D terms except for the term $(2l+2j)\cdots(2l-2j+2)$ in the denominator if $j \geq 1$. Otherwise, l and m occur entirely in the numerator and to integral powers.

Consequently, the only SR-D part appearing, for a fixed k, in any of the coefficients is $(2l+2k+1)/(2l+j+k+1)\cdots(2l-j+k+1)$. Summing over k from $-j$ to j, we see therefore that these coefficients contain the term $(2l+2j)\cdots(2l-2j+2)$ in the denominator if $j \geq 1$, and no such term if $j = 0$. Replacing l and m by R and l_0, and ensuring that R and l_0 appear to the right of Q_μ, we obtain from (6.2) expressions for $[R, Q_\mu]$ in terms of $Q_{\mu'}$, l_0, and R which do not depend for their validity on their actions on any particular state. Furthermore, $[R, Q_\mu](2R+2j)\cdots(2R-2j+2)$ (or $[R, Q_0]$ if $\mu = 0$), is an operator in which R and l_0 occur only to positive powers, together with the $Q_{\mu'}$.

We end this section by giving $[R, Q_\mu]$ for the particular case when $j = 1$, and considering two uses for these commutators. Their values are given by

$$[R, Q_0]L^2(2R+1)$$
$$= (1/\sqrt{2})Q_{+1}(2L^2+l_0) + Q_0(L^2+l_0^2) + (1/\sqrt{2})Q_{-1}(2L^2-l_0) \tag{6.3}$$

and

$$[R, Q_{\pm 1}]L^2(2R+1)$$
$$= \mp \tfrac{1}{2} Q_{\pm 1}(L^2(4l_0 \pm 1) \pm l_0(l_0 \pm 1)) - \tfrac{1}{2}Q_{\mp 1}(L^2 - l_0(l_0 \pm 1))$$
$$+ (1/\sqrt{2})Q_0(L^2(2L^2 - 2l_0^2 \mp l_0) \mp l_0^2(l_0 \pm 1)). \tag{6.4}$$

The first application of these formulas is in the direct verification of the hermiticity relation (4.10). They are

not needed to verify the hermiticity of $0°$ since R does not appear in its expression. However, using (2.1), (2.2), (3.33), and (4.5) in its expression, we have

$$(O^1)^\dagger = -(R - l_0 + 1)Q_{-1} + \sqrt{2}(R^2 - l_0^2 - 1)Q_0 - (R + l_0 + 1)Q_{+1}$$
$$= -Q_{+1}(R + l_0 + 1) + \sqrt{2}Q_0(R^2 - l_0^2 - 1) - Q_{-1}(R - l_0 + 1)$$
$$- [R, Q_{+1}] + \sqrt{2}[R^2, Q_0] - [R, Q_{-1}].$$

(6.3) and (6.4) must now be used in order to proceed further. A straight forward calculation, whose details we omit, does yield $(O^1)^\dagger(2R+1) = O^{-1}(2R-1)$, in agreement with (4.10).

As a final application of (6.3) and (6.4), we write down the commutators of R with the $T(1, \mu)$ themselves. $[R, T(1, 0)]$ can be obtained directly from (6.3), and then one uses

$$[R, T(1, \pm 1)] = (1/\sqrt{2})[l_\mp, [R, T(1, 0)]]$$

The results are

$$[R, T(1, 0)]L^2(2R+1) = (1/\sqrt{2})T(1, -1)l_+(2L^2 + l_0)$$
$$+ T(1, 0)(L^2 + l_0^2) + (1/\sqrt{2})T(1, 1)l_-(2L^2 - l_0) \tag{6.5}$$

and

$$[R, T(1, \pm 1)](2R+1)L^2 = \pm \tfrac{1}{2}T(1, \pm 1)(L^2(4l_0 \pm 3) \mp l_0(l_0 \pm 1))$$
$$+ (1/\sqrt{2})T(1, 0)l_\pm(2L^2 \mp l_0 - 1) - (1/\sqrt{2})T(1, \mp 1)l_\pm^2. \tag{6.6}$$

7. EVALUATION OF $[R, Q_{+\mu}^+]$ AND $[R, Q_{+\mu}^-]$

Here one finds an equation exactly the same as (6.1), but with $B_{l,m}^{k, \pm 1/2}$ replacing B_l^k. Using this and the orthogonality relation for the 3-j symbols, one obtains, for $\mu = \tfrac{1}{2}, \cdots, j$,

$$[R, Q_{\pm\mu}^+] = \sum_{\mu'=1/2}^{j} \left[\frac{(l \pm m + \mu + \tfrac{1}{2})!\,(l \mp m - \mu' - \tfrac{1}{2})!}{(l \mp m - \mu - \tfrac{1}{2})!\,(l \pm m + \mu' + \tfrac{1}{2})!} \right]^{1/2}$$
$$\times \left\{ \sum_{k=-j}^{j} k(2l+2k+1) \begin{pmatrix} j & l & l+k \\ \mp \mu & m \pm \mu \pm \tfrac{1}{2} & -m \mp \tfrac{1}{2} \end{pmatrix} \right.$$
$$\left. \times \begin{pmatrix} j & l & l+k \\ \mp \mu' & m \pm \mu' \pm \tfrac{1}{2} & -m \mp \tfrac{1}{2} \end{pmatrix} \right\} Q_{\pm\mu'}^+$$
$$+ \sum_{\mu'=1/2}^{j} \left[\frac{(l \pm m + \mu + \tfrac{1}{2})!\,(l \mp m + \mu' - \tfrac{1}{2})!}{(l \mp m - \mu - \tfrac{1}{2})!\,(l \pm m - \mu' + \tfrac{1}{2})!} \right]^{1/2}$$
$$\times \left\{ \sum_{k=-j}^{j} k(2l+2k+1) \begin{pmatrix} j & l & l+k \\ \mp \mu & m \pm \mu \pm \tfrac{1}{2} & -m \mp \tfrac{1}{2} \end{pmatrix} \right.$$
$$\left. \times \begin{pmatrix} j & l & l+k \\ \pm \mu' & m \mp \mu' \pm \tfrac{1}{2} & -m \mp \tfrac{1}{2} \end{pmatrix} \right\} Q_{\mp\mu'}^+ \tag{7.1}$$

and

$$[R, Q_{\mp}^\pm] = \sum_{\mu'=1/2}^{j} \left[\frac{(l \pm m - \mu + \tfrac{1}{2})!\,(l \mp m - \mu' - \tfrac{1}{2})!}{(l \mp m + \mu - \tfrac{1}{2})!\,(l \pm m + \mu' + \tfrac{1}{2})!} \right]^{1/2}$$
$$\times \left\{ \sum_{k=-j}^{j} k(2l+2k+1) \begin{pmatrix} j & l & l+k \\ \pm \mu & m \mp \mu \pm \tfrac{1}{2} & -m \mp \tfrac{1}{2} \end{pmatrix} \right.$$
$$\left. \times \begin{pmatrix} j & l & l+k \\ \mp \mu' & m \pm \mu' \pm \tfrac{1}{2} & -m \mp \tfrac{1}{2} \end{pmatrix} \right\} Q_{\pm\mu'}^\pm$$
$$+ \sum_{\mu'=1/2}^{j} \left[\frac{(l \pm m - \mu + \tfrac{1}{2})!\,(l \mp m + \mu' - \tfrac{1}{2})!}{(l \mp m + \mu - \tfrac{1}{2})!\,(l \pm m - \mu' + \tfrac{1}{2})!} \right]^{1/2}$$
$$\times \left\{ \sum_{k=-j}^{j} k(2l+2k+1) \begin{pmatrix} j & l & l+k \\ \pm \mu & m \mp \mu \pm \tfrac{1}{2} & -m \mp \tfrac{1}{2} \end{pmatrix} \right.$$

$$\times \begin{pmatrix} j & l & l+k \\ \pm\,\mu' & m \mp \mu' \pm \tfrac{1}{2} & -m \mp \tfrac{1}{2} \end{pmatrix} Q^{\pm}_{\mp\mu'}.$$

One may again show that the only l- or m-dependent SR-D part appearing in the expression for any of the above commutators is the term $(2l+2j)\cdots(2l-2j+2)$ in the denominator.

The results of specializing (7.1) and (7.2) to the case of $j=\tfrac{1}{2}$ are

$$[R,Q^{\pm}_{\pm 1/2}](2R+1) = \mp\,Q^{\pm}_{\pm 1/2}(2l_0\pm 1) + Q^{\pm}_{\mp 1/2}(R\mp l_0)$$
$$\times (R\pm l_0+1) \qquad (7.3)$$

and

$$[R,Q^{\pm}_{\mp 1/2}](2R+1) = Q^{\pm}_{\pm 1/2} \pm \tfrac{1}{2}Q^{\pm}_{\mp 1/2}(2l_0\pm 1), \qquad (7.4)$$

from which one easily obtains

$$[R,T(\tfrac{1}{2},\pm\tfrac{1}{2})](2R+1) = T(\tfrac{1}{2},\mp\tfrac{1}{2})l_{\pm} \pm \tfrac{1}{2}T(\tfrac{1}{2},\pm\tfrac{1}{2})(2l_0\pm 1).$$
$$(7.5)$$

Finally, from (3.35) and (3.36), we get

$$(\overline{O}^{+1/2,\pm 1/2})^{\dagger} = \mp\,l_{\mp}T(\tfrac{1}{2},\pm\tfrac{1}{2}) \pm (R\pm l_0+1)T(\tfrac{1}{2},\mp\tfrac{1}{2})$$

and

$$(\overline{O}^{-1/2,\pm 1/2})^{\dagger} = \mp\,l_{\mp}T(\tfrac{1}{2},\pm\tfrac{1}{2}) \mp (R\mp l_0)T(\tfrac{1}{2},\mp\tfrac{1}{2}),$$

which yield

$$O^{-1/2,\mp 1/2}(2R) = \pm (\overline{O}^{1/2,\pm 1/2})^{\dagger}(2R+1), \qquad (7.6)$$

$$O^{1/2,\pm 1/2}(2R+2) = \pm (\overline{O}^{-1/2,\pm 1/2})^{\dagger}(2R+1). \qquad (7.7)$$

These agree with the previously obtained expressions (5.4) and (5.5).

8. CONCLUSION

The detailed properties of the shift operators considered in this paper, although rather technical, have already proved extremely useful for the classification and analysis of the IUR of groups such as SU(3),[1,2,6] Sl(3,R),[4] and O(3)$\wedge (T_2 \times \overline{T}_2)$.[5] They are equally applicable to groups such as SU(2,1), a new group SU(2)$\wedge (T_2 \sim \overline{T}_2)$, which is the central extension of O(3)$\wedge (T_2 \times \overline{T}_2)$, and to the graded di-spin Lie algebra of Corwin, Ne'eeman, and Sternberg,[16-18] in all of which cases only the $j=\tfrac{1}{2}$ shift operators occur; the authors hope to consider some of these applications in later papers. Clearly they are applicable to any Lie algebra, graded or otherwise, which contains the O(3) or SU(2) algebra as a sub-Lie algebra, although for $j > 2$ the analysis would be bound to become very complicated. Shift operator techniques have many advantages over other methods of analyzing group representations. First they apply equally well whether or not the group is semisimple; also they do not depend on any specific realization of the IUR in terms of, for instance, a set of functions defined over some space, or based on creation and annihilation operators which are often too specialized to give an exhaustive classification.

Shift operator techniques have been developed by several other authors. Some of these,[7-9] as already mentioned, are just particular examples of our $O^{\pm}_l$ operators for $j=1$. For instance, the $O^{\pm 1}_l$ given in Eqs. (3.33) and (3.34) yield, on change of notation, precisely the third components of the vector operators O_1 and O_2 of Stone[7]; the other two components of Stone's operators, as well as shifting l, also shift m by ± 1.

The most important general classification of shift operators is due to Nagel and Moshinsky,[13] Pang and Hecht,[11] and Wong.[12] In all these cases the shift operators act on the Gel'fand—Zetlin basis states[19]; Nagel and Moshinsky derived shift operators for the canonical chain of groups U(n)$\supset$U($n-1$)$\supset\cdots\supset U(1)$, whereas Pang and Hecht[11] and Wong[12] obtained analogous set of operators for the chain O(n)$\supset$O($n-1$)$\supset\cdots\supset$O(2). These operators which, like the ones constructed in this paper, are contained in the enveloping algebra of the Lie algebras under consideration, act on the Gel'fand—Zetlin states in such a way that a basis state corresponding to the highest weight of an IUR of a subgroup contained in the IUR of the group is taken into a basis state corresponding to the highest weight of a lowered or raised IUR of that subgroup. They therefore also raise or lower the eigenvalues of the Casimirs of that subgroup, just as the ones developed here raise or lower the eigenvalues of the Casimir L^2 of O(3) or SU(2).

The difference between these operators and our O^k_l operators is best illustrated by considering the particular case of the chain O(4)$\supset$O(3). Lohe and Hurst[14] have written down the L^1_4 operator of Wong[12] explicitly, under the name of L_-, in Eq. (4.19) of their paper. L_- acts on the Gel'fand—Zetlin state

$$\left|\begin{matrix} m_1 & m_2 \\ l & \\ l & \end{matrix}\right\rangle \text{ to produce the state } \left|\begin{matrix} m_1 & m_2 \\ l-1 & \\ l-1 & \end{matrix}\right\rangle .$$

Thus L_- lowers both l and m by 1, as does our operator $O^{-1}_l l_-$, where O^{-1}_l is given in Eq. (3.34) of this paper. However, whereas $O^{-1}_l l_-$ lowers l and m by 1 whatever the m value of the state, L_- has this effect only when acting on the state for which $m=l$. For $m \neq l$, L_- does not act on the l value of the state in this clean manner. It could not, since it does not depend explicitly on the l value of the state, as do the O^k_l. One may, in fact, easily check that if in Eq. (3.34) for O^{-1}_l one postmultiplies by l_- and then replaces R by l_0 (which one may only do for the particular case when it acts on a state for which $m=l$), one obtains, apart from an over-all multiplicative constant, precisely the L_- operator of Lohe and Hurst.

Thus, more generally, the difference between the l-shifting operators acting on Gel'fand—Zetlin states and the ones developed in this paper is that, whereas the O^k_l operators shift l in a "clean" manner whatever the m values of the states, the former do so only when acting on states for which $m=l$.

Clearly it should be possible to extend the results of this paper to the construction of analogous shift operators for more general groups, such as O(n) and U(n). For instance, the IUR of O(n) can be uniquely labelled by a set of parameters $\lambda_1,\dots,\lambda_p$, where p is the integral part of $\tfrac{1}{2}n$, which take on either all integral or all half-integral values. Bracken and Green[20] show that an O(n) vector operator θ may be resolved into parts $\theta^{\pm}_1,\dots,\theta^{\pm}_p$

where, for $i = 1, \ldots, p$, $\theta_i^\pm$ shifts the λ_i value of a state upon which it acts by ± 1; when $n = 2p + 1$, θ also has a resolute θ^0 which leaves the values of all the λ_i unchanged. For the case of O(3), these resolutes are just the $j = 1$ operators $O_l^{\pm 1}$ and O_l^0 given in Eqs. (3.32)–(3.34). A true generalization of the present work to arbitrary O(n) would also generalize the results of Bracken and Green to the case where shift operators are constructed using, not just a vector operator, but an arbitrary tensor operator of O(n). One would then expect to obtain shift operators which change the values of the λ_i by amounts other than ± 1.

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On the polynomial expansion of the Hill–Wheeler integral for the rotational energies

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A transformation is used which enables one to carry out exactly the integration of the Hill–Wheeler integral for the rotational energies. The expression for the energies E_J is shown to be the ratio of two polynomials in $J(J+1)$, where J is the total angular momentum. A nice feature of the formulation is that the coefficients in the expansion involve only the matrix elements of the lowest rank independent HJ^{2r} and J^{2r} operators, $0 \leq r \leq N-1$, N being the number of J states contained in the given rotational band, and H being the Hamiltonian of the system.

1. INTRODUCTION

The Hill–Wheeler integral[1] has been very useful in generating a collective nuclear wavefunction. It was shown by Peierls and Yoccoz[2] how this integral can be used to calculate the energies of the collective rotational levels in nuclei. Using the intrinsically deformed Hartree–Fock wavefunctions, Ripka[3] had carried out an extensive series of calculations for the energy levels of $2s - 1d$ shell nuclei. These calculations were quite successful in reproducing the energy levels belonging to various bands. From the analytic point of view one would like to integrate over the Euler angles and obtain a closed form expression for the energies. In the original work of Peierls and Yoccoz,[2] they had expanded the rotation operator in a power series and by keeping the first two nonvanishing terms had derived an expression for the nuclear inertia parameter. Following this approach, Sharon[4] had derived explicit power-series expansions for the projected energies for any nucleus when it is strongly deformed. The purpose of the present work is to show that one can use a linear transformation of the rotation matrices to write this integral as a polynomial in $J(J+1)$, where J is the angular momentum of the rotational level. In Sec. 2 we describe this formulation. The concluding remarks are presented in Sec. 3.

2. FORMULATION

We write the energy E_J of a level in a rotational band as

$$E_J = \text{Num}_J/\text{Den}_J, \tag{1}$$

where Num_J is given by[2,3]

$$\text{Num}_J = \int_0^{\pi} d\beta \, \sin\beta P_J(\cos\beta)\langle\Phi \,|\, H \exp(-i\beta J_y)\,|\, \Phi\rangle, \tag{2}$$

and Den_J is given by the same expression as (2) with the Hamiltonian H replaced by the unit operator. In expression (2) $P_J(\cos\beta)$ is the Legendre polynomial and J_y is the y component of the total angular momentum. The wavefunction $|\Phi\rangle$ denotes an intrinsic wavefunction. The intrinsic wavefunction $|\Phi\rangle$ which we shall consider is made up of deformed single-particle wavefunctions generated either by Hartree–Fock calculations or by Nilsson's method[3] and the values of J contained in such a wavefunction range from some minimum value of J, $J_{\min}$ to some maximum value $J_{\max}$. For simplicity we

have taken the case in which the K-quantum number[3,5] is zero and therefore the various J states included in $|\Phi\rangle$ are $J = 0, 2, 4, \ldots, 2(N-1)$, N being the total number of J states contained in $|\Phi\rangle$. For such a $|\Phi\rangle$, $J_{\min} = 0$ and $J_{\max} = 2(N-1)$. Since our aim is to express Num_J, Den_J as a polynomial in $J(J+1)$, we can first try to see if we can use one of the known polynomial expansions for the Legendre polynomials $P_J(\cos\beta)$ for $(J = 2n)$ which are given by Erdélyi.[6] These expansions which are given in terms of hypergeometric functions do not seem to be suitable for our purpose, since (1) the coefficients of various powers of the trigonometric functions sin and cos are themselves combinations of various powers of $J(J+1)$ $[J = 0, 2, 4, \ldots, 2(N-1)]$ and thus we have to regroup them to pick out the coefficients for each power of $J(J+1)$, (2) each Legendre polynomial is represented by a polynomial of a different degree, and (3) for some representations one cannot work out the matrix element $\langle\Phi \,|\, H\exp(-i\beta J_y)\,|\, \Phi\rangle$ in a simple way.

It seems that the best way to achieve our goal is not to use any of these representations but to make the following linear transformation among the Legendre polynomials: We introduce a set of new functions $U_{2r}(\cos \beta)$ using the following transformation:

$$P_J(\cos \beta) = \sum_{r=0}^{N-1} [J(J+1)]^r \, U_{2r}(\cos \beta), \tag{3}$$

for $J = 0, 2, \ldots, 2(N-1)$. Writing $J = 2n$ ($n = 0, 1, \ldots, N-1$) we can also write (3) as $P_{2n} = \sum_r [2n(2n+1)]^r \, U_{2r}$. Since the determinant of the matrix M formed from the coefficients of expansion in (3) is nonzero, we can write U_{2r} in terms of P_J as

$$\bar{U} = M^{-1} \bar{P}, \tag{4}$$

where $\bar{U}$ denotes the column vector

$$U = \begin{pmatrix} U_0 \\ U_2 \\ \circ \\ \cdot \\ \cdot \\ \circ \\ U_{2(N-1)} \end{pmatrix}$$

and $\bar{P}$ denotes the column

$$\overline{P}=\begin{pmatrix} P_0 \\ P_2 \\ \circ \\ \circ \\ \circ \\ P_{2(N-1)} \end{pmatrix}$$

and the matrix elements of M are given by

$$M_{ij}=[2i(2i+1)]^j, \tag{5}$$

$i, j = 0, 1, \ldots, N-1$; with $M_{00}=1$.

Explicit expressions for some of the U's will be given in Sec. 3.

Since $|\Phi\rangle$ is a linear combination of the wavefunctions $|\Psi_J\rangle$, $J = 0, 2, \ldots, 2(N-1)$, we can write

$$|\Phi\rangle = \sum_{J=0,2}^{2(N-1)} c_J |\Psi_J\rangle, \tag{6}$$

with $\sum_J c_J^2 = 1$.

Let us now consider the evaluation of the matrix element $\langle \Phi | H \exp(-i\beta J_y) | \Phi\rangle$. Using the expansion (6) we get

$$\langle \Phi | H \exp(-i\beta J_y) | \Phi\rangle$$
$$= \sum_{J=0,2}^{2(N-1)} c_J^2 \langle \Psi_J | \exp(-i\beta J_y) | \Psi_J\rangle,$$

since H and $\exp(-i\beta J_y)$ cannot connect different J states. Further, since H is a diagonal in J representation, we can write

$$\langle \Phi | H \exp(-i\beta J_y) | \Phi\rangle$$
$$= \sum c_J^2 E_J \langle \Psi_J | \exp(-i\beta J_y) | \Psi\rangle,$$

where $E_J = \langle \Psi_J | H | \Psi_J\rangle$. Remembering that we are considering the case for which $K=0$ and $\langle \Psi_{J0} | \exp(-i\beta J_y) \times | \Psi_{J0}\rangle$ is just $P_J(\cos\beta)$ we get

$$\langle \Phi | H \exp(-i\beta J_y) | \Phi\rangle = \sum c_J^2 E_J P_J(\cos\beta).$$

Putting in the expansion of $P_J(\cos\beta)$ from (3) in the above expression and writing the double summation as

$$\langle \Phi | H \exp(-i\beta J_y) | \Phi\rangle$$
$$= \sum_{r=0}^{N-1} U_{2r}(\cos\beta) \sum_{J=0,2}^{2(N-1)} C_J^2 E_J [J(J+1)]^r$$

and expressing the summation over J as a matrix element of the operator HJ^{2r} between the intrinsic wavefunction $|\Phi\rangle$ we finally obtain the expression

$$\langle \Phi | H \exp(-i\beta J_y) | \Phi\rangle = \sum_{r=0}^{N-1} \langle | HJ^{2r} | \rangle U_{2r}(\cos\beta). \tag{7}$$

Putting expression (7) in expression (2) we get

$$\text{Num}_J = \sum_{r=0}^{N-1} \langle HJ^{2r}\rangle \int_0^\pi d\beta \sin\beta P_J(\cos\beta) U_{2r}(\cos\beta). \tag{8}$$

Using expression (3) we can rewrite Num_J as

$$\text{Num}_J = \sum_{s=0}^{N-1}\sum_{r=0}^{N-1} [J(J+1)]^s I_{sr} \langle HJ^{2r}\rangle, \tag{9}$$

where

$$I_{sr} = \int_0^\pi d\beta \sin\beta U_{2s}(\cos\beta) U_{2r}(\cos\beta). \tag{10}$$

Using expression (4) and the orthogonality relations

for the Legendre polynomials we get for I_{sr},

$$I_{sr} = \sum_{t=0}^{N-1} (M^{-1})_{st} \frac{2}{4t+1} (M^{-1})_{rt}. \tag{11}$$

Denoting by a_s the following coefficient:

$$a_s = \sum_{r=0}^{N-1} I_{sr} \langle HJ^{2r}\rangle, \tag{12}$$

we find the Num_J is given by the following polynomial:

$$\text{Num}_J = \sum_{s=0}^{N-1} a_s [J(J+1)]^s, \tag{13}$$

while Den_J is given by the polynomial

$$\text{Den}_J = \sum_{s=0}^{N-1} a'_s [J(J+1)]^s. \tag{14}$$

The coefficients a'_s are given by expression (12) with the operator H replaced by unit operator.

Therefore, we have shown that by carrying out the integrations exactly we can write the energies E_J as the ratio of two polynomials in $[J(J+1)]$.

3. CONCLUDING REMARKS

The first remark which we would like to make is that the coefficients a_s, and a'_s in the polynomials expansion involve only the matrix elements of the lowest rank independent operators HJ^{2r}, J^{2r}; $0 \le r \le N-1$. In numerical computations it is certainly advantageous to have expressions for E_J which involve the matrix elements of the lowest rank operators.

The second remark is that the energies E_J can also be written as a polynomial in $[J(J+1)]$. This can be easily done using the operator identity[7,8]

$$\prod_{i=0}^{N-1} [J^2 - (2i)(2i+1)] \equiv 0, \tag{15}$$

and writing

$$E_J = \frac{\text{Num}_J}{\text{Den}_J} = \sum_{v=0}^{N-1} b_v [J(J+1)]^v. \tag{16}$$

The coefficients b_v can be easily found in terms of a_s, a'_s using the identities (15), (16). To see this let us consider the expansion of the following simple function:

$$f(J^2) = (1+\lambda J^2)^{-1},$$

for the case where the intrinsic wavefunction $|\Phi\rangle$ has only two angular momentum states $J = 0$ and 2. For this case the projection operator relation (15) gives $J^4 = 6J^2$. If we now expand f in a power series using binomial theorem, then because of the relation $J^4 = 6J^2$, we can express all powers of J^{2n} higher than J^2 in terms of J^2 only. Thus

$$(1+\lambda J^2)^{-1} = 1 + gJ^2.$$

To determine g, multiply both sides by $(1+\lambda J^2)$ and use the relation $J^4 = 6J^2$, which gives $g = -\lambda(1+6\lambda)^{-1}$. Finally taking the matrix elements of the above polynomial expansion between good angular momentum states gives the desired result:

$$[1 + \lambda J(J+1)]^{-1} = 1 - \lambda(1+6\lambda)^{-1} J(J+1).$$

Because of our remark at the beginning of this section

the coefficients b_ν will also contain only the lowest independent operators HJ^{2r}, J^{2r}; $0 \leqslant r \leqslant N-1$. The polynomial expansion for E_J has also been obtained using different approaches[9,10] with the coefficients involving the matrix elements of various power of the operator $J_+ J_-$.

Finally as a check on our formulation we have calculated E_J's for ^{12}C nucleus[3] which has $K = 0$ and $J = 0, 2, 4$. The functions U_0, U_2, U_4 for this nucleus which can be worked out using expression (4) and (5) are given by

$$\begin{pmatrix} U_0 \\ U_2 \\ U_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{13}{60} & \frac{5}{21} & -\frac{3}{140} \\ \frac{1}{120} & -\frac{1}{84} & \frac{1}{280} \end{pmatrix} \begin{pmatrix} p_0 \\ p_2 \\ p_4 \end{pmatrix}. \tag{17}$$

Using expression (5), (11), (12), (13), and (14) we find that $E_{J=0^+}$ is given by

$$E_{J=0^+} = \frac{\langle H \rangle - \frac{13}{60}\langle HJ^2 \rangle + \frac{1}{120}\langle HJ^4 \rangle}{1 - \frac{13}{60}\langle J^2 \rangle + \frac{1}{120}\langle J^4 \rangle}. \tag{18}$$

The matrix elements HJ^{2r}, J^{2r}; $0 \leqslant r \leqslant 2$ can be easily calculated[11] and when substituted in expression (18) give the same value for $E_{J=0^+}$ as the one obtained by numerically integrating the integral[3] given by expression (2).

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Vector constants and their algebras for classical Hamiltonians

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The explicit form of a vector constant of the motion for an arbitrary relativistic spherically symmetric time independent classical Hamiltonian is obtained by showing that its construction is achieved on solving a linear second order ordinary differential equation. The solution of this equation is presented and the vector, in conjunction with the angular momentum to which it is normal, is used to generate the algebras of the Euclidean group E(3), the orthogonal rotation group O(4) and the special unitary group SU(3). The mass is assumed to be structureless and to move in an externally prescribed scalar potential field.

I. INTRODUCTION

The algebra of scalar functions of position in classical mechanical phase space[1] is an infinite dimensional Lie algebra if the abstract Lie bracket is realized in the form of the usual Poisson bracket. The resulting Lie algebra contains the set of all constants of the motion as a subalgebra of which the set $(0, h)$, h being the Hamiltonian, constitutes an ideal. The relationship of Lie algebras and Lie groups, which are appropriately realized by canonical transformations, to classical mechanics is treated at length by Sudarshan and Mukunda[2] who also examine the group theoretical and symmetry properties of particular dynamical systems. The relationship has also been discussed, for example, by Mukunda[3,4] and Fradkin.[5] The possession by a system of certain symmetries normally implies the existence of scalar constants of the motion. Similarly vector constants of the motion are intimately connected with dynamical symmetries. The existence of a vector constant of the motion, normal to the angular momentum, has been established using group theoretical methods by Mukunda[3] for spherically symmetric Hamiltonians which describe the motion of a point mass in three dimensions. Because the process of evaluating a Poisson bracket invokes first order differential operators it follows that the existence, and more importantly the determination, of vector constants might be amenable to analysis in a differential operator—as distinct from a group theoretical—formulation. The present purpose is to show that this expectation is met and thus to provide a straightforward procedure for determining explicitly the vector constant of a three-dimensional, single-mass spherically symmetric Hamiltonian. The point mass which is structureless, possessing zero spin and no dipole or higher multipole moments, is assumed to be moving under the influence of an externally specified scalar potential field. It is shown in fact that the determination of the vector can be reduced to finding the solution of a second-order ordinary differential equation thereby coincidently establishing its existence. The main result of this paper is the derivation of a simple explicit general expression for the vector constant.[6]

In addition, a method of utilizing this expression, in conjunction with the angular momentum, to generate the algebras of the three groups E(3), O(4), and SU(3)

is presented. The sole requirement in the analysis is that the Hamiltonian be spherically symmetric. The explicit functional form of the Hamiltonian enters only when the vector constant is being derived for a specific dynamical system. Accordingly, both relativistic and nonrelativistic systems are included in the following discussion.

II. DETERMINATION OF THE VECTOR CONSTANT

A time independent vector variable F is a constant of the motion if its Poisson bracket $[F, h]$ with the Hamiltonian is zero. To simplify the notation and in order to emphasize its operational nature the Poisson bracket of two functions $[f, g]$ will also occasionally be written in the form $\lfloor g \rfloor f$. The bracket is evaluated in a phase space of six dimensions, one for each of the cartesian components of the 3-position vector r and one for each of the components of the 3-momentum p. The dynamical scalar variables of the single mass system are functions of the magnitudes r and p and of the angle between r and p only. Consequently, a scalar dynamical function is denoted by $f = f(r, p, l)$ where $l = |r \times p|$ is the magnitude of the particle's angular momentum and the Poisson bracket is

$$[f, g] = [g]f = \frac{(r^2 p^2 - l^2)^{1/2}}{rp}(f_r g_p - f_p g_r), \tag{1}$$

where subscripts denote the partial derivative with respect to the corresponding variable. If the vector F is assumed to be orthogonal to the angular momentum, then its general form is

$$F = f(r, p, l)r + g(r, p, l)p, \tag{2}$$

where the coefficients f and g are arbitrary scalar variables. The condition that F be a constant of the motion is then

$$\frac{f}{p}h_p + [h]g = 0 \tag{3}$$

and

$$\frac{g}{r}h_r - [h]f = 0, \tag{4}$$

which constitute a parabolic system of partial differential equations.[7] In the derivation of Eq. (3) and (4) the spherical symmetry of the Hamiltonian has been assumed. If neither h_r nor h_p is zero these two equa-

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tions lead to the consistency condition

$$(rf + ug_r)(pg + uf_p) - f_r g_p u^2 = 0, \tag{5}$$

where the symbol $u = (r^2 p^2 - l^2)^{1/2}$ is introduced for conciseness.

On using Eq. (3) to eliminate the function f from Eq. (4) one finds that

$$rp[h]_r^2 g + rh_p([h]g)[h] \frac{p}{h_p} + h_r h_p g = 0. \tag{6}$$

Equation (6) is a linear second order partial differential equation which determines the function g once the Hamiltonian is prescribed. Eq. (3) subsequently determines the function f and hence the vector $\mathbf{F}$ is known. It is seen, on using Eq. (1), that the differential equation (6) contains derivatives with respect to r and p only: the angular momentum l appears merely as a parameter. An examination of the coefficients of its second order derivatives shows[7] that the equation is parabolic everywhere independently of the form of the Hamiltonian. Accordingly, it has just one characteristic along which, as is to be expected, the Hamiltonian remains constant. The equation is thus reduced to canonical form by introducing, in place of r and p, two new characteristic variables viz. the Hamiltonian $h(r, p)$ and any arbitrary function $\eta(r, p)$ which is independent of h. The canonical form of Eq. (6) is

$$u^2(h_p \eta_r - h_r \eta_p)^2 g_{\eta\eta} + \left\{ u^2(h_p^2 \eta_{rr} - 2h_r h_p \eta_{rp} + h_r^2 \eta_{pp}) \right.$$

$$- \left(r^2 p h_r - \frac{l^2}{r} h_p \right) h_p \eta_r - \left[\frac{l^2}{r} h_r h_p + u^2 \left(h_p h_{rr} - 2h_r h_{rp} \right. \right.$$

$$\left. \left. + \frac{h_{pp}}{h_p} h_r^2 \right) - r^2 p h_r^2 \right] \eta_p \right\} g_\eta + rp h_r h_p g = 0 \tag{7}$$

which is seen to be a degenerate form in the sense that it contains derivatives with respect to the variable η alone. It is, accordingly, a second order ordinary linear differential equation whose coefficients are functions of η, h, and l. The latter two quantities appear as constant parameters. This equation determines $g(\eta, h, l)$ which in turn determines $g(r, p, l)$ and $f(r, p, l)$ and thus the vector $\mathbf{F}$. Because the equation is second order it has two independent solutions. It is easily seen that one of these solutions determines $\mathbf{F}$ while the other determines a second vector constant which is tantamount to

$$\mathbf{G} = \mathbf{l} \times \mathbf{F} = - (uf + gp^2)\mathbf{r} + (ug + fr^2)\mathbf{p}. \tag{8}$$

III. CONSTRUCTION OF THE VECTOR

The motion of a particle with rest mass m subject to a 3-force derivable from a potential $\phi(r)$ is governed by the Hamiltonian[8]

$$h = (m^2 c^4 + c^2 p^2)^{1/2} + \phi, \tag{9}$$

in which, $\tilde{m}$ being the inertial mass,

$$h = \tilde{m} c^2 + \phi = E \tag{10}$$

is the total energy and consequently a constant. If one chooses the function $\eta(r, p) = r$, Eq. (7) takes the form

$$u^2 g_{rr} + (r^2 \psi_r + l^2/r) g_r - r\psi_r g = 0, \tag{11}$$

where

$$\psi = (E - \phi)^2 / 2c^2. \tag{12}$$

However

$$u = [(2\psi - m^2 c^2)r^2 - l^2]^{1/2}$$

and thus a solution of Eq. (11) is

$$g(r) = r [\exp i\theta(r)], \tag{13}$$

where

$$\theta(r) = \int \frac{l\,dr}{r [(2\psi - m^2 c^2)r^2 - l^2]^{1/2}} \tag{14}$$

and $\theta(r)$ is related to the angular position of the mass in its orbit. Equations (3) and (13) make it possible to express the function f in the form

$$f = -\frac{u}{r} g_r . \tag{15}$$

Therefore, on taking the imaginary part of Eq. (11), one obtains

$$\mathbf{F} = - (u \sin\theta + l \cos\theta)\frac{\mathbf{r}}{r} + r\mathbf{p} \sin\theta \tag{16}$$

$$= \frac{\mathbf{l} \times \mathbf{r}}{r} \sin\theta - \frac{l}{r}\mathbf{r} \cos\theta \tag{17}$$

and thus $\mathbf{\hat{\jmath}} = \mathbf{F}/l$ is a vector constant of unit magnitude.

These expressions generalize the Runge—Lenz vector to include all spherically symmetric relativistic Hamiltonians of the form (9). They include the non-relativistic case if, in Eq. (14), the quantity $(2\psi - m^2 c^2)$ is replaced by $2m(E - \phi)$. Some solutions of the nonrelativistic form of Eq. (7) for gravitational potentials have been presented elsewhere.[9] In particular it has been shown that the usual Runge—Lenz vector corresponds to the solution when the Hamiltonian is that of the nonrelativistic Kepler problem.

IV. GENERATORS OF E(3) AND O(4)

If the vector $\mathbf{F}$ is constructed from the solution pair (f, g) of Eq. (3) and (4) then $\mathbf{M} = R\mathbf{F}$ is also a vector constant determined by the solution pair (Rf, Rg), $R(r, p, l)$ being any scalar constant of the motion. A direct calculation utilizing the consistency condition (5) shows that

$$[M_i, M_j] = - \frac{1}{2l} \epsilon_{ijk} l_k \frac{\partial}{\partial l} (R^2 F^2) \tag{18}$$

and

$$[l_i, M_j] = \epsilon_{ijk} M_k, \tag{19}$$

in which ϵ_{ijk} represents the permutation tensor, the summation convention is implied, and the subscripts range over the values 1, 2, 3.

By choosing $R = S/l$, $S(r, p)$ being an arbitrary constant of the motion independent of l, it is seen that the components of the vector $\mathbf{M} = S\mathbf{\hat{\jmath}}/F$ and the angular momentum $\mathbf{l}$ satisfy the bracket relationships

$$[l_j, l_k] = \epsilon_{jkm} l_m, \tag{20}$$

$$[M_j, M_k] = 0, \tag{21}$$

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$$[l_j, M_k] = \epsilon_{jkm} M_m. \qquad (22)$$

Accordingly, M behaves like a vector under rotations and together with l generates the algebra of the Euclidean group E(3) in three dimensions. The two Casimir invariants are zero and unity. On the other hand if one chooses

$$R(r, p, l) = [S(r, p) - l^2]^{1/2}/l, \qquad (23)$$

$S(r, p)$ being again an arbitrary constant of the motion independent of l, e.g., the Hamiltonian itself, then the relation (21) is replaced by

$$[M_k, M_j] = \epsilon_{kjm} l_m. \qquad (24)$$

Equations (20), (22), and (24) show that this second choice of R produces a vector M which generates, together with l, the algebra of the orthogonal rotation group O(4) in a real four-dimensional space. The Casimir invariants in this case are zero and the function $S(r, p)$.

V. GENERATORS OF SU(3)

A representation of the special unitary group SU(3) can be constructed as follows from the components of the constant unit vectors $\mathcal{J} = F/l$ and $\mathcal{G} = 1 \times \mathcal{J}/l$ together with those of the angular momentum. The vectors F and $\mathcal{G}$ satisfy the bracket relationships

$$[\mathcal{J}_j, \mathcal{J}_k] = [\mathcal{G}_j, \mathcal{G}_k] = 0, \qquad (25)$$

$$[l, \mathcal{J}_j] = -\mathcal{G}_j, \quad [l, \mathcal{G}_j] = \mathcal{J}_j, \qquad (26)$$

$$[\mathcal{J}_j, \mathcal{G}_k] = [\mathcal{J}_k, \mathcal{G}_j] = (\delta_{jk} - \mathcal{J}_j \mathcal{J}_k - \mathcal{G}_j \mathcal{G}_k)/l, \qquad (27)$$

in which δ_{jk} is the Kronecker symbol. Accordingly, the second-order symmetric traceless tensor

$$T_{mn} = il(\mathcal{J}_m \mathcal{J}_n - \mathcal{G}_m \mathcal{G}_n), \quad i = \sqrt{-1}, \qquad (28)$$

establishes, with l, the closed algebra

$$[l_k, T_{mn}] = \epsilon_{kmr} T_{nr} + \epsilon_{knr} T_{mr}, \qquad (29)$$

$$[T_{mn}, T_{rs}] = (\delta_{mr} \epsilon_{nst} + \delta_{ms} \epsilon_{nrt} + \delta_{nr} \epsilon_{mst} + \delta_{ns} \epsilon_{mrt}) l_t. \qquad (30)$$

Thus the five independent components of the tensor T_{mn} together with the three components of l, which generate the group SO(3), form the eight generators of SU(3). In fact, by making the identification

$$I_\pm = \frac{i}{4}(T_{11} \mp 2i T_{12} - T_{22}),$$

$$I_3 = \frac{-i}{2} l_3, \quad Y = \frac{-i}{2}(T_{11} + T_{22}),$$

$$U_\pm = \frac{-i}{2\sqrt{2}}(l_1 \pm i l_2 + T_{13} \pm i T_{23}),$$

$$V_\pm = \frac{-i}{2\sqrt{2}}(l_1 \mp i l_2 - T_{13} \pm i T_{23}),$$

one can establish that

$$[I_3, I_\pm] = \pm I_\pm, \qquad [I_3, U_\pm] = \mp \tfrac{1}{2} U_\pm, \qquad [I_3, V_\pm] = \pm \tfrac{1}{2} V_\pm,$$

$$[I_+, V_+] = [I_+, U_-] = [U_+, V_+] = [Y, I_\pm] = [I_3, Y] = 0,$$

$$[Y, U_\pm] = \pm U_\pm, \qquad [Y, V_\pm] = \pm V_\pm, \qquad [I_+, I_-] = 2I_3,$$

$$[U_+, U_-] = \tfrac{3}{2} Y - I_3, \quad [V_+, V_-] = \tfrac{3}{2} Y + I_3,$$

$$[I_+, V_-] = -U_-, \qquad [I_+, U_+] = V_+, \qquad [U_+, V_-] = I_-,$$

which is a standard representation as presented, e.g., by Fonda and Ghirardi.[10]

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Relativity and deformed Lie groups

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The concept of a deformed Lie group in which the structure coefficients are functions of the group coordinates is defined. Every Lie group can be deformed outside any of its subgroups. The affine group in four dimensions deformed outside $Gl(4,\mathbb{R})$ has a variable group metric that is closely connected with the Ricci tensor of the four-dimensional manifold of translations. An analog of Einstein's vacuum equations expresses the invariance of the metric with respect to the deformation. An enlargement of the affine group leads naturally to the appearance of the energy–momentum terms in the equations, while the gravitational interaction constant plays a role of a fundamental group constant (like c in the Lorentz group) which makes all generators of the gauge group dimensionless.

INTRODUCTION

It is well known that the general relativity theory can be considered as a theory of connections in fiber bundles with Lorentz structure, where some kind of identification of the points in the bundle manifold with linear frames of the base manifold (space—time) must be provided (see, e.g., Ref. 1). In order to make this identification a natural part of a mathematical system, one can enlarge the Lorentz structure to include translations, the additional components of the connection form providing the identification (they are identified with the canonical form of the bundle of linear frames). Such an approach was investigated, e.g., in Refs. 2 and 3. A different approach to the same question is to consider the whole fiber bundle as a homogeneous mathematical system, namely a Lie group, where the identification is provided by the group action. For example, the properties of the Minkowski space are contained in the geometry of the Poincaré group, where the space—time is the subgroup of translations. It is the flat Minkowski space which is described here, because the generators of translations (defining the parallel displacement) commute with each other. The structure coefficients related to translations are zero, and so is the group metric. The questions discussed in the present article are as follows:

Is it possible to introduce a deformation into a Lie group in such a way that the geometry of the group based on a set of variable structure coefficients and the corresponding variable metric describes a space with non-zero curvature? What role is played by the group metric after deformation?

In Sec. 1 the concept of a deformed Lie group is defined. Sections 2 and 3 contain a discussion of the affine group in four dimensions. It is shown in Sec. 3 that the group metric plays the role of the contracted curvature tensor and Einstein's vacuum equations express the requirement of the constant metric in the group space. In Sec. 4 a discussion of possible generalizations designed to include matter fields is presented.

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1. DEFORMED LIE GROUPS

Let G be a Lie group and K a subgroup of G. Select a basis of the Lie algebra of G in such a way that X_a, $a = 1, \ldots, n$, span the Lie algebra $\mathbf{K}$ of K, while vectors X_i, $i = n+1, \ldots, N$, complete the basis. We shall use indices a,b,c,d,e,f for $\mathbf{K}$, and i,j,k,l,m,n for vectors outside $\mathbf{K}$. The elements of the Lie algebra are considered as vector fields (first order differential operators) in the group manifold. The structure coefficients are defined by

$$[X_a, \ X_b] = C_{ab}^c X_c, \quad [X_a, \ X_i] = C_{ai}^b X_b + C_{ai}^k X_k,$$

$$[X_i, \ X_j] = C_{ij}^a X_a + C_{ij}^k X_k, \quad C_{ab}^k = 0.$$

We shall say that G is deformed outside K if the following conditions are satisfied:

(a) There is a set of real valued differentiable functions B_i^a defined on the group manifold.

(b) Vectors X_a and $\tilde{X}_k = X_k + B_k^a X_a$ (summation over repeated upper and low indices) form a basis of the tangent vector space at every point of the group manifold.

(c) Commutation relations between X_a and $\tilde{X}_k$ are the same as in the undeformed group, i.e.,

$$[X_a, \ \tilde{X}_i] = C_{ai}^b X_b + C_{ai}^k \tilde{X}_k. \tag{1}$$

Equation (1) is a set of first order partial differential equations for B_i^a, and it should be shown that they are always integrable. Explicitly

$$[X_a, X_i + B_i^b X_b] = C_{ai}^b X_b + C_{ai}^k X_k + B_i^b C_{ab}^d X_d + (X_a B_i^b) X_b,$$

and also

$$[X_a, X_i + B_i^b X_b] = C_{ai}^b X_b + C_{ai}^k (X_k + B_k^b X_b),$$

where $(X_a B_i^b)$ denotes the application of the differential operator X_a on functions B_i^b.

Equation (1) thus states

$$(X_a B_i^b) = C_{ai}^k B_k^b - C_{ad}^b B_i^d. \tag{2}$$

This system of partial differential equations is consistent if

$$X_c(X_a B_i^b) - X_a(X_c B_i^b) = C_{ca}^d (X_d B_i^b) \tag{3}$$

is satisfied identically. Substitution of (2) into (3) yields

$$C_{ai}^k C_{ck}^l + C_{ic}^k C_{ak}^l + C_{ca}^d C_{id}^l = 0$$

and

$$C^b_{ad} C^d_{ce} + C^b_{cd} C^d_{ea} + C^b_{ed} C^d_{ac} = 0 .$$

Thus the Jacobi identity guarantees integrability of Eq. (2) and condition (c) can be satisfied for every Lie group and each of its subgroups.

2. THE AFFINE GROUP

We select the basis of the Lie algebra of $A(4,\mathbb{R})$ as follows:

L^k_i span the Lie algebra of $Gl(4,\mathbb{R})$, $i,k = 1,\ldots,4$, and satisfy the commutation relations

$$[L^k_i, L^l_j] = \delta^l_i L^k_j - \delta^k_j L^l_i .$$

T_j, $j = 1,\ldots,4$, span the Lie algebra of translations:

$$[T_i, T_j] = 0 , \quad [L^k_i, T_j] = -\delta^k_j T_i . \tag{4}$$

In the case of L^k_i a pair of indices ^k_i is replacing a single subscript a used in Sec. 1.

Consider a deformation of $A(4,\mathbb{R})$ outside $Gl(4,\mathbb{R})$. We introduce

$$\tilde{T}_i = T_i + B^j_{ik} L^k_j , \quad [L^i_k, \tilde{T}_j] = -\delta^i_j \tilde{T}_k . \tag{5}$$

Vectors T_i define coordinates x^i in the subgroup of translations T, in which $T_i = \partial/\partial x^i$. For the purpose of later connection of the scheme with relativity we introduce also a general coordinate system x^μ in T: $\partial/\partial x^\mu = d^i_\mu T_i$, where $d^i_\mu = \partial x^i/\partial x^\mu$.

Now consider vectors

$$\tilde{T}_\mu = d^i_\mu \tilde{T}_i = d^i_\mu T_i + B^k_{\mu l} L^l_k ,$$

where

$$B^k_{\mu l} = d^i_\mu B^k_{il} .$$

Calculate the commutator $[\tilde{T}_\mu, \tilde{T}_\nu]$ at an arbitrary point $x \in T$. Although evaluated on T, the dependence of T_i, L^l_k, and $B^k_{\mu l}$ on the coordinates of $Gl(4,\mathbb{R})$ must be taken into account (we are not projecting vectors $\tilde{T}_\mu$ and $\tilde{T}_\nu$ onto T). Thus, using (4) and (5), we obtain

$$[\tilde{T}_\mu, \tilde{T}_\nu] = (\partial_\mu B^i_{\nu k} - \partial_\nu B^i_{\mu k} + B^i_{\mu j} B^j_{\nu k} - B^i_{\nu j} B^j_{\mu k}) L^k_i \\ + (d^i_\mu B^k_{\nu j} - d^j_\nu B^k_{\mu j}) d^\sigma_k \tilde{T}_\sigma , \tag{6}$$

where d^σ_k is the inverse (matrix) to d^i_μ.

To get still closer to the formalism of the general relativity, we should rewrite (6) for an arbitrary cross section $T \circ a(x)$, obtained from T by an action of an x-dependent element $a(x) \in Gl(4,\mathbb{R})$. Let $a^k_i(x)$ be the 4×4 real matrix representing $a(x)$ [$\exp a^i_i(x) L^i_i = a(x)$], and rewrite $a^k_i(x)$ in a neighborhood of an arbitrary point $x_0 \in T$ as

$$a^k_i(x) = \overline{a}^k_l(x_0) b^l_i(x) , \tag{7}$$

where $\overline{a}^k_i(x_0)$ is a constant matrix, while $b^l_i(x_0) = \delta^l_i$. This helps to derive the correct transformation of $B^k_{\mu i}$ when T is changed to $T \circ a(x)$.

Consider first a special case $\overline{a}^k_i(x_0) = \delta^k_i$, and rewrite

$$\tilde{T}_\mu = d^i_\mu T_i + \frac{\partial b^l_k}{\partial x^\mu} L^k_l - \frac{\partial b^k_k}{\partial x^\mu} L^k_i + B^k_{\mu i} L^i_k .$$

Vector $d^i_\mu T_i + (\partial b^l_i / \partial x^\mu) L^k_i$ projects to the original

derivative ∂_μ in T, but it lies in $T \cdot a(x)$. Thus it takes the role of ∂_μ when the functions to be differentiated are evaluated on $T \cdot a(x)$ instead of T. The transformation of $B^k_{\mu l}$ is then

$$B^k_{\mu l} \to B^k_{\mu l} - \partial_\mu b^k_l . \tag{8}$$

On the other hand, under the action of a constant element of $Gl(4,\mathbb{R})$ functions $B^k_{\mu l}$ transform as

$$B^k_{\mu l} \to \overline{a}^k_i B^i_{\mu j} \overline{a}^{-1j}_l . \tag{9}$$

In fact, when $\overline{a}^k_i$ are chosen as coordinates in $Gl(4,\mathbb{R})$, Eq. (9) induces the dependence of $B^k_{\mu l}$ on $\overline{a}^k_i$ which complies with the requirement (5).

Combining the two transformations (8) and (9) into one general transformation of type (7), one can write

$$B^k_{\mu l} \to a^k_i(x) B^i_{\mu j} a^{-1j}_l(x) - [\partial_\mu a^k_i(x)] a^{-1j}_l(x) . \tag{10}$$

This is in fact a gauge transformation of the "gauge field potentials" $B^i_{\mu j}$. When rewriting Eq. (6) for an arbitrary cross section, it is necessary to keep in mind that the direction of ∂_μ projected to T is not changed, while vectors T_i transform under $ad\ a^{-1}(x)$. Thus d^i_μ is transformed as

$$d^i_\mu \to h^i_\mu(x) \equiv a^i_l(x) d^l_\mu . \tag{11}$$

Substituting (10) and (11) into (6), we obtain

$$[\tilde{T}_\mu, \tilde{T}_\nu] = (\partial_\mu B^i_{\nu k} - \partial_\nu B^i_{\mu k} + B^i_{\mu j} B^j_{\nu k} - B^i_{\nu j} B^j_{\mu k}) L^k_i \\ - (\partial_\mu h^k_\nu - \partial_\nu h^k_\mu + h^j_\nu B^k_{\mu j} - h^j_\mu B^k_{\nu j}) h^\sigma_k \tilde{T}_\sigma , \tag{12}$$

where the effect of a constant transformation of type (9) was included in the dependence of $B^i_{\nu k}$ on the $Gl(4,\mathbb{R})$ coordinates. A general cross section is characterized by functions $h^i_\mu(x)$.

We shall write down the result in the form of the variable structure coefficients:

$$C^i_{\mu\nu} = \partial_\mu B^i_{\nu k} - \partial_\nu B^i_{\mu k} + B^i_{\mu j} B^j_{\nu k} - B^i_{\nu j} B^j_{\mu k} , \tag{13}$$

$$C^\sigma_{\mu\nu} = -(\partial_\mu h^k_\nu - \partial_\nu h^k_\mu + h^j_\nu B^k_{\mu j} - h^j_\mu B^k_{\nu j}) h^\sigma_k . \tag{14}$$

The remaining structure coefficients are easy to evaluate. Requirement (5) implies that

$$[L^i_k, \tilde{T}_\mu] = -h^i_\mu h^\sigma_k \tilde{T}_\sigma ;$$

hence

$$C^\sigma_{k\mu} = -h^i_\mu h^\sigma_k . \tag{15}$$

The structure constants of $Gl(4,\mathbb{R})$ (i.e., $C^{j}_{i\ l}$) are, of course, unchanged.

3. GROUP METRIC AND EINSTEIN'S EQUATIONS

The geometry of the deformed affine group is identical to the geometry of a manifold with an affine connection, once the appropriate identification of the functions is made. Namely, $h^i_\mu(x)$ are the tetrads, connecting arbitrarily chosen linear frames with those determined by a general coordinate system, while $B^i_{\mu k}$ evaluated on a chosen cross section are the components of the connection with respect to a mixed basis. The quantity

$$\Gamma^\sigma_{\mu\nu} = h^\sigma_k(\partial_\mu h^k_\nu + B^k_{\mu j} h^j_\nu) \tag{16}$$

(independent of the choice of the cross section) is interpreted as the usual Christoffel symbols. Equation

(16) expresses the fact that $h_\nu^k(x)$ has a zero covariant derivative if considered as a second order tensor in a mixed basis:

$$\partial_\mu h_\nu^k - \Gamma^\sigma_{\mu\nu} h_\sigma^k + B^k_{\mu i} h_\nu^i = 0 .$$

Then $C^k_{\ i\mu\nu} = R^i_{\ k\mu\nu}$ is the curvature tensor in the mixed basis, and $C^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\mu\ \nu} = -t^\sigma_{\mu\nu}$ is the torsion.

Let us find the components of the group metric from the structure coefficients specified above.

$$G_{\mu\nu} = C^k_{\ i\mu\sigma} C^\sigma_{\nu k} + C^\sigma_{\ i\mu k} C^k_{\nu\sigma} + C^\rho_{\mu\sigma} C^\sigma_{\nu\rho}$$

$$= R^i_{\ k\mu\sigma} h_\nu^k h_i^\sigma + R^i_{\ k\nu\sigma} h_\mu^k h_i^\sigma + t^\sigma_{\mu\sigma} t^\sigma_{\nu\rho} = 2R_{\mu\nu} + t^\sigma_{\mu\sigma} t^\sigma_{\nu\rho} .$$

$R_{\mu\nu}$ is the symmetric contracted curvature tensor (Ricci tensor). Further

$$G_{\mu\ i} = C^\rho_{\mu\sigma} C^\sigma_{i\ \rho} = t^\rho_{\mu\sigma} h_\rho^k h_i^\sigma ,$$

while the remaining components of the group metric (i.e., G_{ik}) are the same as in the undeformed group.

If we want to consider only the class of deformations which do not change the metric of the group at all we put

$$G_{\mu\nu} = 0 \quad \text{and} \quad G_{\mu\ i} = 0 \tag{17}$$

while the equations for G_{ik} are satisfied identically.

Equations (17) reduce to

$$R_{\mu\nu} = 0 \quad \text{and} \quad t^\sigma_{\mu\nu} = 0 , \tag{18}$$

which have the form of Einstein's vacuum equations, except that they concern a general affine connection.

Decomposing the "deformation term" $B^i_{\mu k} L^k_i$ into the Lorentz part and the rest, namely

$$B^i_{\mu k} L^k_i = \tfrac{1}{2} B^{ik}_\mu L_{ik} + \tfrac{1}{2} D^{ik}_\mu K_{ik} ,$$

where

$$L_{ik} = g_{il} L^l_k - g_{kl} L^l_i ,$$
$$K_{ik} = g_{il} L^l_k + g_{kl} L^l_i ,$$

and

$$B^{ik}_\mu = \tfrac{1}{2}(g^{il} B^k_{\mu l} - g^{kl} B^i_{\mu l}) ,$$
$$D^{ik}_\mu = \tfrac{1}{2}(g^{il} B^k_{\mu l} + g^{kl} B^i_{\mu l}) ,$$

we get the conventional Einstein's theory in a special case, when functions D^{ik}_μ can be made identically equal to zero by a selection of a particular cross-section. This happens when functions D^{ik}_μ on T are of the form $D^{ik}_\mu = \partial_\mu \phi^{ik}$. Selection of the gauge that annihilates D^{ik}_μ can be made only up to an arbitrary Lorentz gauge transformation; hence it is characterized by a total of $16 - 6 = 10$ functions of space—time coordinates, in accordance with the number of functions $\phi^{ik} = \phi^{ki}$. Working with the particular class of cross sections makes possible an introduction of the space—time metric

$$g_{\mu\nu}(x) = h^i_\mu(x) h^k_\nu(x) g_{ik} .$$

Let us note at this point that should we begin with a deformed Poincaré group instead of $A(4,\mathbb{R})$; a general deformation outside the Lorentz group would lead at

most to a nonzero torsion, but the space—time metric would be necessarily of Minkowski type. It is thus essential in this approach to begin with a larger group like $A(4,\mathbb{R})$, which in turn brings in a larger gauge invariance like $Gl(4,\mathbb{R})$.

4. INCLUSION OF MATTER FIELDS

The appearance of only $R_{\mu\nu}$ and $t^\sigma_{\mu\nu}$ in the expression for $G_{\mu\nu}$ is closely connected with the character of the commutation relations (4), namely with the fact that the subspace $\langle T_1, \ldots, T_4 \rangle$ is left invariant by the action of the Lie algebra of $Gl(4,\mathbb{R})$. In this section we shall briefly discuss an example in which such a condition is not satisfied.

Consider the group $Gl(5,\mathbb{R})$. This 25-dimensional group contains $A(4,\mathbb{R})$ as a subgroup, e.g., by the following identification.

Let L^a_b span the Lie algebra of $Gl(5,\mathbb{R})$ and have the commutation relations just as in $Gl(4,\mathbb{R})$ except that $a, b = 1, \ldots, 5$. Consider the full set of generators as L^k_i, L^k_5, L^5_i, and L^5_5, where $i, k = 1, \ldots, 4$.

We have

$$[L^k_i, L^5_j] = -\delta^k_j L^5_i , \quad [L^5_j, L^5_k] = 0 ;$$

hence the generators L^5_j can be identified with the translation in $A(4,\mathbb{R})$. Further, the remaining vectors L^k_i, L^k_5, and L^5_5 generate a subgroup, so that we can consider a deformation of $Gl(5,\mathbb{R})$ outside of this subgroup. We have

$$[L^k_5, T_i] = L^k_i - \delta^k_i L^5_5 , \tag{19}$$

and let the deformation be described by

$$\tilde{T}_i = T_i + B^j_{ik} L^k_j + B^5_{ik} L^k_5$$

or

$$\tilde{T}_\mu = d^i_\mu T_i + B^j_{\mu k} L^k_j + B^5_{\mu k} L^k_5 ,$$

where we assumed for simplicity that there is no deformation in L^5_5 direction.

The procedure analogous to that of Sec. 2 yields

$$[\tilde{T}_\mu, \tilde{T}_\nu]$$
$$= R^i_{k\mu\nu} L^k_i + (h^i_\mu B^5_{\nu k} - h^i_\nu B^5_{\mu k}) L^k_i - (h^k_\mu B^5_{\nu k} - h^k_\nu B^5_{\mu k}) L^5_5$$
$$+ (\partial_\nu B^5_{\mu k} - \partial_\mu B^5_{\nu k} + B^5_{\nu l} B^l_{\mu k} - B^5_{\mu l} B^l_{\nu k}) L^k_5 - t^\sigma_{\mu\nu} \tilde{T}_\sigma . \tag{20}$$

Calculating $G_{\mu\nu}$, we obtain

$$G_{\mu\nu} = 2R_{\mu\nu} - 3(B^5_{\mu k} h^k_\nu + B^5_{\nu k} h^k_\mu) + t^\rho_{\mu\sigma} t^\sigma_{\nu\rho} . \tag{21}$$

We see that the new potentials $B^5_{\mu k}$ lead to an extra term that can be connected with the energy—momentum tensor of matter fields.

An interesting point follows from a dimension analysis. In the system of units in which only the length remains (say $h = c = 1$), T_i have dimension [length]$^{-1}$, once interpreted as translations. L^k_i are dimensionless, $B^i_{\mu k}$, again l^{-1}. From (19) it follows that L^k_5 is of dimension l, which leaves l^{-2} for $B^5_{\mu k}$. We may try to express the theory in a form in which only the translations retain the dimension l^{-1} (as physically measurable quantities), while the generators of the whole gauge group are dimensionless. This can be achieved by introducing

a universal constant κ of dimension l^{-1} and redefining the generators

$$\kappa L_5^k = L_5^{\cdot k} ,$$

where $L_5^{\cdot k}$ are dimensionless. The commutation relations (19) change to

$$[L_5^{\cdot \kappa}, \ T_i] = \kappa L_i^k - \delta_i^k \kappa L_5^5 \tag{22}$$

and the expression for $\tilde{T}_\mu$ can be written as

$$\tilde{T}_\mu = d_\mu^i T_i + B_{\mu k}^j L_j^k + B_{\mu k}^{\cdot 5} L_5^{\cdot k} ,$$

where $B_{\mu k}^{\cdot 5} = \kappa^{-1} B_{\mu k}^5$. The "energy—momentum" term in Eq. (21) is then expressed as

$$\kappa \cdot 3(B_{\mu k}^{\cdot 5} h_\nu^k + B_{\nu k}^{\cdot 5} h_\mu^k) .$$

Constant κ can be interpreted as the gravitational interaction constant. It enters the theory as a fundamental constant in the commutation relations defining the group, just like speed of light enters within the Lorentz group. Contracting our group in the Wigner's sense[4] by $\kappa \rightarrow 0$ (like contraction of the Lorentz group to the Galileo group by $1/c \rightarrow 0$) leads to uncoupling the translations from the extra gauge transformations $L_5^{\cdot k}$ [see Eq. (22)], and, physically, uncoupling of the matter fields and the geometry.

The correlation between the appearance of the energy—momentum term and the fundamental constant κ is not limited to the above example only. Anytime the group is enlarged in such a way that commutators of translations and some other generators lie at least in part in Gl(4,$\mathbb{R}$), the energy—momentum term appears, and, at the same time, a constant κ is needed to make the whole gauge group dimensionless.

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Para-Bose coherent states

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In place of the usual commutation relation $[\hat{a}, \hat{a}^{\dagger}] = 1$ we consider the generalized commutation relation characteristic of the para-Bose oscillators, viz., $[\hat{a}, \hat{H}] = \hat{a}$ where $\hat{H}$ is the Hamiltonian $(1/2)(\hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a})$. The number states and the representation of various operators in the basis formed by these states are obtained. We then introduce the para-Bose coherent states defined as the eigenstates of $\hat{a}$ for this generalized case. We consider some of the properties of these coherent states and also show that the uncertainty product $\langle(\Delta\hat{q})^2\rangle \langle(\Delta\hat{p})^2\rangle$ in this case could be made arbitrarily small.

1. INTRODUCTION

The classical Hamiltonian of a harmonic oscillator of mass m and angular frequency ω is given by

$$H' = (1/2m)p'^2 + \tfrac{1}{2}m\omega^2 q'^2, \tag{1.1}$$

with the corresponding equations of motion

$$\dot{q}' = p'/m \quad \text{and} \quad \dot{p}' = -m\omega^2 q'. \tag{1.2}$$

The passage to quantum theory is made in two steps as follows[1]:

(1) We replace in H' the c-number variables q' and p' of the classical theory by the operators $\hat{q}'$ and $\hat{p}'$ respectively. It is being assumed that the operators $\hat{q}'$ and $\hat{p}'$ are Hermitian and that they operate on a Hilbert space with positive definite metric.

(2) We postulate the commutation relation

$$[\hat{q}', \hat{p}'] = i\hbar. \tag{1.3}$$

For the sake of simplicity, we shall be using in place of the quantities q', p', and H' the dimensionless quantities

$$q = (m\omega/\hbar)^{1/2} q', \tag{1.4a}$$

$$p = (m\omega\hbar)^{-1/2} p', \tag{1.4b}$$

$$H = (\hbar\omega)^{-1} H'. \tag{1.4c}$$

We also introduce the quantities α and α^* defined as

$$\alpha = (q + ip)/\sqrt{2}, \quad \alpha^* = (q - ip)/\sqrt{2}. \tag{1.5}$$

The operators corresponding to H, q, p, α, α^* in quantum theory will be denoted as $\hat{H}$, $\hat{q}$, $\hat{p}$, $\hat{a}$, $\hat{a}^{\dagger}$, respectively. The operator $\hat{a}^{\dagger}$ is the Hermitian adjoint of $\hat{a}$. Equations (1.1)–(1.3) then simplify to

$$H = \tfrac{1}{2}(q^2 + p^2) = \alpha^*\alpha, \tag{1.6}$$

$$\dot{q} = p, \quad \dot{p} = -q, \tag{1.7a}$$

or equivalently

$$\dot{\alpha} = -i\alpha \tag{1.7b}$$

and

$$[\hat{q}, \hat{p}] = i, \tag{1.8a}$$

$$[\hat{a}, \hat{a}^{\dagger}] = 1. \tag{1.8b}$$

It may be argued that both of the steps mentioned above for passage to quantum theory are not completely unique. Firstly, since $\hat{q}$ and $\hat{p}$ do not commute with each other, care must be taken in replacing q and p by the corresponding operators $\hat{q}$ and $\hat{p}$, respectively. In fact, depending on different rules of association,[2-4] one may obtain different expressions for $\hat{H}$. Hence for definiteness, we assume Weyl's rule[2] in obtaining the quantum expression for the Hamiltonian, i.e., we write

$$\hat{H} = \tfrac{1}{2}(\hat{q}^2 + \hat{p}^2) \tag{1.9}$$

$$= \tfrac{1}{2}(\hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger}). \tag{1.10}$$

Secondly, the commutation relation

$$[\hat{a}, \hat{a}^{\dagger}] = 1 \tag{1.11}$$

is *not* a consequence of the equations of motion. In fact, if one is only interested in recovering the equations of motion (1.7) for the operators in quantum theory, one must postulate the more general commutation relation

$$[\hat{a}, \hat{H}] = \hat{a}. \tag{1.12}$$

It is readily seen that (1.12) follows from (1.11) but the reverse is in general not true.

The case when the particle operators satisfy the more general commutation relation (1.12) has been referred to as the "para-Bose" case.[5-7] Jordan, Mukunda, and Pepper[5] have obtained the "Fock" representation for the para-Bose operators, i.e., they obtained the eigenvalues, and eigenfunctions of the operator $\hat{H}$ and the representation of the other operators in the basis formed by these eigenstates (see also Ref. 8).

In the present paper we are interested in obtaining the "coherent state" representation of these operators, and discuss some of the properties of these states. In analogy with the usual states,[9,10] we define the para-Bose coherent states as the eigenstates of $\hat{a}$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{1.13}$$

where $\hat{a}$ satisfies (1.12) with $\hat{H}$ given by Eq. (1.10).

In Sec. 2, we give a new derivation for the Fock representation for para-Bose operators. In Sec. 3 we obtain the para-Bose coherent states and discuss some of their properties such as completeness and diagonal coherent state representation in Sec. 4. In Sec. 5 we obtain the uncertainties in the position and momentum variables for the coherent states and observe that in special cases the product of the uncertainties $\langle(\Delta \hat{q})^2\rangle \times \langle(\Delta \hat{p})^2\rangle$ could be made as small as one likes.

2. PARA-BOSE NUMBER STATES

We start with the basic commutation relation (1.12),

$$[\hat{a}, \hat{H}] = \hat{a}, \tag{2.1}$$

where

$$\hat{H} = \tfrac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger). \tag{2.2}$$

From the fact that $\hat{a}^\dagger$ is the Hermitian adjoint of $\hat{a}$ and using the commutation relation (2.1) we readily find that

$$[\hat{a}^\dagger, \hat{H}] = -\hat{a}^\dagger, \tag{2.3}$$

$$[\hat{a}^2, \hat{a}^\dagger] = 2\hat{a}, \tag{2.3a}$$

$$[\hat{a}^{\dagger 2}, \hat{a}] = -2\hat{a}^\dagger. \tag{2.3b}$$

From induction it then follows that

$$[\hat{a}^{2n}, \hat{a}^\dagger] = 2n\hat{a}^{2n-1}, \tag{2.4a}$$

$$[\hat{a}^{\dagger 2n}, \hat{a}] = -2n\hat{a}^{\dagger 2n-1}, \tag{2.4b}$$

whereas

$$[\hat{a}^{2n+1}, \hat{a}^\dagger] = \{2n + [\hat{a}, \hat{a}^\dagger]\}\hat{a}^{2n}, \tag{2.5a}$$

$$[\hat{a}^{\dagger 2n+1}, \hat{a}] = -\hat{a}^{\dagger 2n}\{2n + [\hat{a}, \hat{a}^\dagger]\}. \tag{2.5b}$$

The commutator $[\hat{a}, \hat{a}^\dagger]$ commutes with $\hat{a}^2, \hat{a}^{\dagger 2}$, and $\hat{H}$ but not necessarily with $\hat{a}$ or $\hat{a}^\dagger$.

Starting in a strictly analogous manner as in the case of an ordinary harmonic oscillator we find that the energy eigenvalues differ by integers:

$$h_0, h_0 + 1, \ldots, h_0 + n, \cdots,$$

where h_0 is the lowest eigenvalue of $\hat{H}$. In the present case we take h_0 to be completely arbitrary except for the fact that it has to be nonnegative, since $\hat{H}$ itself is a nonnegative definite operator. We thus label the representation by a parameter h_0, the ground state eigenvalue of $\hat{H}$. The ordinary harmonic oscillator case is obtained when we take $h_0 = \tfrac{1}{2}$.

We now introduce the number operator

$$\hat{N} = \hat{H} - h_0 = \tfrac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger) - h_0, \tag{2.6}$$

and the number states $|n\rangle_{h_0}$ defined by

$$\hat{N}|n\rangle_{h_0} = n|n\rangle_{h_0}, \quad n = 0, 1, 2, \cdots. \tag{2.7}$$

Obviously $|n\rangle_{h_0}$ is also the eigenstate of $\hat{H}$ with the eigenvalue $n + h_0$. Because of the relations

$$[\hat{N}, \hat{a}] = -\hat{a}, \tag{2.8a}$$

$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger, \tag{2.8b}$$

which follow from (2.1) and (2.6), we may interpret $\hat{a}$ and $\hat{a}^\dagger$ as annihilation and creation operators, respectively. In order to obtain the representation for $\hat{a}$ and

$\hat{a}^\dagger$, we write

$$\hat{a}|n\rangle_{h_0} = \lambda_n|n - 1\rangle_{h_0}, \tag{2.9}$$

$$\hat{a}^\dagger|n\rangle_{h_0} = \mu_n|n + 1\rangle_{h_0}, \tag{2.10}$$

where λ_n and μ_n are some constants to be determined. From the Hermiticity requirement $_{h_0}\langle n - 1|a|n\rangle_{h_0} = {}_{h_0}\langle n|a^\dagger|n - 1\rangle_{h_0}^*$ it follows that

$$\lambda_n = \mu_{n-1}^*, \tag{2.11}$$

Further on taking the norms of Eqs. (2.9), (2.10) and adding we obtain

$$|\lambda_n|^2 + |\mu_n|^2 = 2(n + h_0). \tag{2.12}$$

Equations (2.11) and (2.12) then determine λ_n and μ_n apart from the phase factors. For definiteness, we take these constants to be real. We then find that (cf. Refs. 5, 8),

$$\lambda_{2n} = (2n)^{1/2}, \tag{2.13a}$$

$$\lambda_{2n+1} = \{2(n + h_0)\}^{1/2}, \tag{2.13b}$$

$$\mu_n = \lambda_{n+1}, \tag{2.14}$$

$n = 0, 1, 2, \cdots$.

The Para-Bose number states thus satisfy the following properties:

$$\hat{N}|n\rangle_{h_0} = n|n\rangle_{h_0}, \tag{2.15}$$

$$\hat{a}|2n\rangle_{h_0} = (2n)^{1/2}|2n - 1\rangle_{h_0}, \tag{2.16a}$$

$$\hat{a}|2n + 1\rangle_{h_0} = \{2(n + h_0)\}^{1/2}|2n\rangle_{h_0}, \tag{2.16b}$$

$$\hat{a}^\dagger|2n\rangle_{h_0} = \{2(n + h_0)\}^{1/2}|2n + 1\rangle_{h_0}, \tag{2.17a}$$

$$a^\dagger|2n + 1\rangle_{h_0} = \{2n + 2\}^{1/2}|2n + 2\rangle_{h_0}, \tag{2.17b}$$

$$[\hat{a}, \hat{a}^\dagger]|2n\rangle_{h_0} = 2h_0|2n\rangle_{h_0}, \tag{2.18a}$$

$$[\hat{a}, \hat{a}^\dagger]|2n + 1\rangle_{h_0} = 2(1 - h_0)|2n + 1\rangle_{h_0}. \tag{2.18b}$$

Further, we have the completeness relation

$$\sum_{n=0}^{\infty}|n\rangle_{h_0} {}_{h_0}\langle n| = 1 \tag{2.19a}$$

and the orthogonality relation

$$_{h_0}\langle n|m\rangle_{h_0} = \delta_{nm}. \tag{2.19b}$$

Relations (2.17) and (2.18) also give

$$|n\rangle_{h_0} = \left\{\frac{\Gamma(h_0)}{2^n \Gamma([n/2] + 1)\Gamma([(n+1)/2] + h_0)}\right\}^{1/2}\hat{a}^{\dagger n}|0\rangle_{h_0},$$
$$h_0 \neq 0, \tag{2.20}$$

where $[K]$ stands for the largest integer smaller than or equal to K.

It has been mentioned earlier that the constant h_0 is an arbitrary nonnegative number. When $h_0 = \tfrac{1}{2}$, we recover the familiar case of the ordinary oscillator in which case [cf. Eqs. (2.18)] the commutator $[\hat{a}, \hat{a}^\dagger]$ becomes unity. Relation (2.20) is not valid for the case when $h_0 = 0$. In this case

$$\hat{a}^\dagger|0\rangle_0 = 0, \tag{2.21}$$

(and also $\hat{a}|1\rangle_0 = 0$). We therefore find that $|0\rangle_0$ is an isolated state with no possible interaction with any of the other states. For all practical purposes $|1\rangle_0$ is then the ground state. In fact this situation is identical to the

case when $h_0 = 1$, so that we may associate the state $|n\rangle_0$ with the state $|n-1\rangle_1$. Hence without any loss of generality the case $h_0 \equiv 0$ may be ignored.

3. PARA-BOSE COHERENT STATES

The para-Bose coherent states are defined as the eigenstates of the annihilation operator $\hat{a}$,

$$\hat{a}|\alpha\rangle_{h_0} = \alpha|\alpha\rangle_{h_0}. \tag{3.1}$$

The matrix representation of $\hat{a}$ in the para-Bose number states has been obtained in the previous section. We now expand $|\alpha\rangle_{h_0}$ in terms of these number states

$$|\alpha\rangle_{h_0} = \sum_{n=0}^{\infty} C_n |n\rangle_{h_0} \tag{3.2}$$

and use (2.16). From (3.1) we then find the recurrence relations

$$C_{2n} = (\alpha^2/2)\{n(h_0 + n - 1)\}^{1/2} C_{2n-2}, \tag{3.3}$$

$$C_{2n+1} = \alpha\{2(n + h_0)\}^{-1/2} C_{2n}. \tag{3.4}$$

From these relations, we readily obtain

$$C_{2n} = \left\{\frac{\Gamma(h_0)}{n!\,\Gamma(n + h_0)}\right\}^{1/2} \left(\frac{\alpha}{\sqrt{2}}\right)^{2n} C_0 \tag{3.5}$$

and

$$C_{2n+1} = \left\{\frac{\Gamma(h_0)}{n!\,\Gamma(h_0 + n + 1)}\right\}^{1/2} \left(\frac{\alpha}{\sqrt{2}}\right)^{2n+1} C_0. \tag{3.6}$$

Equations (3.5) and (3.6) may equivalently be written in the form

$$C_n = \left\{\frac{\Gamma(h_0)}{2^n \Gamma([n/2] + 1)\Gamma([(n+1)/2] + h_0)}\right\}^{1/2} \alpha^n C_0, \tag{3.7}$$

where $[K]$ as before stands for the largest integer smaller than or equal to K. We now require the coherent state $|\alpha\rangle_{h_0}$ to be normalized,

$$_{h_0}\langle\alpha|\alpha\rangle_{h_0} = 1. \tag{3.8}$$

Equation (3.8) then determines C_0 except for a phase factor

$$C_0 = \left\{\sum_{n=0}^{\infty} \frac{\Gamma(h_0)}{\Gamma([n/2] + 1)\Gamma([(n+1)/2] + h_0)} \left(\frac{1}{2}|\alpha|^2\right)^n\right\}^{-1/2}. \tag{3.9}$$

Let us define

$$f(x) = \sum_{n=0}^{\infty} \frac{\Gamma(h_0)}{\Gamma([n/2] + 1)\Gamma([(n+1)/2] + h_0)} \left(\frac{1}{2}x\right)^n, \tag{3.10}$$

or equivalently

$$f(x) = \Gamma(h_0)(\tfrac{1}{2}x)^{1-h_0}\{I_{h_0-1}(x) + I_{h_0}(x)\}, \tag{3.11}$$

where I_k is the modified Bessel function of the kth order.[11]

From Eqs. (3.2), (3.7), (3.9), and (3.11) we obtain the following expression for the para-bose coherent state

$$|\alpha\rangle_{h_0} = \{f(|\alpha|^2)\}^{-1/2}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(h_0)}{2^n \Gamma([n/2] + 1)\Gamma([(n+1)/2] + h_0)}^{1/2} \alpha^n |n\rangle_{h_0}. \tag{3.12}$$

Using Eq. (2.20) we may also write

$$|\alpha\rangle_{h_0} = [f(\alpha\hat{a}^\dagger)/\{f(|\alpha|^2)\}^{1/2}]|0\rangle_{h_0}. \tag{3.13}$$

The familiar case of the ordinary oscillator, is recovered when we set $h_0 = \frac{1}{2}$. In this case we find from Eq. (3.10) that

$$f(x) = \exp(x). \tag{3.14}$$

Equations (3.13) and (3.14) then give

$$|\alpha\rangle_{1/2} = \exp[-(1/2)|\alpha|^2] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_{1/2}, \tag{3.15a}$$

$$= \exp[-(1/2)|\alpha|^2]\exp(\alpha\hat{a}^\dagger)|0\rangle_{1/2}, \tag{3.15b}$$

which are the well-known expressions for the ordinary coherent states.[9,10]

The Hermitian adjoint of Eq. (3.1) reads

$$_{h_0}\langle\alpha|\hat{a}^\dagger = \alpha^*\,_{h_0}\langle\alpha|. \tag{3.16}$$

However, one may readily show that there are no right eigenstates of $\hat{a}$, i.e., there is no state $|\lambda\rangle$ which satisfies a relation of the type

$$\hat{a}^\dagger|\lambda\rangle_{h_0} = \lambda|\lambda\rangle_{h_0}.$$

We close this section by giving the average values of the various operators in the para-Bose coherent states:

$$_{h_0}\langle\alpha|\hat{a}|\alpha\rangle_{h_0} = {}_{h_0}\langle\alpha|\hat{a}^\dagger|\alpha\rangle_{h_0}^* = \alpha, \tag{3.17}$$

$$_{h_0}\langle\alpha|\hat{q}|\alpha\rangle_{h_0} = \frac{1}{\sqrt{2}}(\alpha + \alpha^*), \tag{3.18}$$

$$_{h_0}\langle\alpha|\,[\hat{a},\hat{a}^\dagger]\,|\alpha\rangle_{h_0} = 2\,\frac{h_0 I_{h_0-1}(|\alpha|^2) + (1 - h_0)I_{h_0}(|\alpha|^2)}{I_{h_0-1}(|\alpha|^2) + I_{h_0}(|\alpha|^2)}, \tag{3.19}$$

$$_{h_0}\langle\alpha|\hat{H}|\alpha\rangle_{h_0} = \frac{(h_0 + |\alpha|^2)I_{h_0-1}(|\alpha|^2) + (1 - h_0 + |\alpha|^2)I_{h_0}(|\alpha|^2)}{I_{h_0-1}(|\alpha|^2) + I_{h_0}(|\alpha|^2)}, \tag{3.20}$$

$$_{h_0}\langle\alpha|\hat{N}|\alpha\rangle = |\alpha|^2 + (1 - 2h_0)\frac{I_{h_0}(|\alpha|^2)}{I_{h_0-1}(|\alpha|^2) + I_{h_0}(|\alpha|^2)}. \tag{3.21}$$

These relations are readily derived from Eqs. (3.1), (1.5), (2.6), (2.18), and (3.12).

4. COMPLETENESS AND THE DIAGONAL COHERENT STATE REPRESENTATION

In this section we show that the para-Bose coherent states introduced in the last section form a complete set, in fact an over-complete set. Analogous to ordinary coherent states, we find the possibility of the diagonal[10] para-Bose coherent state representation.

Since $\hat{a}$ is not Hermitian, the coherent states are not expected to be orthogonal. From Eq. (3.12) we obtain the following expression for the scalar product of two coherent states:

$$_{h_0}\langle\beta|\alpha\rangle_{h_0} = f(\beta^*\alpha)\{f(|\alpha|^2)f(|\beta|^2)\}^{-1/2}. \tag{4.1}$$

We show below that these coherent states form an over-complete set. For this purpose we assume the ex-

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istence of some function $\mu(\alpha)$ such that

$$\int |\alpha\rangle_{h_0 h_0}\langle\alpha|\mu(\alpha)d^2\alpha = 1, \qquad (4.2)$$

where $d^2\alpha = dxdy$, x and y being the real and imaginary parts of α and the integration is performed over the whole complex α plane. We substitute for $|\alpha\rangle_{h_0}$ from Eq. (3.12) and find that

$$\sum_{n,m}\left\{\frac{\Gamma(h_0)}{2^n\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}\right\}^{1/2}$$

$$\times\left\{\frac{\Gamma(h_0)}{2^m\Gamma([m/2]+1)\Gamma([(m+1)/2]+h_0)}\right\}^{1/2}$$

$$\times\int\{f(|\alpha|^2)\}^{-1}\alpha^{*m}\alpha^n\mu(\alpha)d^2\alpha\,|n\rangle_{h_0 h_0}\langle m| = 1. \qquad (4.3)$$

From the orthogonality of the number states [Eq. (2.19b)], we may write

$$\int\{f(|\alpha|^2)\}^{-1}\alpha^{*m}\alpha^n\mu(\alpha)d^2\alpha$$

$$= \frac{2^n\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}{\Gamma(h_0)}\delta_{nm}. \qquad (4.4)$$

If we now use polar coordinates

$$\alpha = r\exp(i\theta), \quad d^2\alpha = r\,dr\,d\theta, \qquad (4.5)$$

we may readily show that μ cannot depend on θ, and is thus a function of $|\alpha|^2$ only. We then write

$$\mu(\alpha) = \mu(|\alpha|^2). \qquad (4.6)$$

Substituting $r^2 = x$, and integrating over θ, we then find that

$$\int_0^\infty \{f(x)\}^{-1}x^n\mu(x)dx = \frac{2^n\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}{\pi\Gamma(h_0)}. \qquad (4.7)$$

Thus our problem of showing the completeness of the para-Bose coherent states reduces to determining $\mu(x)$ whose moments are given by (4.7). If we define a function

$$M(y) = \sum_{n=0}^\infty \frac{(2iy)^n}{n!}\,\frac{\Gamma([n/2]+1)\Gamma([(n+1)/2]+h_0)}{\pi\Gamma(h_0)}, \qquad (4.8)$$

we may write

$$\int_0^\infty \{f(x)\}^{-1}\mu(x)\exp(ixy)\,dx = M(y). \qquad (4.9)$$

The series on the right-hand side of (4.8) is an absolutely convergent series for $|y| < 1$ (and is divergent for $|y| > 1$). For $|y| > 1$, we define $M(y)$ by analytic countinuation. Assuming that $M(y)$ thus defined is well behaved such that we may take the inverse Fourier transform of (4.9), we obtain

$$\mu(x) = \frac{1}{2\pi}\,f(x)\int_{-\infty}^\infty M(y)\exp(-ixy)\,dy. \qquad (4.10)$$

We thus obtain the resolution of the identity operator

$$\int |\alpha\rangle_{h_0 h_0}\langle\alpha|\mu(|\alpha|^2)d^2\alpha = 1, \qquad (4.11)$$

where $\mu(x)$ is given by (4.10), thereby showing that the para-Bose coherent states are complete. Using Eqs. (4.1) and (4.11) we may write

$$|\beta\rangle_{h_0} = \int\frac{f(\alpha^*\beta)\mu(|\alpha|^2)}{\{f(|\alpha|^2)f(|\beta|^2)\}^{1/2}}\,|\alpha\rangle_{h_0}d^2\alpha. \qquad (4.12)$$

Equation (4.12) show that these coherent states in fact form an over complete set. On multiplying (4.12) by $\langle\gamma|$ on the left, we obtain the "self-reproducing" property[12] of $f(x)$:

$$\int\mu(|\alpha|^2)\frac{f(\gamma^*\alpha)f(\alpha^*\beta)}{f(|\alpha|^2)}\,d^2\alpha = f(\gamma^*\beta). \qquad (4.13)$$

The possibility of the existence of the diagonal coherent state representation of an arbitrary operator may also be considered. If we write

$$\hat{G} = \int \phi(\alpha)|\alpha\rangle_{h_0 h_0}\langle\alpha|\mu(|\alpha|^2)d^2\alpha, \qquad (4.14)$$

we obtain using (3.12),

$$\int\frac{\alpha^m\alpha^{*n}\phi(\alpha)\mu(|\alpha|^2)}{f(|\alpha|^2)}\,d^2\alpha$$

$$= {}_{h_0}\langle m|\hat{G}|n\rangle_{h_0}\{\Gamma(h_0)\}^{-1}$$
$$\times\{2^n\Gamma([n/2]+1)\,\Gamma([(n+1)/2]+h_0)$$
$$\times 2^m\Gamma([m/2]+1)\,\Gamma([(m+1)/2]+h_0)\}^{1/2}. \qquad (4.15)$$

Equation (4.15) gives all the moments of $\phi(\alpha)\mu(|\alpha|^2)/f(|\alpha|^2)$.

We close this section by observing that the properties of the usual coherent states are reproduced if we set $h_0 = \frac{1}{2}$. In this case we find from (4.8) that

$$M(y) = \frac{1}{\pi}\,\frac{1}{1-iy}, \qquad (4.16)$$

and if we substitute this expression in (4.10) we obtain $\mu(x) = 1/\pi$. This gives

$$\frac{1}{\pi}\int |\alpha\rangle_{1/2}\,{}_{1/2}\langle\alpha|d^2\alpha = 1. \qquad (4.17)$$

5. UNCERTAINTY RELATIONS

It is well known that for two Hermitian operators $\hat{A}$ and $\hat{B}$ which satisfy the commutation relation

$$[\hat{A},\hat{B}] = i\hat{C}, \qquad (5.1)$$

the uncertainty relation

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \tfrac{1}{4}|\langle\hat{C}\rangle|^2 \qquad (5.2)$$

holds. Relation (5.2) is an equality if the state under consideration is an eigenstate of $\hat{A}+i\lambda\hat{B}$, where λ is some real constant. If we identify $\hat{A}$ and $\hat{B}$ with the position and momentum variables $\hat{q}$ and $\hat{p}$ of the para-Bose operator, we find that

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \geq \tfrac{1}{4}|\langle[\hat{q},\hat{p}]\rangle|^2, \qquad (5.3)$$

We may readily verify that relation (5.3) is an equality for the para-Bose coherent states (being eigenstates of the operator $(\hat{q}+i\hat{p})$. From Eq. (1.5) we may write

$$\hat{q} = (1/\sqrt{2})(\hat{a}+\hat{a}^\dagger), \qquad (5.4)$$

$$\hat{p} = -(i/\sqrt{2})(\hat{a}-\hat{a}^\dagger). \qquad (5.5)$$

The commutator of $\hat{q},\hat{p}$ is therefore given by

$$[\hat{q},\hat{p}] = i[\hat{a},\hat{a}^\dagger]. \qquad (5.6)$$

We also have

$$\hat{q}^2 = \hat{H} + \tfrac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2}),$$ (5.7)

$$\hat{p}^2 = \hat{H} - \tfrac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2}),$$ (5.8)

so that

$$\langle(\Delta\hat{q})^2\rangle = \langle\hat{H}\rangle - \langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle + \tfrac{1}{2}\{\langle(\Delta\hat{a})^2\rangle + \langle(\Delta\hat{a}^\dagger)^2\rangle\}$$ (5.9)

and

$$\langle(\Delta\hat{p})^2\rangle = \langle\hat{H}\rangle - \langle\hat{a}\rangle\langle\hat{a}^\dagger\rangle - \tfrac{1}{2}\{\langle(\Delta\hat{a})^2\rangle + \langle(\Delta\hat{a}^\dagger)^2\rangle\}.$$ (5.10)

For the coherent state $|\alpha\rangle_{h_0}$, $\langle(\Delta\hat{a})^2\rangle = 0$ and we then find from Eqs. (3.20) that

$$\langle(\Delta\hat{q})^2\rangle = \langle(\Delta\hat{p})^2\rangle = \frac{h_0 I_{h_0-1}(|\alpha|^2) + (1 - h_0)I_{h_0}(|\alpha|^2)}{I_{h_0-1}(|\alpha|^2) + I_{h_0}(|\alpha|^2)} .$$ (5.11)

Comparing (5.11) with (3.19) and using (5.6) we obtain

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle = \tfrac{1}{4}|\langle[\hat{q},\hat{p}]\rangle|^2.$$ (5.12)

We thus find that for para-Bose coherent states, the uncertainty relation (5.3) reduces to an equality. However, since $[\hat{q},\hat{p}]$ is in general not a c number the right-hand side of (5.12) itself depends on the given state. Hence the para-Bose coherent states are not the minimum uncertainty states in the absolute sense (except for the case of ordinary oscillator, $h_0 = \tfrac{1}{2}$ when $[\hat{q},\hat{p}]$ becomes a c number). It is obvious that one may find states for which the uncertainty product $\langle(\Delta\hat{q})^2\rangle \times \langle(\Delta\hat{p})^2\rangle$ is in fact less than $\tfrac{1}{4}$, which is the minimum value for the ordinary oscillator. For small values of $|\alpha|$, we know that[11]

$$I_k(|\alpha|^2) \sim (\tfrac{1}{2}|\alpha|^2)^k/\Gamma(k+1), \quad (k \neq -1, -2, \ldots)$$

and hence from (5.11) we obtain

$$\langle(\Delta\hat{q})^2\rangle = \langle(\Delta\hat{p})^2\rangle \sim h_0$$

and

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \sim h_0^2.$$

Thus for para-Bose operators with $h_0 < \tfrac{1}{2}$, we find that the ground state (or the coherent state with $\alpha = 0$) gives the uncertainty product which is less than $\tfrac{1}{4}$.

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Coulomb-like quantization of the electromagnetic field on spacelike hyperboloids

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Maxwell's equations relative to a pseudospherical coordinate system in Minkowski space are cast in a form remarkably similar to the usual Cartesian form. The vector spherical harmonics of the unit hyperboloid derived in an earlier work are then applied to the problem of quantizing the electromagnetic field from the standpoint of the method of quantization on spacelike hyperboloids. A Coulomb-like gauge is introduced for which quantization on spacelike hyperboloids parallels that of ordinary Coulomb gauge quantization. In a kind of "turning around" of the situation which obtains for ordinary Coulomb gauge quantization, our new Coulomb-like gauge condition has manifest covariance with respect to transformations of the homogeneous Lorentz group but not with respect to translations.

I. INTRODUCTION

The method of quantization on spacelike hyperboloids is a form of relativistic quantum theory in which the concept "all space at one time" is replaced by the concept of a "spacelike hyperboloid, " $(x^0)^2 - r \cdot r = \text{const} > 0$. Dynamical variables specified on one spacelike hyperboloid can be extrapolated to a neighboring hyperboloid by means of a scale change generated by a "dilation Hamiltonian"; just as in the usual formalism, dynamical variables specified at one time can be extrapolated forward in time by use of the ordinary Hamiltonian. The Tomonaga—Schwinger equation indicates that the two methods should be equivalent, but it is interesting and potentially fruitful to express quantum field theory in the new form.

Although a number of model systems have been treated earlier using the method of quantization on spacelike hyperboloids, there seems to be no previous application of this method to the full electromagnetic field in four dimensional Minkowski space. Fubini, Hanson, and Jackiw[1] considered a massless scalar particle in a four-dimensional Euclidean spacetime from the standpoint of the new formalism. Their evolution equations enable them to extrapolate field quantities between neighboring spheres $(x^0)^2 + r \cdot r = \text{const}$ in the Euclidean spacetime. The advantage of looking at Euclidean spacetime is that then there is no difficulty in treating all spacetime, whereas in a Lorentzian spacetime the initial quantization provides the fields only within the forward light cone. The means of extrapolating outside the forward light cone in a Lorentzian spacetime has been considered by Sommerfield[2] and di Sessa[3]. Sommerfield gives an extensive investigation of the relationship between the new formalism and the usual Hamiltonian formalism for the specific example of a massive scalar field in one space and one time dimension, and also provides an introduction to the treatment of the same example in four spacetime dimensions. Di Sessa has investigated the propagator in the new formalism and finds it to be the same as the usual one at least in the massive case and provided that the particle—antiparticle distinction is the usual one. This result of di Sessa indicates that the typical lack of manifest translational invariance in applications of the method of quantization on spacelike hyperboloids should correct itself at the level of the

Feynman—Dyson expansion—at least if there are no mass-zero particle present. The Dirac equation in four-dimensional Minkowski space has been treated by the method of quantization on spacelike hyperboloids by Gromes, Rothe, and Stech.[4] Also, quantization on spacelike hyperboloids has been investigated by Fulling[5] in connection with problems of quantization in general relativity. Relevant special functions associated with the unit hyperboloid in Minkowski space have been investigated in Refs. 6—11. Reference 11 contains special tensor methods for physics on spacelike hyperboloids as well as results on vector spherical harmonics of the unit hyperboloid in Minkowski space, both of which will be used extensively in the following. *In the interest of brevity the present paper will be treated as a direct continuation of that earlier paper and will contain no special review of those earlier results.* The prefix C, as in Eq. (C2.1), will signal a reference to a particular equation from this earlier companion paper.

The present work starts with an investigation of Maxwell's equations in a pseudospherical coordinate system for which

$$x^0 = s \cosh\rho,$$
$$x^1 = s \sinh\rho \sin\theta \cos\varphi,$$
$$x^2 = s \sinh\rho \sin\theta \sin\varphi, \qquad (1.1)$$
$$x^3 = s \sinh\rho \cos\theta,$$
$$0 < s < \infty, \quad 0 \leq \rho < \infty,$$
$$0 < \theta \leq \pi, \quad 0 < \varphi \leq 2\pi.$$

Our attention is hereby restricted to the forward light cone. Equations (2.7)—(2.10) obtained in Sec. II provide a convenient representation of Maxwell's equations on spacelike hyperboloids which is very similar to the ordinary Cartesian form. This similarity is achieved by the use of the special tensor methods of Ref. 11. The latter methods exploit the properties of the intrinsic gradient operator

$$\nabla \equiv \hat{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\sinh\rho}\left(\hat{\theta}\frac{\partial}{\partial\theta} + \frac{\hat{\varphi}}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \qquad (1.2)$$

of the unit hyperboloid. A new Coulomb-like gauge for which the vector potential is a solenoidal tangent vector field is introduced [see Eqs. (2.11) through (2.13)].

Quantization in this new gauge is carried out in Sec. III and goes quite parallel to ordinary Coulomb gauge quantization[12]: Generalized coordinates can be found such that the quantum conditions take the ordinary canonical form $[q_\alpha; p_\beta] = i\delta_{\alpha\beta}$. The generalized coordinates $q^\rho_{\alpha l m}$ referred to here are defined through Eqs. (3.7), (3.8) in terms of the expansion coefficients $Q^\rho_{\alpha l m}$ in an eigenfunction expansion of the field involving the solenoidal vector spherical harmonics of the unit hyperboloid developed in Ref. 11.

Due to the Lorentz invariance of the intrinsic gradient operator (1.2) of the unit hyperboloid the new Coulomb-like gauge is invariant under transformations of the homogeneous Lorentz group and our quantization procedure possesses a manifest covariance with respect to that group. However, in a kind of "turning around" of the situation that obtains for ordinary Coulomb gauge quantization the new gauge is not manifestly covariant with respect to translations.

Section IV concerns the photon propagator. Since we are dealing with a theory having zero rest mass, the lack of manifest translational covariance can carry over to the propagator. Indeed it does not seem possible to achieve manifest translational covariance of the photon propagator in the present formalism without the use of some type of limiting procedure. One such limiting procedure discussed in Sec. IV would introduce a small photon rest mass to be set equal to zero at the end of a calculation.

II. MAXWELL'S EQUATIONS ON SPACELIKE HYPERBOLOIDS

We start with Maxwell's equations in the covariant dyadic form

$$\square \cdot F = j \tag{2.1a}$$

and

$$\square \cdot {}^*F = 0 \tag{2.1b}$$

The tensor F is the electromagnetic field tensor having the contravariant components

$$
F^{\mu\nu} =
\begin{pmatrix}
0 & -\mathcal{E}^1 & -\mathcal{E}^2 & -\mathcal{E}^3 \\
\mathcal{E}^1 & 0 & -B^3 & B^2 \\
\mathcal{E}^2 & B^3 & 0 & -B^1 \\
\mathcal{E}^3 & -B^2 & B^1 & 0
\end{pmatrix}.
\tag{2.2}
$$

The dual tensor $^*F$ is defined through the equations $^*F^{\mu\nu} \equiv (1/2)\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ with $\epsilon^{0123} \equiv 1$. The 4-current density j has the contravariant components

$$
j^\mu =
\begin{bmatrix}
\rho \\
J^1 \\
J^2 \\
J^3
\end{bmatrix}
\tag{2.3}
$$

It obeys the differential current conservation law

$$\square \cdot j = 0 \tag{2.4}$$

Since the field tensor is antisymmetric, it has a representation of the form

$$F = -\hat{s} \wedge \mathcal{E} + B, \tag{2.5}$$

in which $\mathcal{E} = e_a \mathcal{E}^a$ is a vector lying in the tangent space of the hyperboloid $x^\mu x_\mu = s^2$ through the field point; and B is a general antisymmetric tensor in the tangent space:

$$B = e_a e_b (-g)^{1/2} \epsilon^{abc} B_c. \tag{2.6}$$

The vector $\mathcal{E}$ can be called the electric field vector in the hyperboloidal space; and $\mathbf{B} \equiv e^a B_a = e_a B^a$, the magnetic induction vector. When the pseudospherical representations (C2.1) and (2.5), (2.6) of the d'Alembertian operator and field tensor respectively are substituted into Maxwell's equations (2.1a), (2.1b); the system of equations

$$-\nabla \cdot \mathcal{E} = s\rho, \tag{2.7}$$

$$\nabla \times \mathbf{B} = sJ + \left(s\frac{\partial}{\partial s} + 2\right)\mathcal{E}, \tag{2.8}$$

$$-\nabla \cdot \mathbf{B} = 0, \tag{2.9}$$

and

$$\nabla \times \mathcal{E} = -\left(s\frac{\partial}{\partial s} + 2\right)\mathbf{B} \tag{2.10}$$

for the tangent vector fields $\mathcal{E}$ and $\mathbf{B}$ results. The similarity of these equations with the usual form of Maxwell's equations—in spite of our unusual coordinate system—is quite striking! We shall refer to Eq. (2.7) as the "electric Gauss's law on spacelike hyperboloids" and similarly for the other equations. The minus sign in the electric Gauss's law is a bit unpleasant. It is a consequence of our negative definite metric on the unit hyperboloid, according to which it is the operator $(-\nabla \cdot)$ which has the meaning usually associated with the divergence.

Vector and scalar potentials can be introduced according to the equations

$$\mathbf{B} \equiv \nabla \times \mathbf{A}/s^2 \tag{2.11}$$

and

$$\mathcal{E} = -s^{-2}\nabla V - s^{-1}(\partial \mathbf{A}/\partial s). \tag{2.12}$$

The factor s^{-2} in Eq. (2.11) has been chosen for later convenience; it is our $\mathbf{A}/s$ which has the dimensions of the usual vector potential. Similarly the explicit power of $1/s$ associated with the ∇V term of Eq. (2.12) has been chosen such that it is our V/s which has the dimensions of the usual scalar potential. The representations (2.11) and (2.12) are designed to satisfy the homogeneous Maxwell's equations (2.9), (2.10) identically. It remains to investigate the consequences of the inhomogeneous Maxwell's equations (2.7) and (2.8) when the fields are written in terms of potentials. At this point it is convenient to make a gauge choice. In the following we shall be concerned with a gauge for which our vector potential on the hyperboloid is solenoidal:

$$\nabla \cdot \mathbf{A} = 0. \tag{2.13}$$

This gauge choice is analogous to the Coulomb gauge of ordinary electrodynamics. The gauge choice (2.13) might be referred to as the "covariant" Coulomb gauge, since Eq. (2.13) has a manifestly covariant form as regards transformations of (only) the homogeneous

Lorentz group. Substituting Eq. (2.12) into Gauss's law (2.7) and using the gauge condition $\nabla \cdot \mathbf{A} = 0$ leads to a Poisson-type equation for V:

$$\nabla^2 V = s^3 \rho. \tag{2.14}$$

The Green's function for this equation has been obtained in Ref. 11 [Eq. (C3.27)]. The vector potential in the relativistic Coulomb gauge is a special example of a solenoidal tangent vector field for which the simplified curl—curl identity (C2.19) holds. Substituting the representation $\mathbf{B} = (\nabla \times \mathbf{A})/s^2$ into the Ampere circuital law (2.8) and using the simplified curl—curl identity and Eq. (2.12); we obtain the differential equation

$$\left\{ \frac{\partial^2}{\partial s^2} + \frac{1}{s} \frac{\partial}{\partial s} + \frac{\nabla^2 - 1}{s^2} \right\} \mathbf{A} = s \mathbf{J}^T, \tag{2.15}$$

in which

$$\mathbf{J}^T \equiv \mathbf{J} - s^{-2}(\partial/\partial s)\nabla V \tag{2.16}$$

can be shown to be the solenoidal part of the current density vector:

$$\nabla \cdot \mathbf{J}^T = 0. \tag{2.17}$$

We note that the closure property of ∇^2 proved in Ref. 11 at the end of Sec. II guarantees that all terms of the wave equation (2.15), including the $\nabla^2 \mathbf{A}$ term, shall refer to the same vector space; viz., the tangent space of the hyperboloid $x^\mu x_\mu = s^2$ through the field point.

III. QUANTIZATION OF THE ELECTROMAGNETIC FIELD ON SPACELIKE HYPERBOLOIDS

The usual Lagrangian density of the electromagnetic field interacting with an external current is $\underline{\mathcal{L}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - (1/s)\mathbf{J} \cdot \mathbf{A} - (1/s)\rho V$. The suspicious minus sign in the vector potential interaction is due to our indefinite metric: $(-1/s)\mathbf{J} \cdot \mathbf{A} = (1/s)(J_x A_x + J_y A_y + J_z A_z)$. When expressed in terms of our field vectors $\mathcal{E}$ and B in the tangent space of the hyperboloid $s = \text{const}$, $\underline{\mathcal{L}}$ becomes $\underline{\mathcal{L}} = \frac{1}{2}(\mathbf{B} \cdot \mathbf{B} - \mathcal{E} \cdot \mathcal{E}) - (1/s)\mathbf{J} \cdot \mathbf{A} - (1/s)\rho V$. Writing $\mathcal{E}$ and B in terms of potentials, we get

$$s^4 \underline{\mathcal{L}} = -\frac{1}{2}\overset{\circ}{\mathbf{A}} \cdot \overset{\circ}{\mathbf{A}} + \frac{1}{2}(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{A}) - s^3 \mathbf{J} \cdot \mathbf{A}$$
$$- (\tfrac{1}{2})s^3 \rho V. \tag{3.1}$$

Here and subsequently a dot over a quantity signifies differentation with respect to the new time evolution parameter

$$\nu = \ln(Ms), \tag{3.2}$$

in which M is a parameter with the dimensions of mass.

We shall quantize in a generalized angular momentum representation, expanding the solenoidal tangent vector field $\mathbf{A}$ as a linear superposition of the solenoidal vector spherical harmonics $\mathbf{Y}^\rho_{\alpha l m}$:

$$\mathbf{A} = \sum_{\rho l m} \int_0^\infty d\alpha \, \mathbf{Y}^\rho_{\alpha l m} Q^\rho_{\alpha l m}. \tag{3.3}$$

The dynamical degrees of freedom of the system are here contained in the generalized Fourier expansion coefficients $Q^\rho_{\alpha l m}$. The $Q^\rho_{\alpha l m}$ do not each represent independent degrees of freedom of the system however; since they obey constraint equations

$$Q^{\rho\,\dagger}_{\alpha l m} = (-1)^m Q^\rho_{\alpha l - m}. \tag{3.4}$$

These constraint equations express the reality $\mathbf{A} = \mathbf{A}^\dagger$ of the field being expanded. The symmetry property

$$\mathbf{Y}^{\rho\,*}_{\alpha l m} = (-1)^m \mathbf{Y}^\rho_{\alpha l, -m} \tag{3.5}$$

of the vector spherical harmonics is involved in the derivation of Eq. (3.4). The constraint equations (3.4) show that the field coordinates $Q^\rho_{\alpha l m}$ for $m \neq 0$ are not generalized coordinates suitable for doing the Hamiltonian dynamics of the field. Such generalized coordinates should be real and independent in order to fit into the usual Hamiltonian formalism. However, the fact that our constraint equations are holonomic[13] indicates that a set of independent real coordinates of the field—referred to subsequently as *canonical coordinates*—must exist. The key to the construction of a set of canonical coordinates lies in the observation that the constraint equations (3.4) impose opposite parities under interchange of m and $-m$ on the real and imaginary parts of $Q^\rho_{\alpha l m}$, $m \neq 0$. This suggests introducing a single real function $q^\rho_{\alpha l m}$ without special symmetry properties—hence not subject to constraint equations—from which the real and imaginary parts of $Q^\rho_{\alpha l m}$ are constructed. These $q^\rho_{\alpha l m}$ will be the canonical coordinates of the field. Explicitly, we have

$$Q^\rho_{\alpha l m} \equiv q^\rho_{\alpha l m}(1 + i) + (-1)^m(1 - i)q^\rho_{\alpha l, -m}, \quad m \neq 0. \tag{3.6}$$

The relations (3.6) could be regarded as a consequence of an explicit defining equation

$$q^\rho_{\alpha l m} \equiv \tfrac{1}{4}\{(1 - i)Q^\rho_{\alpha l m} + (1 + i)Q^{\rho\,\dagger}_{\alpha l m}\}, \quad m \neq 0. \tag{3.7}$$

For $m = 0$ the defining equation is

$$q^\rho_{\alpha l 0} \equiv (\tfrac{1}{2})Q^\rho_{\alpha l 0}. \tag{3.8}$$

The $m = 0$ case requires special consideration; although the details of this will not be indicated explicitly. In terms of canonical coordinates we find a total Lagrangian $L \equiv \int s^3 d\Sigma \underline{\mathcal{L}}$ of

$$L = \mathfrak{L}/s \tag{3.9}$$

where

$$\mathfrak{L} = \sum_{\Lambda m} \{ 2d\alpha \, \dot{q}^2_{\Lambda m} - 2\alpha^2 d\alpha \, q^2_{\Lambda m}$$
$$+ d\alpha q_{\Lambda m} s^3((1 - i)j^T_{\Lambda m} + (1 + i)j^{T\dagger}_{\Lambda m})\} \tag{3.10}$$
$$- \int d\Sigma \, \tfrac{1}{2}s^3 \rho V.$$

We here adopt the shorthand notation $q^\rho_{\alpha l m} \equiv q_{\Lambda m}$, in which Λ signifies the set of quantum numbers consisting of ρ, α, and l. For convenience the integral over α has been replaced in Eq. (3.10) by an approximating discrete sum which is then incorporated in the sum on Λ. The $j^T_{\Lambda m}$ are the generalized Fourier expansion coefficients of the solenoidal part, given by Eq. (2.16), of the current density vector:

$$\mathbf{J}^T \equiv \sum_{\rho l m} \int_0^\infty d\alpha \, \mathbf{Y}^\rho_{\alpha l m} j^{T\rho}_{\alpha l m}. \tag{3.11}$$

We are now in a position to do the Hamiltonian dynamics of the electromagnetic field in a straightforward way. The canonical momenta $p_{\Lambda m}$ and Hamiltonian H are identified through the variation of the action[14]

$$\delta J = \delta \int ds \, L = \left[\sum_{\Lambda m} p_{\Lambda m} \delta q_{\Lambda m} - H\delta s \right] \Big|_{\text{initial}}^{\text{final}}$$

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due to variations in the coordinate values and proper time at the end points. Thus

$$p_{\Lambda m} = \partial \mathcal{L} / \partial \mathring{q}_{\Lambda m} \qquad (3.12)$$

and

$$H = \mathfrak{H}/s, \qquad (3.13)$$

in which

$$\mathfrak{H} = \sum_{\Lambda m} \frac{\partial \mathcal{L}}{\partial \mathring{q}_{\Lambda m}} \mathring{q}_{\Lambda m} - \mathcal{L} \qquad (3.14)$$

is the "dilation Hamiltonian." The canonical momenta are found to be

$$p_{\Lambda m} = 4 d\alpha \, \mathring{q}_{\Lambda m}. \qquad (3.15)$$

Since our coordinates $q_{\Lambda m}$ are independent; canonical quantum conditions can be imposed, the nonzero commutators being

$$[q_{\Lambda_2 m_2}; p_{\Lambda_1 m_1}] = i \, \delta_{\Lambda_2 \Lambda_1} \delta_{m_2 m_1}. \qquad (3.16)$$

The canonical $q_{\Lambda m}$, $p_{\Lambda m}$ having served their purpose of providing the basic commutator structure of the theory, we now return to our original description in terms of the not independent variables $Q_{\Lambda m}$. Commutation relations associated with the $Q_{\Lambda m}$ can be derived by expressing the original variables in terms of the canonical variables, and then using the canonical commutation relations (3.16). These calculations involve the use of Eqs. (3.6), (3.8), and (3.15) in addition to (3.16). The nonzero commutators between pairs of variables $Q_{\Lambda m}$, $Q^\dagger_{\Lambda m}$, $\mathring{Q}_{\Lambda m}$, and $\mathring{Q}^\dagger_{\Lambda m}$ thereby obtained are

$$[Q_{\Lambda_2 m_2}; \mathring{Q}^\dagger_{\Lambda_1 m_1}] = [Q^\dagger_{\Lambda_2 m_2}; \mathring{Q}_{\Lambda_1 m_1}]$$
$$= i \, \delta_{\rho_2 \rho_1} \delta(\alpha_2 - \alpha_1) \delta_{l_2 l_1} \delta_{m_2 m_1} \qquad (3.17)$$

and

$$[Q_{\Lambda_2 m_2}; \mathring{Q}_{\Lambda_1 m_1}] = [Q^\dagger_{\Lambda_2 m_2}; \mathring{Q}^\dagger_{\Lambda_1 m_1}]$$
$$= i \, \delta_{\rho_2 \rho_1} \delta(\alpha_2 - \alpha_1) \delta_{l_2 l_1} (-1)^{m_1} \delta_{m_2, -m_1}. \qquad (3.18)$$

To free ourselves of the need of any further reference to the rather awkward canonical variables, we require the dilation Hamiltonian in terms of the original variables. This is $\mathfrak{H} = \mathfrak{H}_0 + \mathfrak{H}_{int}$, where

$$\mathfrak{H}_0 = \sum_{\rho l m} \int_0^\infty d\alpha \, \tfrac{1}{2} (\mathring{Q}^\dagger_{\Lambda m} \mathring{Q}_{\Lambda m} + \alpha^2 Q^\dagger_{\Lambda m} Q_{\Lambda m}) \qquad (3.19)$$

and

$$\mathfrak{H}_{int} = - \sum_{\rho l m} \int_0^\infty d\alpha \, Q_{\Lambda m} s^2 j^{T\dagger}_{\Lambda m} + \int d\Sigma \, \tfrac{1}{2} s^3 \rho V. \qquad (3.20)$$

The dilation equations of motion

$$\mathring{O} = (1/i)[O; \mathfrak{H}] \qquad (3.21)$$

can be derived from the Heisenbergh equations of motion $(d/ds)O = (1/i)[O; H]$ by making the change of variables (3.2) and noting the relation (3.13) between the Hamiltonian H and the dilation Hamiltonian $\mathfrak{H}$. By use of the commutation relations (3.17) and (3.18) it can be checked that the dilation equations of motion (3.21) lead to the correct proper-time evolution in our theory: the dilation equations of motion for $Q_{\Lambda m}$ are obeyed trivially,

and the equations of motion for $\mathring{Q}_{\Lambda m}$ express the original wave equation (2.15) for A in the generalized angular momentum representation.

For applications it is convenient to make a further transformation to a representation in terms of creation and annihilation operators $a_{\Lambda m}$, $a^\dagger_{\Lambda m}$. These are defined in such a way as to lead to a factorization of the integrand of Eq. (3.19) for $\mathfrak{H}_0$:

$$a_{\Lambda m} \equiv (\alpha/2)^{1/2} Q_{\Lambda m} + i(1/2\alpha)^{1/2} \mathring{Q}_{\Lambda m}. \qquad (3.22)$$

The commutation relations for pairs of operators from the set $a_{\Lambda m}$, $a^\dagger_{\Lambda m}$ are the usual ones, the only nonzero commutators being

$$[a_{\Lambda_2 m_2}; a^\dagger_{\Lambda_1 m_1}] = \delta_{\rho_2 \rho_1} \delta(\alpha_2 - \alpha_1) \delta_{l_2 l_1} \delta_{m_2 m_1}. \qquad (3.23)$$

Other important quantities expressed in this representation are

$$\mathbf{A} = \sum_{\rho l m} \int_0^\infty d\alpha \, (2\alpha)^{-1/2} \{ \mathbf{Y}_{\Lambda m} a_{\Lambda m} + \mathbf{Y}^*_{\Lambda m} a^\dagger_{\Lambda m} \}, \qquad (3.24)$$

$$-\dot{\mathbf{A}} = \sum_{\rho l m} \int_0^\infty d\alpha \, (2\alpha)^{-1/2} \{ \mathbf{Y}_{\Lambda m} i \alpha a_{\Lambda m} - \mathbf{Y}^*_{\Lambda m} i \alpha a^\dagger_{\Lambda m} \}, \qquad (3.25)$$

$$\mathfrak{H}_0 = \sum_{\rho l m} \int_0^\infty d\alpha \, \alpha a^\dagger_{\Lambda m} a_{\Lambda m}, \qquad (3.26)$$

and

$$\mathfrak{H}_{int} = - \sum_{\rho l m} \int_0^\infty d\alpha \, (2\alpha)^{-1/2} \{ a^\dagger_{\Lambda m} s^3 j^T_{\Lambda m} + a_{\Lambda m} s^3 j^{T\dagger}_{\Lambda m} \}$$
$$+ \int d\Sigma \, \tfrac{1}{2} s^3 \rho V. \qquad (3.27)$$

One-proper-time commutation relations for the full fields,

$$[A(2)_\mu; -\dot{A}(1)_\nu] = i \{ g_{\mu\nu} - \hat{s}_{2\mu} \hat{s}_{2\nu}$$
$$+ \nabla_{2\mu} (\nabla_2^2)^{-1} (\nabla_{2\nu})^\dagger \} \delta(21), \qquad (3.28)$$

in which

$$(\nabla)^\dagger = -\nabla - 3\hat{s}, \qquad (3.29)$$

can be derived by the use of representations (3.24) and (3.25) of the full fields and by the use of algebra (3.23) of the creation and annihilation operators.[15] The one-proper-time commutator (3.28) is seen to have a general structure analogous to the corresponding object in ordinary Coulomb gauge quantization[16] except that Eq. (3.28) is formed entirely from objects invariant under the homogeneous Lorentz group. Having arrived at the commutator (3.28), we are now in a position to make a complete return to coordinate space. To this end the further relations

$$\mathfrak{H}_0 = \int d\Sigma : \{ -\tfrac{1}{2} \dot{\mathbf{A}} \cdot \dot{\mathbf{A}} - \tfrac{1}{2} (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{A}) \} : \qquad (3.30)$$

in which the colons signify normal ordering, and

$$\mathfrak{H}_{int} = \int d\Sigma \{ s^3 \mathbf{J} \cdot \mathbf{A} + \tfrac{1}{2} s^3 \rho V \} \qquad (3.31)$$

should be noted.

IV. FREE PHOTON PROPAGATION FUNCTION

The proper-time evolution of the annihilation and creation operators $a_{\Lambda m}$, $a^\dagger_{\Lambda m}$ is quite simple in the free field case. When the appropriate commutators are evaluated the dilation equation of motion (Eq. (3.21) with $\mathfrak{H} = \mathfrak{H}_0$) for $a_{\Lambda m}$ is found to be

$\mathfrak{H} \ \dot{a}_{\Lambda m} + i\alpha \ a_{\Lambda m} = 0.$ (4.1)

The solution of Eq. (4.1) can be written

$a_{\Lambda m}(\nu) = e^{-i\alpha\nu} b_{\Lambda m},$ (4.2)

in which $b_{\Lambda m} = a_{\Lambda m}(0)$ is a proper-time-independent set of harmonic oscillator annihilation operators obeying the algebra (3.23). When it is substituted in Eq. (3.24), we obtain the following representation of the free field

$A = \sum_{\rho l m} \int_0^\infty d\alpha \ (2\alpha)^{-1/2} \{ \mathbf{Y}_{\Lambda m} e^{-i\alpha\nu} b_{\Lambda m}$

$\qquad + \mathbf{Y}_{\Lambda m}^* e^{i\alpha\nu} b_{\Lambda m}^\dagger \},$ (4.3)

which exhibits the proper-time dependence explicitly.

In the present formalism the appropriate defining equation of the photon propagation function

$D_{\mu\nu}(21) \equiv \langle 0 | S(A(2)_\mu A(1)_\nu) | 0 \rangle$

$\qquad = \theta(\nu_2 - \nu_1)\langle 0 | A(2)_\mu A(1)_\nu | 0 \rangle$

$\qquad + \theta(\nu_1 - \nu_2)\langle 0 | A(1)_\nu A(2)_\mu | 0 \rangle$ (4.4)

makes use of an ordering with respect to proper time (S-ordering), in place of ordinary time ordering.[17] Our first photon propagator will be obtained by simply using this defining equation in conjunction with the field expansion (4.3). The result of substituting Eq. (4.3) in Eq. (4.4) and using the algebra (3.23) of creation and annihilation operators is the eigenfunction expansion

$D_{\mu\nu}(21) = \sum_{\rho l m} \int_0^\infty d\alpha \ (2\alpha)^{-1} Y_{\Lambda m}(2)_\mu Y_{\Lambda m}^*(1)_\nu e^{-i\alpha|\nu_2-\nu_1|}.$ (4.5)

This eigenfunction expansion provides the boundary condition that in the remote future ($\nu_2 \to +\infty$) the function (4.5) contains only "positive-α" waves, $\exp(-i\alpha\nu_2)$ while near the light cone ($\nu_2 \to -\infty$) it contains only "negative-α" waves, $\exp(i\alpha\nu_2)$. Armed with this boundary condition, we can calculate the function (4.5) as the solution of an appropriate differential equation.

The differential equation in question can be obtained by investigating the effect of the operator$[(\partial^2/\partial\nu_2^2) + \nabla_2^2 - 1)]$ on both sides of Eq. (4.4). This equation

$[(\partial^2/\partial\nu_2^2) + \nabla_2^2 - 1)]D_{\mu\nu}(21)$

$\qquad = (g_{\nu\mu} - \hat{s}_{1\nu}\hat{s}_{1\mu} + \nabla_{1\nu}(\nabla_1^2)^{-1}(\nabla_{1\mu})^\dagger)i\delta(\nu_2 - \nu_1)\delta(21)$ (4.6)

assumes only the free field equation of motion [cf. Eq. (2.15)]

$((\partial^2/\partial\nu^2) + \nabla^2 - 1)A = 0$ (4.7)

and the one-proper-time commutation relations (3.28). Comparing Eq. (4.6) with the similar equation

$((\partial^2/\partial\nu_1^2) + \nabla_1^2 - 1)D_{\mu\nu}(21)$

$\qquad = (g_{\mu\nu} - \hat{s}_{2\mu}\hat{s}_{2\nu} + \nabla_{2\mu}(\nabla_2^2)^{-1}(\nabla_{2\nu})^\dagger)i\delta(\nu_2 - \nu_1)\delta(21)$ (4.8)

leads to the symmetry property

$D_{\mu\nu}(21) = D_{\nu\mu}(12).$ (4.9)

From Eqs. (4.6) and (4.8) we infer the representations

$D_{\mu\nu}(21) = (g_{\nu\mu} - \hat{s}_{1\nu}\hat{s}_{1\mu} + \nabla_{1\nu}(\nabla_1^2)^{-1}(\nabla_{1\mu})^\dagger)D(21)$ (4.10)

and

$D_{\mu\nu}(21) = (g_{\mu\nu} - \hat{s}_{2\mu}\hat{s}_{2\nu} + \nabla_{2\mu}(\nabla_2^2)^{-1}(\nabla_{2\nu})^\dagger)D(21)$ (4.11)

of $D_{\mu\nu}$ in terms of a "scalar photon propagator," $D(21)$, obeying the differential equation

$[(\partial^2/\partial\nu_2^2) + \nabla_2^2 - 1]D(21) = i\delta(\nu_2 - \nu_1)\delta(21).$ (4.12)

We now seek a solution of equation (4.12) obeying the boundary conditions of the function (4.5): that there shall be only "positive-α" waves present as $\nu_2 \to +\infty$ and only "negative-α" waves present as $\nu_2 \to -\infty$. To this end, we write down an eigenfunction expansion of $D(21)$ in a series of the scalar spherical harmonics of the unit hyperboloid discussed in Ref. 11:

$D(21) = \sum_{lm} \int_0^\infty d\alpha \ Y_{\alpha lm}(2)Y_{\alpha lm}^*(1)C_\alpha(\nu_2, \nu_1).$ (4.13)

Substituting in Eq. (4.12) leads to the following equation for the Fourier amplitude $C_\alpha(\nu_2, \nu_1)$

$[(\partial^2/\partial\nu_2^2) + \alpha^2]C_\alpha(\nu_2, \nu_1) = i\delta(\nu_2 - \nu_1),$ (4.14)

whose solution obeying the above boundary conditions is

$C_\alpha(\nu_2, \nu_1) = (-1/2\alpha)e^{-i\alpha|\nu_2-\nu_1|}.$ (4.15)

With this result Eq. (4.13) can be written

$D(21) = -\sum_{lm} \int_0^\infty d\alpha(2\alpha)^{-1}Y_{\alpha lm}(2)Y_{\alpha lm}^*(1)e^{-i\alpha|\nu_2-\nu_1|},$ (4.16)

a result for the scalar photon propagator which parallels Eq. (4.5) for the full tensor propagator. The discrete sum in Eq. (4.16) can be performed by means of the following special case:

$\sum_{l=0}^\infty \sum_{m=-l}^l Y_{\alpha lm}(2)Y_{\alpha lm}^*(1) = \frac{\alpha}{2\pi^2}\frac{\sin(\alpha\rho_{21})}{\sinh\rho_{21}},$ (4.17)

$\cosh\rho_{21} \equiv \hat{s}_2 \cdot \hat{s}_1,$

of the general addition theorem for Gegenbauer functions of Durand, Fishbane, and Simmons.[18] The quantity ρ_{21} in Eq. (4.17) can be viewed as the ρ coordinate of one of the two timelike vectors $\hat{s}_{1,2}$, measured in a Lorentz frame of reference for which the other vector is aligned along the time axis. Using the addition theorem (4.17) Eq. (4.16) goes over into

$D(21) = -\int_0^\infty \frac{d\alpha}{4\pi^2}\frac{\sin(\alpha\rho_{21})}{\sinh\rho_{21}}e^{-i\alpha|\nu_2-\nu_1|}.$ (4.18)

This integral is elementary. Introducing a convergence factor $\exp(-\epsilon\alpha)$, $\epsilon > 0$, we find (new ϵ)

$D(21) = \frac{1}{4\pi^2}\frac{\rho_{21}}{\sinh\rho_{21}}\frac{1}{(\nu_2 - \nu_1)^2 - \rho_{21}^2 - i\epsilon}, \quad \epsilon > 0.$ (4.19)

This is the scalar photon propagator which is to be used in conjunction with Eq. (4.10) or (4.11) to give the tensor propagation function (4.5). Remarkably, the denominator of Eq. (4.19) has the property of vanishing only on the light cone $(\mathbf{x}_2 - \mathbf{x}_1)^2 = 0$ through the point of emission or absorption of the photon.[19] The propagator (4.19) is invariant under transformations of the homogeneous Lorentz group but not under translations. This lack of translational invariance indicates that the propagator (4.19) would provide a rather awkward description of the known physics of the electromagnetic field.

It is clear from the work of di Sessa[3] that the propa-

gator (4.19) is not an inevitable result. We could go over to a new description of the field in terms of creation and annihilation operators c_{Am} related to the b_{Am} by means of a Bogoliubov transformation, thereby changing the vacuum state to be used in Eq. (4.4) and effectively selecting a different solution of the propagator equation (4.12). To obtain a physically useful propagator, we should choose a c representation whose vacuum can be identified with the usual physical photon vacuum. A criterion for selecting such a "good" representation which works for massive particles is given by di Sessa, who then finds that the criterion leads to the usual form of the propagator. One outcome of this result is that the lack of manifest translational invariance typical of theories of quantization on spacelike hyperboloids corrects itself in the propagator and hence also in the Feynman—Dyson expansions, at least for particles having a nonzero rest mass. Since we are here dealing with a zero mass theory, di Sessa's criterion is not applicable. Perhaps it could be replaced in the massless case by the criterion that the above mentioned restoration of manifest translational invariance shall occur in the propagator. At the present time however there is no known c representation for which translational invariance is actually restored to the propagator. However, translational invariance could be incorporated in the present formalism by allowing the photon to have a small rest mass, which would be allowed to approach zero at the end of a calculation. By di Sessa's result above the net effect of this would evidently be simply the replacement of our $D(21)$ by the usual solution of Eq. (4.12):

$$D(21) \rightarrow s_2 s_1 / 4\pi^2 \left[(\mathbf{x}_2 - \mathbf{x}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1) - i\epsilon \right]. \qquad (4.20)$$

In such an approach the expansion

$$\frac{s_2 s_1}{4\pi^2 \left[(\mathbf{x}_2 - \mathbf{x}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1) - i\epsilon \right]}$$

$$= \sum_{lm} \int_0^\infty \frac{d\alpha}{4i\alpha \sin(\pi i\alpha)} Y_{\alpha lm}(2) Y^*_{\alpha lm}(1) \{ e^{\pi i i \alpha + i \alpha | \nu_2 - \nu_1 |} + e^{-\pi i i \alpha - i \alpha | \nu_2 - \nu_1 |} \} \qquad (4.21)$$

of the usual propagator in a series of Lorentz harmonics may prove useful.

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$$\cosh \rho_{21} = \hat{s}_2 \cdot \hat{s}_1 = \cosh |\nu_2 - \nu_1|$$
$$= \tfrac{1}{2} \exp(\nu_> - \nu_<) + \tfrac{1}{2} \exp[-(\nu_> - \nu_<)].$$

This is equivalent to the relation $2\hat{s}_> \cdot \hat{s}_< = (s_>/s_<) + (s_</s_>)$. Multiplying through by $s_> s_<$ and transposing gives

$$0 = (s_>)^2 - 2s_> s_< \hat{s}_> \cdot \hat{s}_< + (s_<)^2 = (s_> \hat{s}_> - s_< \hat{s}_<)^2.$$

Since $\mathbf{x} = s\hat{s}$, the result $(\mathbf{x}_2 - \mathbf{x}_1)^2 = 0$ follows.

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An inverse scattering transform for potentials of compact support[a]

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We consider the inverse scattering problem for the one-dimensional Schrödinger equation on the whole line,

$(d^2/dx^2)\phi(x,k)+[k^2-V(x)]\phi(x,k)=0.$

In some applications, as for example in the synthesis of electromagnetic media, it is important to have *sufficient* conditions on the scattering data such that the corresponding potential has compact support in some prescribed interval. The scattering data traditionally used in connection with the above ISP have been either one of the reflection coefficients (we are assuming that the potential V does not support bound states) r and $\tilde{r}$—from the left and right, respectively. Although it is easy to obtain simple conditions on r ($\tilde{r}$) to ensure cutoff of the potential on the left (right), conditions on r ($\tilde{r}$) that guarantee cutoff on the right (left) are too complicated to be of any practical value. In this paper, we propose to use new scattering data, namely the ratio r/t (where r is either one of the reflection coefficients and t is the transmission coefficient), and give necessary and sufficient conditions for the corresponding potential to have support contained in $[-a, a]$.

1. INTRODUCTION

The inverse scattering problem (ISP) for the Schrödinger equation on the entire real line,

$$\frac{d^2}{dx^2}\varphi(x,k)+[k^2-V(x)]\varphi(x,k)=0, \quad x\in(-\infty,\infty),$$

$$k\in\mathbb{R},$$ (1)

has received renewed attention, since the discovery of its application to nonlinear evolution equations.[1,2]

The ISP consists of deriving all possible information about the potential V from knowledge of quantities—called scattering data—related to the asymptotic form of certain solutions of (1). We refer to Faddeev's survey[3] for a detailed exposition of the ISP main results (for a bibliography on inverse problems, complete up to 1974, the interested reader might consult Refs. 4 and 5).

We shall assume throughout our exposition that the potential V does not support bound states.

In this paper, we shall be interested in finding necessary and sufficient conditions on some suitable scattering data for the corresponding potential V to have compact support.

Relations between the analyticity of the scattering data and the range of V have been discussed in the ISP literature, both for the radial and for the whole line equation. However, to the best of our knowledge, up to the present time no simple necessary *and sufficient* conditions *on primordial* scattering data have been obtained that ensure cutoff of V at *both* sides.

We propose to show that, for the purpose stated above, the function

$$\rho(k)=2ik\,r(k)/t(k)$$ (2)

is more convenient, as scattering data, than the usual reflection coefficient $r(k)$, and that it provides a simple answer to our problem.

2. THE CHOICE OF CONVENIENT SCATTERING DATA

We begin by establishing some terminology for our discussion. Let r and $\tilde{r}$ be the left and right reflection coefficients, respectively. Consider:

$$h(k)=r(k)\exp(-2ika),$$

$$\tilde{h}(k)=\tilde{r}(k)\exp(2ika),$$

and define

$$\partial H^2_\pm(\mathbb{R})=\{f\colon\mathbb{R}\to\mathbb{C}\mid f \text{ is the boundary value of some}$$

$$f_c\in H^2_\pm(\mathbb{R})\},$$

where

$$H^2_\pm(\mathbb{R})=\{f\colon\text{Im}z\gtrless 0\to\mathbb{C}\mid f \text{ is analytic in Im}z\gtrless 0$$

$$\text{and }\sup_{y\gtrless 0}\int_{-\infty}^{\infty}|f(x+iy)|^2dx<\infty\}$$

are the usual Hardy—Lebesgue spaces.

Then we can assemble known results[6] to state the following. (See *Note added in proof* at end of paper.)

Lemma: Let

$$V=\{V\in L^2(\mathbb{R})\mid\int_{-\infty}^{+\infty}(1+|x^2|)|V(x)|\ dx<\infty \text{ and}$$

$$\frac{-d^2}{dx^2}+V \text{ has an empty point spectrum in }L^2(\mathbb{R})\}.$$

Then $V\in V$ has compact support contained in $[-a,a]$ if and only if

$$h\in\partial H^2_+(\mathbb{R})$$ (3)

and

$$\tilde{h}\in\partial H^2_-(\mathbb{R}).$$ (3')

Apparently, this lemma would settle the question we raised in the Introduction—we now have necessary and sufficient conditions on some scattering data, ensuring that $V \in \cancel{V}$ has compact support.

However, these conditions involve *both* r and $\tilde{r}$, and we know that each of these, separately, determines V uniquely. One could argue that, since each reflection coefficient determines the other one, the condition on $\tilde{r}$ could be transposed as a condition on r. The problem is that the explicit relationship between r and $\tilde{r}$ is quite involved and we would be left with two conditions rather asymmetrical, one being very simple—condition (3)—the other very complicated.

At this point, we suggest discarding r and $\tilde{r}$ altogether and looking for some more convenient scattering data.

From the unitarity of the S-matrix, we have

$$\text{(for all real } k) \quad \frac{r(k)}{t(k)} = -\frac{\tilde{r}(-k)}{t(-k)}. \tag{4}$$

This relationship provides the hint needed to guide our choice. If we set:

$\rho(k) = 2ik\, r(k)/t(k)$, (left transition coefficient)

$\tilde{\rho}(k) = 2ik\, \tilde{r}(k)/t(k)$, (right transition coefficient)

we see that (4) implies,

$\tilde{\rho}(k) = \rho(-k)$,

and it is evident that, for $V_1(x) \equiv V(-x)$, the left transition coefficient is $\rho_1(k) = \tilde{\rho}(k)$, where $\tilde{\rho}$ is the right transition coefficient for V.

Hence, if we find a condition on $\rho(k)$ that implies $V(x) \equiv 0$ for $x < -a$, then this same condition on $\rho(-k)$ ensures $V(x) \equiv 0$ for $x > a$.

Therefore, the transition coefficient seems to be a convenient set of scattering data (we note that this was conjectured[7] in 1966) as far as ensuring simultaneous cutoff on both sides through symmetrical conditions—on $\rho(\pm k)$, a "single set" of scattering data— provided we can find such conditions.

Now define the application

$\cancel{L} : \cancel{V} \longrightarrow L^2(\mathbb{R}), \quad \cancel{L}(V) = \rho.$

It is a consequence of known results[6] (see *Note added in proof*) that $\cancel{L}$ is well defined and injective. In the next section we show that $\cancel{L}$ is indeed a suitable inverse scattering transform for our purpose of controlling the support of the potential.

3. FINITE RANGE POTENTIALS IN $\cancel{V}$

Theorem: $V \in \cancel{V}$ has support in $[-a, a]$ if and only if, $\cancel{L}(V) = \rho \in E_a$, where,

$E_a = \{ \rho \in L^2(\mathbb{R}) \,|\, \rho$ is the restriction to the real line

of an entire function of exponential type $\leq 2a\}$.

Outline of Proof: The "if" part of the theorem is an immediate consequence of the results in Ref. 6.

Let us assume that $\rho \in E_a \cap \cancel{L}(\cancel{V})$. Notice that if we can show that $\cancel{L}(V) = \rho \in E_a$ implies $V(x) \equiv 0$ for $x < -a$,

then it follows from the symmetry considerations made in Sec. 2 that $V(x) \equiv 0$ for $x > a$ also holds [indeed, ρ is in E_a if and only if, $k \longmapsto \rho(-k)$ is in E_a].

Therefore, from the lemma in Sec. 2, it suffices to show that $\rho \in E_a$ implies $h \in \partial H_+^2(\mathbb{R})$.

We begin by observing that if $g \in E_a$ then from a classical Paley—Wiener theorem, it follows that the function

$$f(k) = g(k) \exp(-2ika)$$

belongs to $\partial H_+^2(\mathbb{R})$. Now let us consider the two cases (i) and (ii).

(i) $t(0) \neq 0$: If we set $\rho(k) = 2ikg(k)$, then $g \in E_a$, and

$h(k) = g(k)\, t(k) \exp(-2ika)$.

Furthermore, from the observation above, $g \in E_a$ implies

$k \longmapsto g(k) \exp(-2ika) \in \partial H_+^2(\mathbb{R})$.

Since t is the boundary value of a function t_c, analytic in $\text{Im} k > 0$, continuous for $\text{Im} k \geq 0$, and bounded in $\text{Im} k \geq 0$, it follows that $h \in \partial H_+^2(\mathbb{R})$.

(ii) $t(0) = 0$: In this case, let $t(k) = 2ik\, q(k)$. Then

$h(k) = \rho(k)\, q(k) \exp(-2ika)$.

Mutatis mutandis the same argument used in case (i), substituting q for t, completes the proof of the theorem.

4. FINAL REMARKS

This result has some applications: (i) in the numerical treatment of the ISP, where it is of interest to have an estimate of the support of V.

(ii) in the synthesis of nonhomogeneous electromagnetic media. It has been shown[8,9] that this ISP constitutes an adequate model for many of these problems, such as the synthesis of nonuniform transmission lines, dielectric filters, etc. In such questions, realizability conditions are important—among these, conditions that ensure the device to be constructed to have finite length, which in the ISP model, means that V must have compact support.

As far as applications to the KdeV equation are concerned, our result, as it stands, seems to offer little interest. Indeed it is well known that if the solution to the KdeV initial-value problem is in $\cancel{V}$ at some instant t_0 and has compact support, then at any $t > t_0$ it no longer has compact support (this result is also an immediate consequence of the theorem of Sec. 3).

However, we are in the process of obtaining, using similar methods, conditions on ρ that ensure V to have a prescribed *decay* when $|x| \to \infty$. We note that, when considering the solution to the KdeV initial-value problem through ISP methods, one is not really restricted to working with the usual inverse scattering transform $V \longmapsto r(k)$. Indeed, any scattering data will do, as long as it determines V uniquely and its evolution in time is known. Our choice $V \longmapsto \rho(k)$ responds to both of these conditions and, in addition,

appears to be convenient for the description of the solution's behavior as $|x| \to \infty$, in terms of decay properties of the initial data.

Note added in proof: After the completion of this work we read a preprint of P. Deift and E. Trubowitz entitled "Inverse Scattering on the Line" which presents a thorough and deep analysis of the subject. In particular the authors point out some errors in Faddeev's,[6] and show that the characterization problem for the class of potentials

$$L_1^1 = \left\{ V \mid \int_{-\infty}^{+\infty} (1 + |x|) \, |V(x)| \, dx < \infty \right\}$$

is still open—this has also been noted[10] by K. Chadan and P. C. Sabatier. In the Deift—Trubowitz paper, the characterization problem for the class

$$L_2^1 = \left\{ V \mid \int_{-\infty}^{+\infty} (1 + x^2) \, |V(x)| \, dx < \infty \right\}$$

is completely solved. For this reason, we have used in our work the class L_2^1, instead of L_1^1.

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Super Clifford algebra

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An extension of su(2,2;2n) is presented which is based upon a theory of Z_4 gradings of Clifford algebras. In the example given su(2,2;4) is extended and the Clifford algebra of an internal six-dimensional space is manifested. It is suggested that super Clifford algebras of this type can give valuable insight into the study of su(2,2;2n) supersymmetry theories.

I. INTRODUCTION

A type of graded algebra with indices graded over Z_4 and Z is introduced. The example given contains as subalgebras the Clifford algebra[1] of the real orthogonal space $R^{2,4}$ and the graded Lie algebra[2] u(2, 2; 4). Taken together these subalgebras generate a super Clifford algebra which contains the Clifford algebra of an internal space $R^{6,0}$ (six-dimensional extensions of space—time have been considered in other contexts[3]).

In Secs. I through V the mathematics of the super Clifford algebra D is developed and in Table I a representation is presented over the 16-dimensional Dirac algebra. The representation is bijective so the dimension of D is $(4)(4)(16) = 256$.

In Sec. VI a copy of the Poincare algebra is identified with the aid of the imbeddings of 4-vectors in D. This is then used to construct objects which behave formally like twistors[4] but bear hidden internal indices. In Sec. VIII a Lagrangian is presented and its invariances shown.

As in Ref. 5 no use is made of a commuting imaginary scalar i except to demonstrate connections with more familiar treatments of spinors. All algebras presented are real and generators for U(1) phase transformations are drawn from D itself.

II. THE CLIFFORD ALGEBRAS

The Clifford algebra of the Minkowski space $R^{1,3}$ [signature $(- + + +)$] is the familiar algebra of Dirac operators. The sixteen basis elements of the algebra are here denoted $1, \gamma_\mu, \sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \gamma_\mu\gamma_5, \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_0$, and are obtained by taking antisymmetric products of k of the γ_μ for $k = 0, 1, 2, 3, 4$. (Greek indices will run from 0 to 3 throughout.) The generators of this algebra satisfy $\frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = -\eta_{\mu\nu}$ $= -(\text{diag}(- + + +))_{\mu\nu}$. Henceforth, unless a statement to the contrary is supplied, reference to any $n \times n$ representation will be understood to be over the Dirac algebra.

The Clifford algebra of the real orthogonal space $R^{2,4}$ has generators Γ_n, $n = 0, 1, 2, 3, 4, 5$ which are chosen to satisfy $\frac{1}{2}(\Gamma_i\Gamma_j + \Gamma_j\Gamma_i) = -(\text{diag}(- + + + - +))_{ij}$. The following 2×2 representation of this algebra will be used here:

$$\Gamma_\mu = \begin{bmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{bmatrix}, \quad \Gamma_4 = \begin{bmatrix} 0 & -\gamma_5 \\ \gamma_5 & 0 \end{bmatrix}, \quad \Gamma_5 = \begin{bmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{bmatrix}. \tag{1}$$

Again a complete basis for this Clifford algebra can be generated by taking antisymmetric products of k of the Γ_i for $k = 0, \ldots, 6$. The fifteen matrices $\Gamma_{ij} = \frac{1}{2}(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i)$ and $\Delta^{\text{ex}} = \Gamma_1\Gamma_2\Gamma_3\Gamma_0\Gamma_4\Gamma_5$ constitute a basis of the Lie algebra su(2, 2)×u(1).

The final Clifford algebra we need to consider is that of the real orthogonal space $R^{6,0}$. It has generators Λ_n, $n = 0, 1, 2, 3, 4, 5$ which satisfy $\frac{1}{2}(\Lambda_i\Lambda_j + \Lambda_j\Lambda_i)$ $= -(\text{diag}(- - - - - -))_{ij}$. The 2×2 representation of this algebra used here is determined by

$$\Lambda_0 = \begin{bmatrix} \gamma_0 & 0 \\ 0 & \gamma_0 \end{bmatrix}, \quad \Lambda_i = \begin{bmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{bmatrix} (i = 1, 2, 3),$$

$$\Lambda_4 = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}, \quad \Lambda_5 = \begin{bmatrix} C & 0 \\ 0 & -C \end{bmatrix}, \tag{2}$$

where $C = \gamma_0\gamma_5$. The fifteen matrices $\Lambda_{ij} = \frac{1}{2}(\Lambda_i\Lambda_j - \Lambda_j\Lambda_i)$ together with $\Delta^{\text{in}} = \Lambda_0\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5$ constitute a representation of the Lie algebra su(4)×u(1)≈so(6) ×u(1) which was the internal part of the second supersymmetry of Ref. 5.

III. Z_4 GRADINGS OF CLIFFORD ALGEBRAS

Let C be a Clifford algebra for the real orthogonal space $R^{p,q}$ (signature $p(-)q(+)$) and let g_i, $i = 1, 2, \ldots, p + q = n$ be the generators of C satisfying

$$\frac{1}{2}(g_ig_j + g_jg_i) = -(\text{diag}(p(-)q(+)))_{ij}. \tag{3}$$

Let C_k be the $\binom{n}{k}$ dimensional subspace of C with a basis consisting of the antisymmetric products of k of the g_i. For $s = 0, 1, 2, 3$ define $A_s = C_s \oplus C_{s+4} \oplus C_{s+8} \oplus \cdots$ and let x be in A_i and y in A_j. Define a product on C by $x \circledast y = \frac{1}{2}(xy + (-)^{ij}yx)$. It is not difficult to show that $x \circledast y$ is an element of A_m where $m = (i + j) \mod 4$ so that C becomes a Z_4 graded algebra with this product which we shall call the $Z_4^\circledast$ algebra on $R^{p,q}$ [or $(Z_4^\circledast)^{p,q}$].

Define $B_0 = A_2, B_1 = A_1, B_2 = A_0, B_3 = A_3$ and let x be in B_i and y in B_j. Define another product on C by $x * y = \frac{1}{2}(xy - (-)^{ij}yx)$. $x * y$ is an element of B_m where $m = (i + j) \mod 4$ so that this product also defines a Z_4 graded structure on C which we shall call the Z_4^* algebra on $R^{p,q}$ [or $(Z_4^*)^{p,q}$ with graded subspaces $((Z_4^*)^{p,q})_s$ for $s = 0, 1, 2, 3$].

IV. SUPER CLIFFORD ALGEBRA

D is a real algebra partitioned into subspaces D_{ij} the indices of which are graded over Z_4 and Z (although D_{ij} is trivial for $|j| > 2$). A 4×4 representation of D is presented in table I in which the graded product (also denoted *) is determined by

$$x * y = \tfrac{1}{2}(xy - (-)^{ik+jm}yx)$$

for x in D_{ij} and y in D_{km} [so that $x * y$ is an element of D_{pq} for $p = (i + k)\bmod 4$ and $q = j + m$]. Another product on D is determined by

$$x \oplus y = \tfrac{1}{2}(xy + (-)^{ik+jm}yx).$$

$x \oplus y$ is an element of D_{ab} for $a = (i + k + 2)\bmod 4$ and $b = j + m$ so that it respects a different grading on the Z_4 index.[6] Let D_k and $D_{,m}$ be the unions of the subspaces D_{km} for fixed k and m, respectively. (D_0 is the supersymmetry of Ref. 5 characterized by an internal subalgebra su(4)×u(1) [see Table I]).

Two important subalgebras of D are D^{ex} (external) and D^{in} (internal). D^{ex} is essentially the Z_4^* algebra on $R^{2,4}$. In Table I one sees how the 2×2 representation of $(Z_4^*)^{2,4}$ determined by (1) is mapped onto D^{ex}. The adjoining of the odd components $D_{0,\pm1}$ from Ref. 5 to D^{ex} determines the rest of D. In particular the internal subalgebra D^{in} of D appears. It is isomorphic to the Z_4^* algebra on $R^{6,0}$. The isomorphism between the 2×2 representation of $(Z_4^*)^{6,0}$ determined by (2) and the 4×4 representation of D^{in} is given in Table I [note that the respective images of $((Z_4^*)^{6,0})_s$, $s = 1, 3$ are the subspaces D_{j0}, $j = 3, 1$ of D]. Finally note that D^{ex} and D^{in} cannot interact through the algebra product without an intermediate in $D_{,\pm1}$.

V. CONJUGATION

Let C be the Clifford algebra of the general space $R^{p,q}$ as in Sec. III. Conjugation on C is an anti-involution determined by the conditions $g_i^- = - g_i$ for the generators of C, and $(uv)^- = v^- u^-$ for arbitrary elements u and v of C. This implies that if u is in $((Z_4^*)^{p,q})_s$ then $u^- = - u$

TABLE I. The basic of D is presented. The subspaces D^{ex}, D^{in}, and $D_{,\pm1}$ are separately displayed.

D^{ex}. In the table below the 2×2 matrices listed correspond to 4×4 matrices in D^{ex} in the following way:

$$\begin{bmatrix} x & u \\ v & y \end{bmatrix} \rightarrow \begin{bmatrix} x & 0 & 0 & u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ v & 0 & 0 & y \end{bmatrix}.$$

D^{ex}_{nm}

	$m = -2$	$m = 0$	$m = 2$
$n = 0$	$\begin{bmatrix} 0 & 0 \\ \gamma_5\gamma_\mu & 0 \end{bmatrix}$	$\begin{bmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \sigma_{\mu\nu} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{bmatrix}$	$\begin{bmatrix} 0 & \gamma_5\gamma_\mu \\ 0 & 0 \end{bmatrix}$
$n = 1$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \gamma_5 & 0 \end{bmatrix}$	$\begin{bmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{bmatrix}\begin{bmatrix} \gamma_5\gamma_\mu & 0 \\ 0 & -\gamma_5\gamma_\mu \end{bmatrix}$	$\begin{bmatrix} 0 & \gamma_5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
$n = 2$	$\begin{bmatrix} 0 & 0 \\ \gamma_\mu & 0 \end{bmatrix}$	$\begin{bmatrix} \sigma_{\mu\nu} & 0 \\ 0 & -\sigma_{\mu\nu} \end{bmatrix}\begin{bmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & \gamma_\mu \\ 0 & 0 \end{bmatrix}$
$n = 3$	$\begin{bmatrix} 0 & 0 \\ \sigma_{\mu\nu} & 0 \end{bmatrix}$	$\begin{bmatrix} \gamma_5\gamma_\mu & 0 \\ 0 & \gamma_5\gamma_\mu \end{bmatrix}\begin{bmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{bmatrix}$	$\begin{bmatrix} 0 & \sigma_{\mu\nu} \\ 0 & 0 \end{bmatrix}$

Action of conjugation on D^{ex}:

$$\begin{bmatrix} x & u \\ v & y \end{bmatrix}^- = \begin{bmatrix} y^- & -u^- \\ -v^- & x^- \end{bmatrix}$$

D^{in}. The correspondence between the following 2×2 matrices and 4×4 matrices in D^{in} is:

$$\begin{bmatrix} x & u \\ v & y \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x & u & 0 \\ 0 & v & y & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

D^{in}_{nm} (subscripts i and j range from 1 to 3)

$m = 0$

$n = 0$	$\begin{bmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{bmatrix}$	$\begin{bmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{bmatrix}$	$\begin{bmatrix} \sigma_{ij} & 0 \\ 0 & \sigma_{ij} \end{bmatrix}$	$\begin{bmatrix} \sigma_{ij} & 0 \\ 0 & -\sigma_{ij} \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \sigma_{ij} \\ \sigma_{ij} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \sigma_{0i} \\ -\sigma_{0i} & 0 \end{bmatrix}$
$n = 1$	$\begin{bmatrix} \gamma_i & 0 \\ 0 & \gamma_i \end{bmatrix}$	$\begin{bmatrix} \gamma_i & 0 \\ 0 & -\gamma_i \end{bmatrix}$	$\begin{bmatrix} \gamma_5\gamma_i & 0 \\ 0 & \gamma_5\gamma_i \end{bmatrix}$	$\begin{bmatrix} \gamma_5\gamma_i & 0 \\ 0 & -\gamma_5\gamma_i \end{bmatrix}$	$\begin{bmatrix} 0 & C \\ -C & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \gamma_0 \\ -\gamma_0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \gamma_5\gamma_i \\ \gamma_5\gamma_i & 0 \end{bmatrix}$
$n = 2$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} \sigma_{0i} & 0 \\ 0 & \sigma_{0i} \end{bmatrix}$	$\begin{bmatrix} \sigma_{0i} & 0 \\ 0 & -\sigma_{0i} \end{bmatrix}$	$\begin{bmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \sigma_{ij} \\ -\sigma_{ij} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \sigma_{0i} \\ \sigma_{0i} & 0 \end{bmatrix}$

$n = 3$
$$\begin{bmatrix} \gamma_0 & 0 \\ 0 & \gamma_0 \end{bmatrix} \quad \begin{bmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \quad \begin{bmatrix} C & 0 \\ 0 & -C \end{bmatrix} \quad \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \quad \begin{bmatrix} 0 & \gamma_5\gamma_i \\ -\gamma_5\gamma_i & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \gamma_0 \\ \gamma_0 & 0 \end{bmatrix} \quad \begin{bmatrix} \gamma_0 & 0 \\ 0 & -\gamma_0 \end{bmatrix}$$

Action of conjugation on $\mathcal{D}^{\mathrm{in}}(x^{\hat{}} = \gamma_0 x^{-} \gamma_0)$:

$$\begin{bmatrix} x & u \\ v & y \end{bmatrix}^{-} = \begin{bmatrix} x^{\hat{}} & v^{\hat{}} \\ u^{\hat{}} & y^{\hat{}} \end{bmatrix}$$

$\mathcal{D}_{,*1}$. The indices μ and α are independent so that the matrices below each signify eight basis elements.

$\mathcal{D}_{n,+1}$

$n = 0$
$$\begin{bmatrix} 0 & \gamma_\mu & \gamma_\alpha & 0 \\ 0 & 0 & 0 & C\gamma_\mu \\ 0 & 0 & 0 & C\gamma_\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \gamma_5\gamma_\alpha & \gamma_5\gamma_\alpha & 0 \\ 0 & 0 & 0 & -C\gamma_5\gamma_\mu \\ 0 & 0 & 0 & -C\gamma_5\gamma_\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$n = 1$
$$\begin{bmatrix} 0 & \gamma_\mu C & \gamma_\alpha C & 0 \\ 0 & 0 & 0 & \gamma_\mu \\ 0 & 0 & 0 & \gamma_\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -\gamma_5\gamma_\mu C & -\gamma_5\gamma_\alpha C & 0 \\ 0 & 0 & 0 & \gamma_5\gamma_\mu \\ 0 & 0 & 0 & \gamma_5\gamma_\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$n = 2$
$$\begin{bmatrix} 0 & \gamma_\mu & \gamma_\alpha & 0 \\ 0 & 0 & 0 & -C\gamma_\mu \\ 0 & 0 & 0 & -C\gamma_\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \gamma_5\gamma_\mu & \gamma_5\gamma_\alpha & 0 \\ 0 & 0 & 0 & C\gamma_5\gamma_\mu \\ 0 & 0 & 0 & C\gamma_5\gamma_\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$n = 3$
$$\begin{bmatrix} 0 & -\gamma_\mu C & -\gamma_\alpha C & 0 \\ 0 & 0 & 0 & \gamma_\mu \\ 0 & 0 & 0 & \gamma_\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \gamma_5\gamma_\mu C & \gamma_5\gamma_\alpha C & 0 \\ 0 & 0 & 0 & \gamma_5\gamma_\mu \\ 0 & 0 & 0 & \gamma_5\gamma_\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{D}_{,-1}$. The map
$$\begin{bmatrix} 0 & x & y & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ x^{-} & 0 & 0 & 0 \\ y^{-} & 0 & 0 & 0 \\ 0 & -u^{-} & -v^{-} & 0 \end{bmatrix} \text{ of } \mathcal{D}_{n,+1} \rightarrow \mathcal{D}_{n,-1}$$

is a linear isomorphism.

Action of conjugation on $\mathcal{D}_{,\pm1}$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & u & v & 0 \end{bmatrix}^{-} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \gamma_0 u^{-} & 0 & 0 & 0 \\ \gamma_0 v^{-} & 0 & 0 & 0 \\ 0 & -x^{-}\gamma_0 & -y^{-}\gamma_0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & x & y & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \end{bmatrix}^{-} = \begin{bmatrix} 0 & u^{-}\gamma_0 & v^{-}\gamma_0 & 0 \\ 0 & 0 & 0 & -\gamma_0 x^{-} \\ 0 & 0 & 0 & -\gamma_0 y^{-} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for $s = 0, 1$ and $u^{-} = u$ for $s = 2, 3$. If $x = x^i g_i$ then Eq. (3) implies $(x, y) = x * y^{-}$ where (x, y) is the $R^{\rho,q}$ inner product of x and y.

Conjugation thus defined on $\mathcal{D}^{\mathrm{ex}}$ and $\mathcal{D}^{\mathrm{in}}$ can be extended to all of $\mathcal{D}$ (uniquely up to sign). Its action on $\mathcal{D}$ is presented in Table I.

VI. 4-VECTORS

In this section 4-vectors will be imbedded in $\mathcal{D}^{\mathrm{ex}}$. These imbeddings can be used to assign explicit conformal transformations to elements of $\mathcal{D}^{\mathrm{ex}}$. We shall concentrate on the Poincare group here. Since $\mathcal{D}^{\mathrm{in}}$ and $\mathcal{D}_{,\pm1}$ play no part in this the 2×2 representation of $\mathcal{D}^{\mathrm{ex}}$ in Table I and Sec. II will be used and the imbedding into $\mathcal{D}$ will be understood.

Let x^μ be a space—time 4-vector and $x = x^\mu \gamma_\mu$. Define

$$X_1(x) = \begin{bmatrix} x & \gamma_5 x^2 \\ \gamma_5 & x \end{bmatrix} \text{ in } \mathcal{D}_1^{\mathrm{ex}},$$

$$X_2(x) = \begin{bmatrix} \frac{1}{2}\gamma_5 & x \\ 0 & \frac{1}{2}\gamma_5 \end{bmatrix} \text{ in } \mathcal{D}_2^{\mathrm{ex}}.$$

The Poincare group is then generated by the matrices $\Gamma_{\mu\nu} = \Gamma_\mu \otimes \Gamma_\nu$ (Lorentz transformations) and $T_\mu = \frac{1}{2}(\Gamma_{5\mu} - \Gamma_{4\mu})$ (translations). For example, if $v = v^\mu \gamma_\mu$, then

$$\exp(v^\mu T_\mu) X_1(x) \exp(-v^\nu T_\nu)$$

$$= \begin{bmatrix} 1 & \gamma_5 v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & \gamma_5 x^2 \\ \gamma_5 & x \end{bmatrix} \begin{bmatrix} 1 & -\gamma_5 v \\ 0 & 1 \end{bmatrix} = X_1(x + v).$$

Similarly $X_2(x) \mapsto X_2(x + y)$. X_2 has the advantage that it commutes with Δ^{ex} which is the generator of phase transformations in $\mathcal{D}$ (that is, Δ^{ex} takes the place of the imaginary unit i).

Momentum 4-vectors obviously cannot be imbedded in the same way. Let p^μ be the momentum 4-vector of a particle and $p = p^\mu \gamma_\mu$. Let $J = J^{\mu\nu}\sigma_{\mu\nu}$, where $J^{\mu\nu}$ is the antisymmetric angular momentum tensor of the parti-

cle defined so that if the particle is spinless with 4-position $x = x^\mu \gamma_\mu$ then $J = \frac{1}{2}(xp - px)$.

Define

$$P_1(p, J) = \begin{bmatrix} \gamma_5 p & 2J \\ 0 & \gamma_5 p \end{bmatrix} \text{ in } D_3^{ex},$$

$$P_2(p) = \begin{bmatrix} 0 & \gamma_5 p \\ 0 & 0 \end{bmatrix} \text{ in } D_0^{ex}.$$

These matrices are consistent with the actions of $\Gamma_{\mu\nu}$ and T_μ defined above. For example,

$$\exp(y^\mu T_\mu) P_1(p, J) \exp(-y^\nu T_\nu) = P_1(p, J + \tfrac{1}{2}(vp - py)).$$

Note that P_2 commutes and P_1 anticommutes with the phase generator Δ^{ex}.

It should be obvious that the choice of T_μ to represent translations was to some extent arbitrary. We can obtain a representation of translations and 4-vector imbeddings dual to that given above by applying a conjugate transpose anti-involution on elements of D. For example,

$$P_1^d(p, J) = \begin{bmatrix} \gamma_5 p & 0 \\ -2J & \gamma_5 p \end{bmatrix} \text{ and } P_2^d(p) = \begin{bmatrix} 0 & 0 \\ \gamma_5 p & 0 \end{bmatrix}.$$

The dual translation generator is $T_\mu^d = -\frac{1}{2}(\Gamma_{5\mu} + \Gamma_{4\mu})$.

VII. SPINORS AND TWISTORS

A spinor in D is an element of $D_{\pm 1}$. Henceforth π will refer to an object linear in γ_μ and $\gamma_5 \gamma_\nu$ and ω will be linear in 1, γ_5, and $\sigma_{\mu\nu}$. Represent the γ-matrices by the following 4×4 complex matrices:

$$\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma_k = \begin{bmatrix} 0 & i\sigma_k \\ i\sigma_k & 0 \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

where the Pauli matrices are

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

$$\sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(Note—the algebra is still real despite the complex representation.) The matrices π and ω take the forms

$$\pi = \begin{bmatrix} \pi_1 & -\pi_2^* & \pi_3 & -\pi_4^* \\ \pi_2 & \pi_1^* & \pi_4 & \pi_3^* \\ \pi_3 & -\pi_4^* & -\pi_1 & \pi_2^* \\ \pi_4 & \pi_3^* & -\pi_2 & -\pi_1^* \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_1 & -\omega_2^* & -\omega_3 & \omega_4^* \\ \omega_2 & \omega_1^* & -\omega_4 & -\omega_3^* \\ \omega_3 & -\omega_4^* & \omega_1 & -\omega_2^* \\ \omega_4 & \omega_3^* & \omega_2 & \omega_1^* \end{bmatrix}.$$

These are Weyl spinors with internal indices.

The Hermitian transposes of these matrices satisfy

$\pi^\dagger = -\pi^\wedge$ and $\omega^\dagger = \omega^\wedge$ so that the adjoint spinors are $\bar{\pi} = \pi^\dagger \gamma_0 = -\pi^\wedge \gamma_0 = -\gamma_0 \pi^-$ and $\bar{\omega} = \omega^\dagger \gamma_0 = \omega^\wedge \gamma_0 = \gamma_0 \omega^-$.

Therefore,

$$\begin{bmatrix} 0 & \omega & \pi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \pi' & \omega' & 0 \end{bmatrix}^- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\bar{\pi}' & 0 & 0 & -\bar{\omega} \\ \bar{\omega}' & 0 & 0 & \bar{\pi} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(see Table I).

We can construct in D objects that behave at least formally like twistors. Define

$$S_1(\pi, \omega) = \begin{bmatrix} 0 & \omega & 0 & 0 \\ \gamma_5 \bar{\pi} & 0 & 0 & \gamma_5 \bar{\omega} \\ 0 & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \end{bmatrix} \text{ in } D_1,$$

$$S_2(\pi) = \begin{bmatrix} 0 & \pi & 0 & 0 \\ 0 & 0 & 0 & \gamma_5 \bar{\pi} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ in } D_2,$$

and let $p = p^\mu \gamma_\mu$ and $J = J^{\mu\nu} \sigma_{\mu\nu}$ be as in the last section with the added condition that $p^2 = 0$ (massless particle). The translation generator T_μ generates the following actions on the matrices S_k:

$$S_1(\pi, \omega) \mapsto S_1(\pi, \omega + \gamma_5 v \pi) \quad \text{and} \quad S_2(\pi) \mapsto S_2(\pi), \qquad (4)$$

where $v = v^\mu \gamma_\mu$ defines the translation.

Suppose $\pi\bar{\pi} = p$ and $\bar{\pi}\pi = 0$ [these conditions are met by $\pi = \exp(\gamma_5 \theta) p(2p_0)^{-1/2}$], and $-\frac{1}{2}(\omega\bar{\pi} - \pi\bar{\omega})\gamma_5 = J$. Then $S_1(\pi, \omega) * S_2(\pi) = -\frac{1}{2}P_1(p, J)$ and $S_2(\pi) * S_2(\pi) = -P_2(p)$. (5)

The transformations (4) and the relations (5) are formally similar to transformations and relations on the components of twistors as they are commonly presented.[4]

VIII. A LAGRANGIAN

The operator $D = P_2(\not{p})(\not{p} = \partial^\mu \gamma_\mu)$ and its dual will be used in the construction of a simple Lagrangian for a massless spinor field. The field $\psi(x)$ in plane-wave form with momentum spinor π is imbedded in D as follows:

$$\Psi = \exp(\Delta^{ex} p^\mu x_\mu) S_2(\pi) \exp(-\Delta^{ex} p^\nu x_\nu)$$

$$= S_2(\exp(\gamma_5 p^\mu x_\mu)\pi) = S_2(\psi(x)).$$

The Lagrangian will be self-dual $(L^d = L)$ and strictly contained in D^{in}. Its invariances include the Poincare group and internal $SU(2) \times U(1)$. Define

$$L = (-1/4)\Delta^{in}[\Psi(D^d\Psi) - (\Psi D^d)\Psi + \Psi^d(D\Psi^d) - (\Psi^d D)\Psi^d]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $L = (-1/4)\gamma_5[\bar{\psi}(\not{\partial}\psi) - (\overline{\not{\partial}\psi})\psi - \gamma_0(\bar{\psi}(\not{\partial}\psi) - (\overline{\not{\partial}\psi})\psi)\gamma_0]$.

If $u = \exp(\theta^{ij}\sigma_{ij})$ (sums from 1 to 3) and $v = \exp(\gamma_5\phi)$ then

$$W = UV = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & uv & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an SU(2)×U(1) transformation which acts nontrivially on Ψ [i.e., $WS_2(\psi)W^{-1} = S_2(\psi v^{-1}u^{-1})$]. The action of W on $\Psi^d(D\Psi^d)$ is $(W^d)^{-1}\Psi^d W^d (WDW^{-1}(W^d)^{-1}\Psi^d W^d)$. Since external

and internal elements commute and V cancels this action reduces to $U(\Psi^d(D\Psi^d))U^{-1}$. The action of W on $\underline{L}$ is therefore $U\underline{L}U^{-1}$ or $L \to uLu^{-1}$. L is of the form $A + A^{\wedge} + \gamma_0 A\gamma_0 + A^-$ which implies L is a real-valued function multiplying the identity. Therefore, $uLu^{-1} = L$. Poincaré invariance is similarly shown. The equation of motion of the field ψ is $\gamma_5\not{\partial}\psi = 0$.

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The linear representations of $\overline{SL}$(3,R) on a Banach space

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All continuous irreducible linear representations of $\overline{SL}$(3,R) on a Banach space are studied and classified, using successively infinitesimal methods and methods of induced representations. Irreducibility and equivalence criteria are given in terms of the values of the invariants of the representation as well as in terms of the inducing characters. The reduction into irreducible factors D^j of the restriction to SU(2) is carried out and the multiplicity of each D^j is determined. A necessary and sufficient condition for an irreducible representation to be equivalent to a unitary representation is also established.

1. INTRODUCTION

We are interested in studying the irreducible representations (IR's) of the universal (twofold) covering group $\overline{SL}(3,\mathbb{R})$ of the unimodular linear group SL(3, $\mathbb{R}$) (or, equivalently, the projective representations of the latter) in a Banach space. This problem has quite often been treated from different viewpoints,[1-5] but it has not been completely solved. The general theory of representations of noncompact Lie groups, developed by Gel'fand, Graev, Naĭmark, and their collaborators (see, for example, Ref. 6) has solved the problem of determining the principal series of representations of semisimple groups by constructive methods, using functional spaces over coset spaces of the group; unfortunately, there is no method known so far which yields systematically all representations of such groups. One of the most powerful results in the domain is due to Harish-Chandra[7] who has proved that all unitary IR's of a noncompact semisimple Lie group appear as factors of induced representations on functional spaces over the maximal compact subgroup. However this is not a very amenable tool because it is not immediate, technically speaking, to determine all possible unitary kernels of a given compact group. Indeed, in the case of $\overline{SL}(3,\mathbb{R})$, in the most exhaustive study of its UIR's due to Šijački,[4] the author is led to make a restrictive conjecture about the form of the SU(2) kernels; so that in spite of the abundance of his results there is no proof of their exhaustiveness.

Our method of investigation is based on the study of the Harish-Chandra module of the Lie algebra which decomposes to a direct sum of so(3) modules and to the study of the centralizer of so(3) as well as of ladder operators from one so(3) module to another. This procedure is justified by the fact that an irreducible module must be monogeneous; so that if the IR of $G = \overline{SL}(3,\mathbb{R})$ operates on the Banach space β, and if f is a C^ω vector of β, then $\mathcal{U}f$ is dense in β ($\mathcal{U}$ denoting the universal enveloping algebra of our group). Using this remark we see that one needs only to consider the Lie algebra action on a dense subspace of β, which we shall denote by β_{alg}, and which is isomorphic to a quotient module of $\mathcal{U}$ by some left ideal $\mathcal{I}$ (in fact the ideal which annihilates f). The action of $\mathcal{U}$ on β_{alg} defines the representation within the so-called *infinitesimal equivalence*; once it is known, one can define β as a completion of β_{alg} by introducing topology.

This procedure leads to a lack of information if stopped here; indeed, even if the ideal $\mathcal{I}$ is sufficiently large (in particular if the center of $\mathcal{U}$ is represented by scalars), there is no standard method to decide whether $\mathcal{U}/\mathcal{I}$ is an irreducible module; so one must axiomatically identify β_{alg} not to $\mathcal{U}/\mathcal{I}$ but to $\mathcal{U}/\mathcal{I}'$, where $\mathcal{I}'$ is the maximal proper ideal containing $\mathcal{I}$. On the other hand, it is not sure at all, and in fact not true, that the representation obtained for general $\mathcal{I}$ is integrable to the group, topology having been explicitly excluded from the beginning.

The consideration of the maximal compact subgroup provides us with a strong classification principle: Since all Banach representations of a compact group are known to be finite dimensional, and since, in our case, the dimension defines the representation, the classification criterion will be the dimension $2l + 1$ of the lowest dimensional irreducible component. The number l will be called (improperly) *the lowest weight of the representation* (of G, or $\mathcal{U}$, with which we are dealing). The lowest weight considerations combined with the fixed values taken by the invariants, enable us to classify the irreducible modules β_{alg} and to obtain equivalence and irreducibility criteria (Theorem 1).

To obtain necessary conditions for unitarity (which prove in fact to be sufficient) when working inside β_{alg} we first grant β_{alg} with a pre-Hilbert structure, then write down the inequalities due to the positiveness of the scalar product for a lowest weight vector φ and its immediate successors $X\varphi$ through the ladder operators X, following the method used by Bargmann[8] for SL(2, $\mathbb{R}$) and Naĭmark[9] for SL(2, $\mathbb{C}$). These criteria are established in Theorem 2 ($=$ 2A + 2B + 2C + 2D).

It remains to show (1) that the representations defined on β_{alg} are indeed integrable to group representations and (2) that the unitarity criteria are not only necessary but also sufficient. Although general theorems are available for (1) (Lepowsky has recently proved that Harish-Chandra modules yield integrable representations), and though a critical survey of UIR's already constructed by the various authors would suffice, we chose to construct again from the beginning the induced representations of G and seek irreducible components with the help of Theorem 1. This method has the advantage of giving the j-multiplicity for every UIR D^j of SU(2), a result which is quite complicated to

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obtain using infinitesimal methods only. The sufficient condition for unitarity is established by considerations on a two-step induction.

This paper is organized as follows: Sec. 2 contains the study of the properties of an algebra of so(3) tensors and introduces ladder operators through the techniques of reduction of a tensor product of so(3) modules, without any considerations on sl(3, $\mathbb{R}$). In Sec. 3 we study the universal enveloping algebra U of sl(3, $\mathbb{R}$) as an so(3) module. In Sec. 4 the results of the two previous sections are combined to yield lowest weight considerations which are summarized in Theorem 1. The sufficient conditions for unitarity, separated into four cases according to lowest weights, are established in Sec. 5 and Theorem 2A + 2B + 2C + 2D. The study of induced representations is carried out in Sec. 6 and all related results are established there. In the concluding Sec. 7 we compare our results with previous ones and we give some explicit formulas in the Appendix.

2. BEHAVIOR OF SO(3) TENSORS

Let U be an associative algebra which contains U_c, the universal enveloping algebra of so(3). The generators of so(3) can be identified with the three independent components $J_{kl} = - J_{lk}$ $(k,l = 1,2,3)$ of the skew symmetric two-rank tensor J, and we shall denote by w the Casimir element of U_c putting $w = \frac{1}{2}J_{kl}J_{lk}$ (we adopt the Einstein summation condition for *latin* letters only, throughout this paper). We assume furthermore that the elements of U are linear combinations of tensors, so that the bracket representation of so(3) on U decomposes into the direct sum of finite odd dimensional representations. The bracket representation (b.r.) on tensors T of rank one is given by

$$[J_{kl}, T_j] = \delta_{lj}T_k - \delta_{kj}T_l \qquad (2.1)$$

and the b.r. on tensors of rank n is the tensor product of n times the b.r. on tensors of rank one.

Let R be a representation of U on some space S. If the decomposition of S into eigenspaces of w is known and if one wants to know how U/U_c acts on these eigenspaces, one has to diagonalize the operator Adw defined by $\text{Adw}(X) = [w, X]$; this problem is of course equivalent to the reduction of the tensor product of two representations of so(3).

We shall restrict ourselves to tensors of rank two (noticing however that similar formulas can be derived for tensors of any rank). From 2.1 one obtains

$$\tfrac{1}{2}[w, E_{kl}] = E'_{kl} = J_{ki}E_{il} - E_{ki}J_{il}. \qquad (2.2)$$

Since we expect contracted tensors, traceless symmetric tensors, and skew symmetric tensors to behave differently, and since rank two tensors form an algebra under contraction, we introduce the following notations: E will denote the tensor with components E_{kl}, ^tE the transposed one, with components $(^tE)_{kl} = E_{lk}$, EF will denote the contracted product with components $(EF)_{kl}$ $= E_{ki}F_{il}$, and the Kronecker δ_{kl} symbol will be written 1 or, most often, omitted. Contracted indices shall also be omitted and we shall write $\langle E \rangle$ instead of $\frac{1}{2}E_{kk}$. We shall also put

$$\mathbf{E} \times \mathbf{F} = \mathbf{EF} + {}^t(\mathbf{EF}) - \tfrac{4}{3}\langle \mathbf{EF} \rangle, \quad \mathbf{E} \wedge \mathbf{F} = \mathbf{EF} - {}^t(\mathbf{EF}).$$
$$(2.3)$$

If, in particular, we put $\mathbf{E} = 1$ (or $\mathbf{F} = 1$) in (2.3), we obtain the traceless symmetric and skew-symmetric parts of the tensor $\mathbf{F}$ (or $\mathbf{E}$). Finally we shall denote the contracted bracket by

$$[E;F] = \mathbf{EF} - {}^t({}^t\mathbf{F}{}^t\mathbf{E}) = - {}^t[{}^t\mathbf{F}; {}^t\mathbf{E}]. \qquad (2.4)$$

The following two formulas can be derived by considering the range $\{1,2,3\}$ of the indices. First let A, B, C be traceless tensors of rank two; one has

$$A_{\alpha i}B_{ik}C_{k\beta} + A_{ik}B_{k\beta}C_{\alpha i} + A_{k\beta}B_{\alpha i}C_{ik}$$
$$\quad + A_{k\beta}B_{ik}C_{\alpha i} + A_{ik}B_{\alpha i}C_{k\beta} + A_{\alpha i}B_{k\beta}C_{ik}$$
$$= \delta_{\alpha\beta}(A_{ii}B_{ik}C_{kl} + A_{il}B_{ki}C_{lk}) + A_{\alpha\beta}B_{kl}C_{lk}$$
$$\quad + A_{ik}B_{\alpha\beta}C_{kl} + A_{kl}B_{lk}C_{\alpha\beta}. \qquad (2.5)$$

Next let K, L be skew-symmetric tensors of rank two; one has

$$K_{ij}L_{mn} = \delta_{jm}U_{in} - \delta_{jn}U_{im} - \delta_{im}U_{jn} + \delta_{in}U_{jm}$$

with $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.6)$

$$U_{jm} = K_{jh}L_{hm} - \delta_{jm}\tfrac{1}{2}\langle KL \rangle.$$

Let us now examine Adw. We first observe:

Fact 1: The set of elements of U which commute with so(3) is a subalgebra X of U. The center C of X contains the algebraic span of Z and Z_c, where Z is the center of U, and Z_c is that of U_c. The elements of X are exactly the tensors of rank zero (i.e., the fully contracted ones).

Fact 2: There is an so(3) module isomorphism between tensors of rank one and skew-symmetric tensors of rank two given by

$$K_{ij} = \epsilon_{ijh}k_h, \quad k_h = \tfrac{1}{2}\epsilon_{hij}K_{ij}. \qquad (2.7)$$

Now, having introduced K' in (2.2), one easily establishes for skew-symmetric K:

$$\mathbf{K}' = \mathbf{J} \wedge \mathbf{K} - \mathbf{K} = - \mathbf{K} \wedge \mathbf{J} + \mathbf{K}, \qquad (2.8a)$$

$$\mathbf{K}'' = \mathbf{K}w + \mathbf{K}' - \mathbf{J}\langle \mathbf{KJ} \rangle, \qquad (2.8b)$$

$$\mathbf{K}' \wedge \mathbf{J} = - \mathbf{K}w + \mathbf{J}\langle \mathbf{KJ} \rangle. \qquad (2.8c)$$

Assume now that U acts on S by the representation R, S being the direct sum of S_j such that S_j is the eigenspace of w for the eigenvalue $j(j+1)$. (Of course there is no distinction algebraically possible between j and $-j-1$.) Then $\frac{1}{2}\text{Adw}$ acting on K is represented by the matrix

$$\begin{pmatrix} 0 & j^2 + j & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

which has the eigenvalues $j+1$, 0, and $-j$. Except for the two values $w = 0$ and $w = -\frac{1}{4}$, the three eigenvalues are distinct, so Adw is diagonalizable and K splits to three components $\mathbf{K}^+$, $\mathbf{K}^-$, $\mathbf{K}^0 = w^{-1}\langle \mathbf{KJ} \rangle \mathbf{J}$, such that $\mathbf{K}^\alpha S_j \subset S_{j+\alpha}$. The two components $\mathbf{K}^+$, $\mathbf{K}^-$ are j-dependent and transform to one another by the transformation $j \to -j-1$. We have the following formulas for $w(w + \tfrac{1}{4}) \neq 0$:

$$K = K^+ + xw^{-1}J + K^- \quad \text{with } x = \langle KJ \rangle, \tag{2.9a}$$

$$K' = (j+1)K^+ - jK^-, \tag{2.9b}$$

and conversely

$$(2j+1)K^+ = jK + K' - (j+1)^{-1}xJ$$

$$= (j+1)K - K \wedge J - (j+1)^{-1}xJ, \tag{2.10a}$$

$$(2j+1)K^- = (j+1)K - K' - j^{-1}xJ$$

$$= jK + K \wedge J - j^{-1}xJ. \tag{2.10b}$$

If Z is another skew-symmetric tensor with $\langle ZJ \rangle = y$ we can form the quantities $\langle Z^+K^- \rangle$ and $\langle Z^-K^+ \rangle$ which commute with w:

$$\langle Z^-K^+ \rangle = (2j+1)^{-1}[(j+1)\langle ZK \rangle - \langle (Z \wedge K)J \rangle - (j+1)^{-1}yx], \tag{2.11a}$$

$$\langle Z^+K^- \rangle = (2j+1)^{-1}[j\langle ZK \rangle + \langle (Z \wedge K)J \rangle - j^{-1}yx]. \tag{2.11b}$$

If we restrict ourselves to integrable representations of so(3), the exceptional value $w = -\frac{1}{4}$ will never occur and the value $w = 0$ will correspond to the trivial representation, for which $J = 0$. Choosing arbitrarily nonnegative values for j, we have, for $j = 0$, $K = K' = K^+$ and $K^0 = K^- = x = 0$. We can include this case in formulas (2.9)—(2.11) by adopting the convention that xw^{-1} is finite for $j = 0$.

Finally, using (2.6) we find, for skew-symmetric K and Z:

$$(Z \wedge K)' = \langle ZJ \rangle K - Z \langle KJ \rangle, \tag{2.12}$$

$$[\langle KJ \rangle, \langle KK \rangle] = \langle (K \wedge K)'K \rangle + \langle K(K \wedge K)' \rangle, \tag{2.13}$$

$$(K \times J) \wedge J = K(w-1) + K' + \tfrac{1}{3}Jx. \tag{2.14}$$

We now repeat the same procedure with the traceless symmetric tensor H; if $K = H \wedge J = J \wedge H$, we obtain:

$$H' = J \times H - 3H = -H \times J + 3H, \tag{2.15a}$$

$$H'' = H(4w - 3) + 4H' - 3K \times J, \tag{2.15b}$$

$$H' \times J = -4Hw + H \times J + 3K \times J. \tag{2.15c}$$

Since $(K \times J)' = K' \times J$ we see that, if H acts on S_j, $\frac{1}{2}Adw$ is represented by the matrix M, the eigenvalues of which are $-2j+1$, $-j$, 0, $j+1$, $2j+3$,

$$M = \begin{pmatrix} 0 & (2j+3)(2j-1) & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & -3 & 0 & j(j+1) & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The eigenvalues of M are distinct for

$$(w + \tfrac{1}{4})w(w - \tfrac{3}{4})(w - 2) \neq 0.$$

Except for these values H splits into five components:

$$H = H^{++} + H^+ + H^0 + H^- + H^=, \tag{2.16a}$$

$$H' = (2j+3)H^{++} + (j+1)H^+ - jH^- - (2j-1)H^=, \tag{2.16b}$$

such that H^α carries S_j into $S_{j+\alpha}$. From (2.14) we obtain

$$(K^\pm \times J) \wedge J = K^\pm (j^2 + j - \tfrac{1}{2} \pm (j + \tfrac{1}{2})),$$

and

$$\langle (J \times J)(J \times J) \rangle = \tfrac{4}{3}w^2 - w, \tag{2.17}$$

so that

$$H^0 = (\tfrac{4}{3}w^2 - w)^{-1}xJ \times J \quad \text{with } x = \langle KJ \rangle, \tag{2.18a}$$

$$(2j+1)H^+ = (j+2)^{-1}(K \times J + j^{-1}K' \times J - (\tfrac{4}{3}w - 1)H^0), \tag{2.18b}$$

$$(2j+1)H^- = (j-1)^{-1}(K \times J - (j+1)^{-1}K' \times J - (\tfrac{4}{3}w - 1)H^0), \tag{2.18c}$$

$$(4j+2)H^{++} = 2(j+1)H - H \times J + (j+2)^{-1}$$
$$\times (-2K \times J - (j+1)^{-1}K' \times J + \tfrac{1}{3}j(2j-1)H^0), \tag{2.18d}$$

$$(4j+2)H^= = 2jH + H \times J + (j-1)^{-1}$$
$$\times (-2K \times J + j^{-1}K' \times J + \tfrac{1}{3}(j+1)(2j+3)H^0). \tag{2.18e}$$

If G is another traceless symmetric tensor such that $G \wedge J = L$ we have:

$$\langle GH^+ \rangle = (j^2 + 2j)^{-1}\langle LK^+ \rangle, \quad \langle GH^- \rangle = (j^2 - 1)^{-1}\langle LK^- \rangle, \tag{2.19a}$$

$$\langle GH^{++} \rangle = -\langle GH^+ \rangle + (4j+2)^{-1}[2(j+1)\langle GH \rangle$$
$$- \langle (G \wedge H)J \rangle + w^{-1}\langle LK' \rangle - (2j-1)\langle GH^0 \rangle], \tag{2.19b}$$

$$\langle GH^= \rangle = -\langle GH^- \rangle + (4j+2)^{-1}[2j\langle GH \rangle$$
$$+ \langle (G \wedge H)J \rangle - w^{-1}\langle LK' \rangle - (2j+3)\langle GH^0 \rangle]. \tag{2.19c}$$

Examining the exceptional values of w (or j) for integrable representations of so(3) we drop $w = -\frac{1}{4}$. For $w = 0$, $j = 0$ the representation is trivial. From (2.15a) we get $H' = 3H$, hence $H = H^{++}$ and the formulas (2.16)—(2.19) are valid under the convention that $j^{-1/2}J$ is finite for $j = 0$.

For $w = \frac{3}{4}$, $j = \frac{1}{2}$, the representation is a multiple of the two-dimensional one, for which $J \times J = 0$. From (2.5) and (2.15a) one obtains

$$2H'' = H(4w - 15) + 11H' + 3H \times (J \times J), \tag{2.20}$$

hence $\frac{1}{2}Adw$ has the eigenvalues 4 and $\frac{3}{2}$, corresponding to H^{++} and H^+. We leave it to the reader to check that the above formulas hold under the convention $H^0 = 0$ and that they yield $H^= = H^- = 0$.

For $w = 2$, $j = 1$, H^- and $H^=$ cannot be separately defined, but their sum can be defined by $\phi = H - H^{++} - H^+ - H^0$, as well as $A = \phi \wedge J$. We then obtain $[w, \phi] = -2\phi + J \times A$, $[w, A] = -2A$. If we admit only integrable representations, $J \times A$ must vanish since $AS_1 \subset S_0$ on which so(3) must be trivially represented, and H is again the sum of eigenvectors of Adw. We leave it to the reader to verify that in fact ϕ vanishes, by using the fact that the contraction-free symmetric tensor of rank three of U_c (which is a polynomial of degree three in the J's) vanishes on S_1.

3. THE ENVELOPING ALGEBRA OF SL(3,ℝ)

We shall use now the results of the previous section, assuming that U is the universal enveloping algebra of sl(3,ℝ). The generators of the eight-dimensional Lie algebra form a traceless tensor E, the commutation relations being

$$[E_{ij}, E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj} \, . \tag{3.1}$$

We identify so(3) inside sl(3, $\mathbb{R}$) by putting $\mathbf{J} = \mathbf{E} - {}^t\mathbf{E}$. We shall denote by Θ the involutive automorphism of $\mathcal{U}$, the restriction of which on sl(3, $\mathbb{R}$) is

$$\Theta\mathbf{E} = -{}^t\mathbf{E}, \tag{3.2}$$

and the restriction of Θ to $\mathcal{U}_c$ is the identity.

Let us now examine some invariant subspaces of $\mathcal{U}$ under the adjoint representation. We shall denote by $\mathbf{F}$ the traceless rank two tensor $\mathbf{F} = \mathbf{EE} + {}^t({}^t\mathbf{E}\,{}^t\mathbf{F}) - \frac{4}{3}\langle\mathbf{E}^2\rangle$, and by $T^{(r)}$ and $T^{(l)}$ the symmetric tensors of rank three (with ten components each) defined by

$$T^{(l)}_{\alpha\alpha\alpha} = \epsilon_{\alpha kl}(E_{\alpha k}F_{\alpha l} - F_{\alpha k}E_{\alpha l}), \tag{3.3a}$$

$$T^{(r)}_{\alpha\alpha\alpha} = \epsilon_{kl\alpha}(F_{k\alpha}E_{l\alpha} - E_{k\alpha}F_{l\alpha}). \tag{3.3b}$$

Notice that $\Theta F = {}^tF$ and $\Theta T^{(l)} = T^{(r)}$.

Using (2.5) and (3.1) one easily proves:

Fact 3: A necessary and sufficient condition for an sl(3, $\mathbb{R}$) submodule of $\mathcal{U}$ to be isomorphic to the sl(3, $\mathbb{R}$) module $\mathbf{E}$ is to be a linear combination of $\mathbf{E}$ and $\mathbf{F}$ with coefficients in the center Z of $\mathcal{U}$.

More precisely, we have

$$[(F + \lambda_1 E)_{ij}, (F + \lambda_2 E)_{kl}]$$
$$= \epsilon_{ikh}T^{(r)}_{hjl} + \epsilon_{jlh}T^{(l)}_{hlk} + \delta_{kj}((a + \lambda_1\lambda_2)E$$
$$+ (\lambda_1 + \lambda_2)F)_{il} - \delta_{il}((a + \lambda_1\lambda_2)E + (\lambda_1 + \lambda_2)F)_{kj},$$
$$\tag{3.4}$$

where $a = 1 + \frac{4}{3}\langle\mathbf{E}^2\rangle$ lies in Z; and

$$2E^2 = F + a - 1 + 3E, \tag{3.5a}$$

$$2EF = 2FE = aE + b + 3F, \tag{3.5b}$$

$$2F^2 = (-a + 2)F + 2bE + a^2 - a + 3aE. \tag{3.5c}$$

The elements a and $b = \frac{4}{3}\langle\mathbf{EF}\rangle$ span Z algebraically. Notice that $\Theta a = a$, $\Theta b = -b$.

The bracket of E and $T^{(r)}$, $T^{(l)}$ is:

$$[E_{kl}, T^{(l)}_{\alpha\beta\gamma}] = \delta_{l\alpha}T^{(l)}_{k\beta\gamma} + \delta_{l\beta}T^{(l)}_{k\gamma\alpha}$$
$$+ \delta_{l\gamma}T^{(l)}_{k\alpha\beta} - \delta_{kl}T^{(l)}_{\alpha\beta\gamma}, \tag{3.6a}$$

$$[E_{kl}, T^{(r)}_{\alpha\beta\gamma}] = -\delta_{\alpha k}T^{(r)}_{l\beta\gamma} - \delta_{\beta k}T^{(r)}_{l\gamma\alpha}$$
$$- \delta_{\gamma k}T^{(r)}_{l\alpha\beta} + \delta_{lk}T^{(r)}_{\alpha\beta\gamma} \, . \tag{3.6b}$$

Let us now introduce the symmetric and skew-symmetric components of $\mathbf{E}$ and $\mathbf{F}$, as well as the contracted components of $T^{(r)}$ and $T^{(l)}$ which yield skew-symmetric two rank tensors. Putting

$$\mathbf{H} = \mathbf{E} + {}^t\mathbf{E}, \quad \mathbf{G} = \mathbf{F} + {}^t\mathbf{F}, \quad \mathbf{K} = \mathbf{F} - {}^t\mathbf{F},$$

we obtain from (3.5):

$$\mathbf{H}^2 + \mathbf{J}^2 = \mathbf{G} + 2(a - 1) + 3\mathbf{J}, \tag{3.7a}$$

$$\mathbf{HJ} + \mathbf{JH} = \mathbf{K} + 3\mathbf{H}, \tag{3.7b}$$

$$\mathbf{HG} + \mathbf{JK} = \mathbf{GH} + \mathbf{KJ} = a\mathbf{H} + 2b + 3\mathbf{K}, \tag{3.7c}$$

$$\mathbf{JG} + \mathbf{HK} = \mathbf{GJ} + \mathbf{KH} = a\mathbf{J} + 3\mathbf{G}, \tag{3.7d}$$

$$\mathbf{GG} + \mathbf{KK} = (2 - a)\mathbf{G} + b\mathbf{H} + 2a(a - 1) + 3a\mathbf{J}, \tag{3.7e}$$

$$\mathbf{GK} + \mathbf{KG} = (2 - a)\mathbf{K} + b\mathbf{J} + 3a\mathbf{H}. \tag{3.7f}$$

Putting

$$\mathbf{Z} = \mathbf{HK} - \mathbf{GJ} = \mathbf{KH} - \mathbf{JG}, \tag{3.8}$$

$$\mathbf{K}' = \mathbf{JK} - \mathbf{KJ} = \mathbf{GH} - \mathbf{HG},$$

we have $Z_{\alpha\beta} + K'_{\alpha\beta} = \epsilon_{\alpha\beta h}T^{(r)}_{h l l}$, $Z_{\alpha\beta} - K'_{\alpha\beta} = \epsilon_{\alpha\beta h}T^{(l)}_{h l l}$ and (3.4)—(3.6) yield:

$$[H; H] = 5J, \quad [H; J] = [J; H] = 3H, \quad [J; J] = J,$$

$$[H; G] = [G; H] = 5K, \quad [H; K] = [K; H] = 3G,$$

$$[J; K] = [K; J] = K, \tag{3.9}$$

$$[G; G] = 5aJ + 2Z, \quad [G; K] = 3aH - 2K',$$

$$[K; G] = 3aH + 2K', \quad [K; K] = aJ - 2Z,$$

$$[H; Z] = -[Z; H] = 3K', \quad [H; K'] = -[K'; H] = 3Z. \tag{3.10}$$

We shall now examine the commutator $\mathcal{X}$ of so(3) in $\mathcal{U}$. We have:

Fact 4: $\mathcal{X}$ is algebraically spanned by the five elements

$$\langle\mathbf{E}^2\rangle = \tfrac{3}{4}(a - 1), \quad \langle\mathbf{EF}\rangle = \tfrac{3}{4}b,$$

$$\langle\mathbf{J}^2\rangle = w, \quad \langle\mathbf{JK}\rangle = x, \quad \langle\mathbf{JZ}\rangle = y.$$

The proof of this fact lies on the decomposability properties of contracted products of tensors of rank two given by (2.5) and (2.6) and on the remark that a tensor of rank zero is the contracted product of a rank two tensor by either H or J. Note that $\Theta = 1$ on a, w, y and $\Theta = -1$ on b and x.

Now let $\mathbf{K} = \mathbf{H} \wedge \mathbf{J}$, $\mathbf{L} = \mathbf{G} \wedge \mathbf{J}$. From (3.7d) and (3.8) we obtain

$$\mathbf{L} + \mathbf{Z} = a\mathbf{J}, \quad \mathbf{H} \wedge \mathbf{K} = \mathbf{Z} + a\mathbf{J}, \quad \mathbf{K} \wedge \mathbf{K} = 2\mathbf{L} - a\mathbf{J},$$

so that

$$\langle\mathbf{K}^2\rangle = aw + y, \quad \langle\mathbf{KK}\rangle = 3y, \quad \langle\mathbf{LJ}\rangle = aw - y. \tag{3.11}$$

Using (3.7), (3.8), (3.9), and (3.11) we obtain immediately:

$$\langle\mathbf{H}^2\rangle = 3(a - 1) - w, \quad \langle\mathbf{HG}\rangle = \langle\mathbf{GH}\rangle = 3b - x,$$
$$\langle\mathbf{G}^2\rangle = 3a(a - 1) - aw - y, \tag{3.12a}$$

$$\langle\mathbf{HH}'\rangle = 9(a - 1) - 8w, \quad \langle\mathbf{HG}'\rangle = \langle\mathbf{GH}'\rangle = 9b - 8x,$$
$$\langle\mathbf{GG}'\rangle = 9a(a - 1) - 8aw - 5y. \tag{3.12b}$$

Next, we establish

$$\mathbf{G} \wedge \mathbf{K} + 2\mathbf{K}' = \mathbf{K} \wedge \mathbf{G} - 2\mathbf{K}' = 2b\mathbf{J} - (a - 2)\mathbf{K}, \tag{3.13}$$

hence

$$\langle\mathbf{LK}\rangle = \langle\mathbf{KL}\rangle = 2bw - (a - 2)x. \tag{3.14}$$

Now by means of (2.6) one establishes that for $\phi - {}^t\phi = A + {}^tA = 0$ one has:

$$(\phi \times A)\wedge\mathbf{J} + (\phi \wedge A)' = 3\phi \wedge A - 2\phi \wedge A', \tag{3.15a}$$

$$\mathbf{J}\wedge(A \times \phi) - (A \wedge \phi)' = 3A \wedge \phi + 2A' \wedge \phi, \tag{3.15b}$$

hence, putting $A = Z$, $\phi = H$, and in view of $Z \times H + H \times Z = 0$, one obtains

$$\langle KL'\rangle + \langle LK'\rangle = 6(bw - (a - 1)x). \tag{3.16}$$

Finally, using (2.13) one obtains

$$\langle KL'\rangle - \langle LK'\rangle = \tfrac{1}{2}[x, y]. \tag{3.17}$$

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4. CLASSIFICATION OF IR'S BY SU(2) LOWEST WEIGHT

Let us now examine an irreducible representation of $\overline{SL}(3,\mathbb{R})$ on a Banach space $\mathcal{B}$, or, more exactly, an irreducible representation of $\mathcal{U}$ on $\mathcal{B}_{alg}$, a dense subspace of $\mathcal{B}$. Irreducibility implies that $\mathcal{B}_{alg}$ is monogeneous and isomorphic to a quotient module of $\mathcal{U}$: For every $\varphi \in \mathcal{B}_{alg} \mathcal{U} / \varphi$ must be either $\mathcal{B}_{alg}$ itself or zero. If, in particular $\mathcal{U}\varphi \neq \mathcal{B}_{alg}$, then φ must itself be equal to zero.

On the other hand, since we are dealing with representations integrable to the whole group, they must be integrable to the maximal compact subgroup SU(2). For this reason there must be a lowest value l taken by j, such that $j(j+1)$ is an eigenvalue of the Casimir operator of SU(2). If $\mathcal{B}_{alg}$ decomposes into the discrete sum $\mathcal{B}_{alg} = \oplus_{j \geq l}\mathcal{B}^j$, $2j$ taking integral values, then, for any j, $\mathcal{B}_{alg} = \mathcal{U}\mathcal{B}^j$; in view of what immediately precedes, $\mathcal{B}^j$ itself must be monogeneous, i.e., generated by the action of $\mathcal{X}\mathcal{U}_c$ on any one of its elements, $\mathcal{X}$ denoting as previously the set of zero rank tensors. Notice also that the center Z must be represented by scalar operators.

In what follows we shall use the results established in the previous sections to express that $H^=$, $G^=$, K^-, L^- vanish on $\mathcal{B}^l$ (and also that $H^=$, $G^=$ vanish on $\mathcal{B}^{l+1}$). This property will suffice to establish criteria of irreducibility, equivalence and unitarity.

By straightforward substitution we obtain:

$$(4j+2)\langle HH^=\rangle_j = (j^2-j)^{-1}\{(2j-1)^{-1}x_j^2$$
$$- (2j-3)(y_j - aw) + j(2j-1)[a(j-3)$$
$$- j(j-1)^2]\}, \tag{4.1a}$$

$$(2j+1)\langle HG^= + GH^=\rangle_j = (j^2-j)^{-1}\{(2j-1)^{-1}$$
$$\times [ax_j w - \tfrac{1}{2}(x_j y_j + y_j x_j)]$$
$$+ (2j-3)ax_j - (2j-1)(j^2-3j+3)x_j$$
$$+ (j-3)j(2j-1)b\}, \tag{4.1b}$$

$$(4j+2)\langle HG^= - GH^=\rangle_j = (j^2-j)^{-1}(2j-1)^{-1}(j-\tfrac{3}{2})[x_j,y_j], \tag{4.1c}$$

$$(2j^2+j)\langle KK^-\rangle_j = -x_j^2 + y_j(j^2-2j) + aw^2, \tag{4.2a}$$

$$\tfrac{1}{2}(2j^2+j)\langle KL^- + LK^-\rangle_j = \tfrac{1}{2}(x_j y_j + y_j x_j) + (2j^2-j)$$
$$\times (bw - ax_j + x_j), \tag{4.2b}$$

$$(2j^2+j)\langle KL^- - LK^-\rangle_j = -\tfrac{1}{2}(j-2)[x_j,y_j]. \tag{4.2c}$$

We may also introduce the linear combinations:

$$(2j+1)j[2(2j-1)\langle HH^=\rangle_j + (j-1)^{-1}\langle KK^-\rangle]$$
$$= (j-1)[9j^2 a - j^2(2j-1)^2 - 3y_j], \tag{4.3a}$$

$$(2j+1)j[(2j-1)\langle HG^= + GH^=\rangle_j + \tfrac{1}{2}(j-1)^{-1}\langle KL^- + LK^-\rangle_j]$$
$$= (j-1)[3bj(2j-1) + x_j(3a+1 - 4(j-1)^2)], \tag{4.3b}$$

$$2(2j+1)j[2(2j-1)\langle HG^= - GH^=\rangle_j - (j-1)^{-1}\langle KL^- - LK^-\rangle_j]$$
$$= [x_j,y_j]. \tag{4.3c}$$

Setting both sides of these relations to zero we see that $[x,y]$ vanishes. The remaining four relations are not independent. We shall examine them separately:

All factors $G^=$, $H^=$, K^-, L^- are well defined. Notice that if $j = \tfrac{3}{2}$, relations (4.1) are not sufficient to establish $[x,y] = 0$, while in all other cases they suffice; this means that $[x,y]$ vanishes on $\mathcal{B}^{l+1}$ as well, unless $l = \tfrac{1}{2}$. If all relations hold, we have, for $j = l$:

$$y_l = l^2[3a - \tfrac{1}{3}(2l-1)^2], \tag{4.5}$$

$$ax_l + l(2l-1)b - \tfrac{1}{3}(2l-1)(2l-3)x_l = 0, \tag{4.6}$$

$$x_l^2 = l^2(2l-1)^2(a - \tfrac{1}{3}(l^2 - 2l)). \tag{4.7}$$

The eigenvalues of the four operators a, b, x, y can be given in terms of two independent parameters, λ and l:

$$y_l = l^2[3\lambda^2 - \tfrac{1}{3}(l+1)^2], \tag{4.8a}$$

$$x_l = l(2l-1)\lambda, \tag{4.8b}$$

$$a = \lambda^2 + \tfrac{1}{3}(l^2 - 2l), \tag{4.8c}$$

$$b = -\lambda^3 + \lambda(l-1)^2. \tag{4.8d}$$

We see that if the lowest weight $l > 1$ of the representation is imposed, a and b cannot be arbitrarily chosen, but they must obey the relation

$$[(3a+1) - (l-1)^2][(3a+1) - 4(l-1)^2]^2 - 27b^2 = 0. \tag{4.9}$$

Conversely if two arbitrary values of a and b are given, the above equation is an equation on $\mu^2 = (l-1)^2$, which admits, in general, three solutions. Rewriting it as an equation on μ we have

$$[4\mu^3 - 3(3a+1)\mu]^2 - [(3a+1)^3 - 27b^2] = 0, \tag{4.10}$$

and this equation must admit an integral or half-integral positive number as a solution.

As for the number λ, it is the root of the equation

$$4\lambda^3 - (3a+1)\lambda + b = 0. \tag{4.11}$$

We point out that, for fixed λ and μ, the other two roots of (4.11) are $\lambda' = \tfrac{1}{2}(-\lambda + \mu)$, $\lambda'' = \tfrac{1}{2}(-\lambda - \mu)$, while the other two roots of (4.10) are $\mu'^2 = \tfrac{1}{4}(3\lambda - \mu)^2$, $\mu''^2 = \tfrac{1}{4}(3\lambda + \mu)^2$.

We see also that if a, b, and l are given, satisfying (4.9), then λ is fixed as easily seen on (4.6), unless

$$b = 3a - (2l-1)(2l-3) = 0, \tag{4.12}$$

in which case we have two possible values for λ, the values $\lambda = \pm(l-1)$.

We shall call "relevant lowest weight" and abbreviate by RLW the representations described in this case.

(B) $j = 1$

In this case the two sets of relations (4.2) and (4.3) yield only two independent equations; as shown in the previous section there is no problem with the $j - 1$ in the denominator since $J \times K^-$ must vanish; the left-hand side of (4.3a), (4.3b) is defined and the right-hand side vanishes, so we obtain only

$$x_1^2 + y_1 - 4a = 0, \tag{4.13a}$$

$$x_1 y_1 - (a-1)x_1 + 2b = 0. \tag{4.13b}$$

It is clear that we have the choice of two independent parameters to label the representation. If x_1 is chosen as one of the labeling parameters, the other being any linear combination of a and y, the labeling is one to one; but if, say, a and b are fixed, x is given by an equation of degree three,

$$x_1^3 - (3a+1)x_1 - 2b = 0. \qquad (4.14)$$

We check that (4.14) can easily be derived from (4.11) by taking

$$x_1 = -2\lambda_1, \qquad (4.15a)$$

λ being one of the three roots of (4.11). In fact if one parametrizes a and b by λ and μ as in (4.8),

$$a = \lambda^2 + \tfrac{1}{3}(\mu^2 - 1), \quad b = \lambda(\mu^2 - \lambda^2), \qquad (4.15b)$$

we obtain in general three values for x_1, namely -2λ, $\lambda + \mu$, $\lambda - \mu$. We shall use the abbreviation LW1 for representations of lowest weight $l = 1$.

(C) $j = \tfrac{1}{2}$

In this case Eqs. (4.1), (4.2) give rise to

$$x = y - aw = 0, \qquad (4.16)$$

which expresses that $J \times J = 0$, hence $\langle KJ \rangle = \langle LJ \rangle = 0$. Representations with lowest weight $l = \tfrac{1}{2}$, in abbreviation LW$\tfrac{1}{2}$, are thus completely characterized by the quantitites a and b, so that Eqs. (4.11) and (4.14) play no role here. We recall that as seen in case (A), representations LW$\tfrac{1}{2}$ need not have an irreducible restriction to SU(2) for the next to the lowest weight, $j = \tfrac{3}{2}$.

(D) $j = 0$

In this case (4.16) holds again with $w = 0$, so that now

$$x = y = 0. \qquad (4.17)$$

Representations of lowest weight zero, LW0 in abbreviation, are, like LW$\tfrac{1}{2}$, determined by a and b alone. In what concerns the next-to-the-lowest weight, we see that $\beta^1 = \{0\}$, since all skew-symmetric rank two tensors in U involve J at least once; so that if $\varphi \in \beta^0$, $K\varphi = L\varphi = 0$.

Now, examining (4.11) closer we see it has in general three distinct roots; two of them will coincide if the discriminant is zero, or, equivalently, if $\mu = 0$ is a root of (4.10); that is, if

$$b^2 - (a + \tfrac{1}{3})^3 = 0. \qquad (4.18)$$

In this case (4.11) becomes

$$(2\lambda - b^{1/3})^2(\lambda + b^{1/3}) = 0. \qquad (4.19)$$

Notice that there is a triple root $\lambda = 0$ if $b = 3a + 1 = 0$.

Coming now to (4.10) to seek RLW representations, we are searching for roots μ of (4.10) such that $4\mu^2$ is a strictly positive square integer. The six roots of (4.10) can be separated into two triplets $\pm(\mu_1, \mu_2, \mu_3)$ such that $\mu_1 + \mu_2 = \mu_3$. Thus either one or three of the numbers $2\mu_i$ are relative integers (or none of them!). Discarding the solution $\mu = 0$, if any, because it leads to a LW1 representation, and which is obtained if and only if (4.18) holds, we are now able to give the *a priori*

equivalence criteria for all integrable representations of U, within, of course, infinitesimal equivalence.

Theorem 1: All strongly continuous irreducible representations $\mathbb{R}$ of $\overline{SL}(3, \mathbb{R})$ on a Banach space β can be defined by the action of the infinitesimal generators on the dense subspace β_{alg}; β_{alg} is the irreducible factor generated by the action of U on $\varphi_l \in \beta^l$. The space β^l is $(2l+1)$-dimensional and the scalar representation χ_l of χ on β^l determines $\mathbb{R}$ up to equivalence. There are four series of representations according to the value of the lowest weight $l : 0, \tfrac{1}{2}, 1,$ or > 1. For given values (a, b) of the invariants of U, there exist the following irreducible representations in the different series (up to equivalence):

LW0: one.

LW$\tfrac{1}{2}$: one;

LW1: if $b^2 - (a + \tfrac{1}{3})^3 \neq 0$: three,

if $b^2 - (a + \tfrac{1}{3})^2 = 0$, $b \neq 0$: two,

if $b = a + \tfrac{1}{3} = 0$: one.

RLW: There are as many inequivalent IR's for fixed a, b, corresponding to the weight l, as many roots (each one counted with its multiplicity) of (4.20) are squared nonzero integers:

$$s(s - 9a - 3)^2 - 4(3a+1)^3 + 108b^2 = 0, \qquad (4.20)$$

the weight l being equal to $\tfrac{1}{2}\sqrt{s} + 1$. If there are more than one such roots, either $b^2 = (a + \tfrac{1}{3})^3$, and in this case there are two IR's, with $l = 1 + \tfrac{3}{2}(a + \tfrac{1}{3})^{1/2}$; or $b^2 \neq (a + \tfrac{1}{3})^3$, and there are three of them, corresponding to l_1, l_2, l_3, such that

$$1 < l_1 \leq l_2 < l_1 + l_2 - 1.$$

Either two of them are half-integer and the third one integer, or all three are integers, in which case there are two even ones and one odd one, or three odd ones.

5. NECESSARY CONDITIONS FOR UNITARITY

Let us now introduce a scalar product $(\varphi \mid \psi) = \overline{(\psi \mid \varphi)}$ which gives a pre-Hilbert structure to β_{alg}, and ask the question whether the IR $\mathcal{R}$ is equivalent to a unitary one. We shall establish here the necessary conditions for unitarity.

If $\mathcal{R}$ is unitary, iE_{kl} must be an essentially self-adjoint operator. Completely symmetrized polynomials in the E_{kl}'s must have real (imaginary) eigenvalues if their degree is even (odd). We thus immediately obtain

$$x_l^2 \leq 0, \quad b^2 \leq 0. \qquad (5.1)$$

Since representations of the RLW series depend on a single parameter λ other than l, and we have $x_l = \lambda(2l - 1)l$, one has:

Theorem 2A: A necessary condition for a representation (λ, l) of the RLW series to be unitary is that λ be an imaginary number.

Since β_j and $\beta_{j'}$ are orthogonal for $j \neq j'$, and since $H = H^{++} + H^+ + H^0 + H^- + H^{--}$, we have

$$(H_{kh}\varphi \mid H_{kh}\varphi) = \sum_{\alpha=-2}^{2} (H_{kh}^{+\alpha}\varphi \mid H_{kh}^\alpha\varphi) = -\sum_{\alpha=-2}^{2} (\varphi \mid H_{kh}H_{kh}^\alpha\varphi).$$

Summing over the five components, we see that $\langle \mathrm{HH}^\alpha \rangle$ (and not only $\langle \mathrm{HH} \rangle$) must take as eigenvalues negative real numbers because of positive definiteness of the scalar product. The same holds for $\langle KK^* \rangle$ (the minus sign is due to the inner product $\langle \ \rangle$ for a skew-symmetric tensor, and $K^* = K$ with respect to the scalar product), hence for $\langle KK' \rangle = (l+1)\langle KK^* \rangle_l$. Thus we have from (3.11)

$$y_l \le 0 \qquad (5.2)$$

which is of course a stronger condition than $\langle \mathrm{HH} \rangle \le 0$. Check through (4.8) that (5.2) is fullfilled for imaginary λ in the RIW case. For $l = 1$, (5.2) can be written in the (x_1, a) or the (λ, μ) parametrization as

$$4a - x_1^2 = \tfrac{4}{3}(\mu^2 - 1) \le 0 \qquad (5.3)$$

hence $a \le \tfrac{1}{4}x_1^2 \le 0$.

In terms of the (a, b) labeling, we see that for real numbers a and ib either one or three of the solutions of (4.11) are imaginary according to whether $b^2 - (a + \tfrac{1}{3})^3$ is negative or positive. Since, anyhow,

$$[(\mu + 3\lambda)^2 - 4][(\mu - 3\lambda)^2 - 4] = [(2+\mu)^2 - 9\lambda^2][(2-\mu)^2 - 9\lambda^2]$$

is strictly positive for $\mu^2 \le 1$, λ imaginary, we can, without loss of generality, multiply both sides of the inequality (5.3) with the above expression to obtain

$$b^2 - a(a-1)^2 \ge 0 \quad \text{and} \quad a \le 0. \qquad (5.4)$$

Thus we have:

Theorem 2B: A necessary condition for a representation of the LW1 series to be unitary is $x_1^2 \le 0$ and $y_1 \le 0$, or equivalently, λ imaginary and $\mu^2 \le 1$; with respect to the (a, b) labeling, this condition for at least one of the IR's defined in Theorem 1 is expressed by $a \le 0$, $b^2 \le 0$, and $b^2 \ge a(a-1)^2$, while the condition for all three of them is $a \le 0$, $b^2 \le 0$, and $b^2 \ge (a + \tfrac{1}{3})^3$.

We note that $(a + \tfrac{1}{3})^3 = a(a-1)^2 + \tfrac{1}{3}(3a - \tfrac{1}{3})^2 \ge a(a-1)^2$.

We also note that for $l = \tfrac{1}{2}$, (5.2) yields $a \le 0$ and for $l = 0$ it adds nothing.

Coming now to the case $l = 0$, we see that for a nonzero $\varphi \in \beta_0$, $H\varphi$ and $G\varphi$ lie in β_2 (in fact they span β_2, as easily established with the help of the relations in U). The positive-definiteness of the scalar product in β_2 can be expressed in terms of the Schwarz inequality,

$$(H\varphi \mid H\varphi)(G\varphi \mid G\varphi) - (H\varphi \mid G\varphi)(G\varphi \mid H\varphi) \ge 0,$$

where H and G stand for any (the same for both) linear combination of the H_{kl}'s and G_{kl}'s. By summation over all indices and transposition of the operators, we obtain the condition

$$\langle \mathrm{HH} \rangle_0 \langle \mathrm{GG} \rangle_0 - \langle \mathrm{HG} \rangle_0 \langle \mathrm{GH} \rangle_0 \le 0 \qquad (5.5)$$

which gives, using (3.12a),

$$b^2 - a(a-1)^2 \ge 0, \qquad (5.6)$$

$$\langle \mathrm{HH} \rangle_0 = 3(a-1) \le 0, \quad \langle \mathrm{GG} \rangle_0 = 3a(a-1) \ge 0. \qquad (5.7)$$

Conditions (5.7) yield either $a \le 0$, or $a = 1$, in which case both $\langle HH \rangle_0$ and $\langle GG \rangle_0$ vanish. So β_2 contains only the zero vector, b vanishes as well, and we are dealing with the trivial representation; the corresponding λ, μ

values are $\lambda = 0$, $\mu^2 = 4$, and $\lambda = \pm 2$, $\mu^2 = 1$. We thus have:

Theorem 2C: Besides the trivial representation corresponding to the parametrization $a = 1$, $b = 0$, the other unitary representations of the LW0 series must fullfill the conditions $a \le 0$, $b^2 \le 0$, $b^2 \ge a(a-1)^2$; or, equivalently, in the (λ, μ) labeling, there must be at least one labeling couple such that $\lambda^2 \le 0$, $\mu^2 \le 1$.

Finally, for the case $l = \tfrac{1}{2}$, we again use the Schwarz inequality, separating $H^+\varphi$ and $G^+\varphi$ which lie in $\beta_{3/2}$ (in fact they span it). Since $H = H^+ + H^{++}$ and $H' = \tfrac{3}{2}H^+ + 4H^{++}$, we have $5H^+ = 8H - 2H'$. We obtain

$$5\langle HH^+ \rangle_{1/2} = 6a, \quad 5\langle HG^+ \rangle_{1/2} = 6b, \quad 5\langle GG^+ \rangle_{1/2} = 6a(a + \tfrac{1}{4}),$$

$$(5.8)$$

hence

$$a \le 0, \quad a(a + \tfrac{1}{4}) \ge 0, \qquad (5.9)$$

and

$$b^2 - a^2(a + \tfrac{1}{4}) \ge 0. \qquad (5.10)$$

The condition (5.9) gives either $a \le -\tfrac{1}{4}$ or $a = 0$. In this last case $\beta_{3/2}$ reduces to the zero subspace and $b = 0$. This representation is known to be multiplicity free on su(2) and the spectrum of j is $\tfrac{1}{2} + 2\mathbf{N}$. So we have:

Theorem 2D: A necessary condition for an IR of the $\mathrm{LW}\tfrac{1}{2}$ series to be unitary is either $b^2 \le 0$, $a \le -\tfrac{1}{4}$, $b^2 \ge a^2(a + \tfrac{1}{4})$, or $a = b = 0$. Equivalently, in the (λ, μ) labeling, the condition is the existence of one among the three possible labelings satisfying $\lambda^2 \le 0$, $\mu^2 \le \tfrac{1}{4}$ or $\lambda = 0$, $\mu^2 = 1$.

Remark: If the equality sign holds in (5.10), then $\beta_{3/2}$ is of multiplicity one, and there is one μ equal to $\tfrac{1}{2}$. Notice that there exists then an IR of the RLW series with $l = \mu + 1 = \tfrac{3}{2}$.

6. INDUCED REPRESENTATIONS OF $\overline{\mathrm{SL}}(3; \mathrm{IR})$

A. Generalities

The construction of induced representations of $\overline{\mathrm{SL}}(3, \mathrm{IR})$ has been carried out by many authors[1,4,6] under different assumptions, so we do not give all the details. We point out that we speak of Hilbert just for the sake of commodity: Any Banach space which is the closure of a dense subspace containing the good eigenfunctions would suit our purpose; and, indeed, when speaking of unitarity of the complementary or "semi-discrete" series (principal series with one compact generator in the Cartan subalgebra) the final Hilbert space is not a subspace of the initial one, because of the renormalization procedure (completion with respect to a different scalar product).

We are mainly interested in establishing that all representations discussed in Theorem 1 are effectively group representations and not merely local ones. We shall also treat the questions of equivalence, unitarity, and of j-multiplicity (i.e., decomposition on the maximal compact subgroup).

Let Γ be a regularly embedded (in the Mackey–Bruhat sense) subgroup of a Lie group G, and L a repre-

sentation of Γ on a Hilbert space $\mathcal{J}$. Let K be the homogeneous space $\Gamma\backslash G$; G acts on the right on K by $u\mapsto u\circ g$, τ denotes a smooth section from K to G, such that G factorizes uniquely by $G=\Gamma\circ\tau(K)$, and $\eta(u,g)$ will denote the square root of the Jacobian $D(u\circ g)/D(u)$. Let $\mathcal{H}=L^2(K;\mathcal{J};du)$ be the space of square-integrable functions from K to $\mathcal{J}$, with respect to the quasi-invariant measure du. The induced representation U^L of G on $\mathcal{H}$ can be defined by

$$U^L(g)f(u)=\eta(u,g)\,L(\tau_u g\tau_{u\circ g}^{-1})f(u\circ g). \qquad (6.1)$$

We recall the following well-known facts:

Fact 5: If U^L is irreducible, then L is irreducible (but not conversely).

Fact 6: If L is unitary, then U^L is equivalent to a unitary representation (but not conversely).

Fact 7: If L is itself induced by the representation M of $\Delta\subset\Gamma$, then U^M is equivalent to U^L.

B. Subgroups of $\overline{\mathrm{SL}}(3,\mathbb{R})$

Let $G=NAK$ be an Iwasawa decomposition of the group $G=\overline{\mathrm{SL}}(3,\mathbb{R})$; within a null-measure set, almost every element g of G factorizes uniquely to $g=n(g)\cdot a(g)\cdot k(g)$ with $n(g)\in N$, $a(g)\in A$, $k(g)\in K$. The subgroup N is a real connected nilpotent group, isomorphic to the group of 3×3 triangular matrices with diagonal elements equal to unity. The subgroup A is a real connected Abelian group of dimension two; it is isomorphic to the group of diagonal 3×3 matrices with diagonal elements $\exp\alpha_i$ with $\alpha_1+\alpha_2+\alpha_3=0$. The subgroup K is isomorphic to $\mathrm{SU}(2)$.

Let $\Delta=\mathcal{N}_G(N)$ denote the normalizer of N in G; it is the semidirect product of $A\times Q$ by N, Q being the finite group (of order 8) of quaternions, which is a subgroup of $\mathrm{SU}(2)$.

Let Φ denote the homomorphism which brings G on $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{SU}(2)$ on $\mathrm{SO}(3)$. The image $\Phi\Delta$ determines canonically an orthogonal basis in $\mathbb{R}^3$; let T_1^* be the one-parameter subgroup of $\mathrm{SU}(2)$, with elements $\gamma_i(\psi)$, such that $\Phi(T_1^*)$ is the stabilizer of the ith component of the basis. One can parametrize $\mathrm{SU}(2)$ by the subgroups T_1^*, writing $\mathrm{SU}(2)=T_3^*T_2^*T_3^*$, the factorization being unique up to a null-measure set. There are of course six such factorizations possible. The three parameters (ψ,φ,ϑ), may be chosen in the cube $[0,4\pi[^3$ with the convention that $(\psi+2\pi,\varphi+2\pi,\vartheta)$, $(\psi+2\pi,\varphi,\vartheta+2\pi)$, $(\psi+\pi,-\varphi,\vartheta-\pi)$, and (ψ,φ,ϑ) denote the same element. One may also limit the range of the parameters to $0\le\psi<4\pi$, $0\le\varphi<\pi$, $0\le\vartheta<2\pi$, in order to have a one-to-one parametrization. Notice that $\gamma_i(2\pi)=-1$ is the element of the center which is not 1, the same for all i, and that Q contains the elements $(\sigma_i)^l=\gamma_i(l\pi)$ for integer l. We have for $i\ne j$

$$\sigma_i\gamma_j(\psi)=\gamma_j(-\psi)\sigma_i \qquad (6.2)$$

so that T_i^* and Q generate the nonconnected subgroup T_i which is the normalizer of T_i^* in $\mathrm{SU}(2)$.

Now let Γ_i^* be the subgroup of G generated by T_i and Δ; we have $\Gamma_i^*\approx\Phi^{-1}(\mathrm{GL}(2,\mathbb{R}))\circ\mathcal{T}_2$ where $\mathcal{T}_2$ is a two-dimensional Abelian subgroup of N. The "homogeneous"

subgroup Γ_i of Γ_i^* is the direct product $\Gamma_i=A_i^c\times B_i$, where A_i^c is the intersection of A and of the center of Γ_i and $\Phi(B_i)$ is isomorphic to the set of 2×2 real matrices with determinant ±1. The connected component B_i^* of B_i is a twofold covering of $\mathrm{SL}(2,\mathbb{R})$ and its center is the cyclic group of order 4 generated by σ_i; we shall write $A_i^s=B_i^*\cap A$, so that $A=A_i^c\times A_i^s$. Notice that $T_i\subset B_i$ and $T_i^*\subset B_i^*$ and that $B_i^*=(N\cap B_i^*)A_i^sT_i^*$ is an Iwasawa decomposition of B_i^*.

C. The induced representations

We are *a priori* interested in representations L of Δ which are trivial on N, which will induce representations U^L of G; thus L is defined by its restrictions L_A on A and L_Q on Q. The irreducible representations of A are all one-dimensional; they may be labeled by a triplet $\vec{\lambda}=(\lambda_1,\lambda_2,\lambda_3)$ of complex numbers modulo the translation by $(1,1,1)$, so that we can impose, say, $\lambda_1+\lambda_2+\lambda_3=0$. If $a\in A$ and $\varphi(a)$ has diagonal elements $\exp\alpha_i$ such that $\alpha_1+\alpha_2+\alpha_3=0$, we will write

$$L(a)=\exp(\textstyle\sum\lambda_i\alpha_i).$$

There are five IR's of Q labeled by (k), with $k=0,1,2,3,\frac{1}{2}$. The trivial representation will be denoted by (0), the two-dimensional one by $(\frac{1}{2})$; the remaining three are one-dimensional, such that $(i)(\sigma_j)=-1$ if $i\ne j$ and $(j)(\sigma_j)=1$.

The carrier space $\mathcal{H}^{(k)}$ of the induced representation U^L, for $L=(\vec{\lambda},k)$, is a space of functions from the coset space $X=\Delta\backslash G=Q\backslash\mathrm{SU}(2)$ to $\mathbb{C}^1$ if $k\ne\frac{1}{2}$ and to $\mathbb{C}^2$ if $k=\frac{1}{2}$. One easily sees in (6.1) that, for any L,

$$U^L(g)L(\sigma_i)=L(\sigma_i)U^L(g)\quad g\in G,\ \sigma_i\in Q \qquad (6.3)$$

so that, in case $k=\frac{1}{2}$, $U^L(g)$ leaves invariant the eigenspaces of all linear combinations of the $L(\sigma_i)$; if $\mathcal{H}^{(1/2)}=\mathcal{H}^{1/2,+}\oplus\mathcal{H}^{1/2,-}$, then both subspaces are invariant and the two restrictions of U^L on them are equivalent. Notice that $(\frac{1}{2})$ is an induced representation of Q by a faithful representation of any one of its order four cyclic subgroups, so that $\mathcal{H}^{1/2}$ can be considered as a functional space from $\mathbb{Z}_4\backslash\mathrm{SU}(2)$ to $\mathbb{C}^1$.

One may identify $\mathcal{H}^{(k)}$ with a closed subspace of $\mathcal{H}$, the space of complex-valued functions on $\mathrm{SU}(2)$, by requiring

$$\mathcal{H}^{(k)}=\{f\in\mathcal{H};\, f(\sigma_iu)=L(\sigma_i)f(u)\}. \qquad (6.4)$$

By means of the parametrization $\mathrm{SU}(2)=T_3^*T_2^*T_3^*$ and in view of (6.2) we may write Table I more explicitly.

To find the multiplicity $N(j;k)$ of the isotypic component D^j of the restriction of U^L to $\mathrm{SU}(2)$, one uses the fact that $i\partial/\partial\psi$ takes the values $\{-j,\dots,j-1,j\}$ once each for fixed j, so that we have Table II, with the function $E(x)$, defined by

$$E(x)=\mathrm{Sup}\,k\quad(k\in\mathbb{Z};\, k\le x),$$

being the "integer part" of x.

D. Induction by steps

Now we shall use Fact 7 to obtain information on U^L by constructing the intermediate representations V_i^L of Γ_i^*. Because of the choice of $L=(\vec{\lambda},k)$ and of the fact that T_i^* normalizes $\mathcal{T}_2$, V_i^L is determined by its restric-

TABLE I. Dependence on k of carrier spaces of $U^{(\vec{\lambda},k)}$.

$\mathcal{H}^{(0)}$	$f(\psi,\varphi,\vartheta)$	$= f(-\psi+\pi,\ \varphi+\pi,\ \vartheta)$	$= f(-\psi,\ \varphi+\pi,\ \vartheta)$	$= f(\psi+\pi,\ \varphi,\ \vartheta)$
$\mathcal{H}^{(1)}$	$f(\psi,\varphi,\vartheta)$	$= f(-\psi+\pi,\ \varphi+\pi,\ \vartheta)$	$= -f(-\psi,\ \varphi+\pi,\ \vartheta)$	$= -f(\psi+\pi,\ \varphi,\ \vartheta)$
$\mathcal{H}^{(2)}$	$f(\psi,\varphi,\vartheta)$	$= -f(-\psi+\pi,\ \varphi+\pi,\ \vartheta)$	$= f(-\psi,\ \varphi+\pi,\ \vartheta)$	$= -f(\psi+\pi,\ \varphi,\ \vartheta)$
$\mathcal{H}^{(3)}$	$f(\psi,\varphi,\vartheta)$	$= -f(-\psi+\pi,\ \varphi+\pi,\ \vartheta)$	$= -f(-\psi,\ \varphi+\pi,\ \vartheta)$	$= f(\psi+\pi,\ \varphi,\ \vartheta)$
$\mathcal{H}^{(1/2)}$	$f(\psi,\varphi,\vartheta)$	$= -f(\psi+2\pi,\varphi,\vartheta)$		

tion on $\Gamma_i = A_i^c \times B_i$; the restriction to A_i^c is given by the number λ_i (or $\lambda_i - \frac{1}{3}\sum_j \lambda_j$ if we do not impose the sum to vanish). Putting, for sake of brevity, $\lambda = \lambda_3$ and $\mu = \mu_3$ $= \frac{1}{2}(\lambda_1 - \lambda_2)$, the restriction $M_3^{\mu,k}$ to B_3 of V_3^L depends only on μ and k and not on λ. The representation space $\mathcal{J}^{(k)}$ is a space of complex-valued functions of the variable $\psi \in [0,4\pi[$, satisfying

$$f(\psi+\pi) = L(\sigma_3)f(\psi) = M_3^{\mu,k}(\sigma_3)f(\psi) \qquad (6.5)$$

or, more precisely

$$\mathcal{J}^{(0)} = \mathcal{J}^{(3)} = \{f; f(\psi+\pi) = f(\psi)\}, \qquad (6.6a)$$

$$\mathcal{J}^{(1)} = \mathcal{J}^{(2)} = \{f; f(\psi+\pi) = -f(\psi)\}, \qquad (6.6b)$$

$$\mathcal{J}^{(1/2)} = \{f; f(\psi+2\pi) = -f(\psi)\}, \qquad (6.6c)$$

and we also must have for $j \neq 3$,

$$M_3^{\mu,k}(\sigma_j)f(\psi) = L(\sigma_j)f(-\psi),$$

or, more precisely,

$$M_3^{\mu,k}(\sigma_2)f(\psi) = (i)^{2k}f(-\psi). \qquad (6.7)$$

It is immediate to see that $L^2(T\backslash SU(2),\ \mathcal{J}^{(0)},\ du)$ $= \mathcal{H}^{(0)} \oplus \mathcal{H}^{(3)}$ and that the same happens for the couple $\{1,2\}$; for $k = \frac{1}{2}$ there is no new splitting.

Calculating the infinitesimal generators X, Y, J of the connected subgroup B_3^+, which satisfy the commutation relations

$$[X,Y] = 2J, \quad [J, X \pm iY] = \mp 2i(X \pm iY)$$

we obtain

$$X^\mu \pm iY^\mu = \exp(\mp 2i\psi)\left(1 - \mu \pm i\frac{\partial}{\partial\psi}\right), \quad J^\mu = \frac{\partial}{\partial\psi}. \qquad (6.8)$$

The properties of these representations are quite well known, so we limit ourselves to giving the main results without details.

Let τ be the transformation $1 \longleftrightarrow 2$ on the indices of $\vec{\lambda}$ and σ; τ brings μ to $-\mu$, it leaves invariant the representations (0), (3), $(\frac{1}{2})$ of Q, and it permutes (1) and (2). V_3^L like $M_3^{\mu,k}$ is irreducible except for a discrete set of cases, for which it is indecomposable and contains a finite number of irreducible factors. So if one denotes by $\overline{V}_3^L$ (resp. $\overline{U}^L$) the completely reducible representation which has the same factors and factor multiplicities as V_3^L (resp. U^L) one easily proves that

TABLE II. Multiplicity of D^j in $U^{(\vec{\lambda},k)}$.

k	Range j	$N(j;k)$
0	integer	$E(\frac{1}{2}j+\frac{1}{2})+(-1)^j$
1, 2, 3	integer	$E(\frac{1}{2}j+\frac{1}{2})$
$\frac{1}{2}$	half-integer	$2 \times (j+\frac{1}{2})$

$\overline{V}_3^L$ and $\overline{V}_3^{\tau L}$ are equivalent so that $\overline{U}^L$ is equivalent to $\overline{U}^{\tau L}$. Since the index 3 has been arbitrarily chosen we deduce:

Theorem 3: Let $L = (\vec{\lambda},k)$ be an irreducible representation of Δ, trivial on N, and τ be a permutation of S_3 [with $\tau(0) = (0)$, $\tau(\frac{1}{2}) = (\frac{1}{2})$]. The irreducible factors of the representations $\overline{U}^L$ and $\overline{U}^{\tau L}$ induced by L and τL $= (\tau\vec{\lambda},\ \tau k)$ are in one-to-one correspondence and the corresponding factors are equivalent to each other.

In order to study the factors of $M_3^{\mu,k}$ we introduce, for fixed k and $l \in 1 + \frac{1}{2}\mathbb{N}$, the space $\mathcal{J}^{[l}$ and $\mathcal{J}^{]l}$ by

$$\frac{\partial}{\partial\psi} \mathcal{J}^{]l} \subset \mathcal{J}^{]l} \quad \text{and} \quad \frac{\partial}{\partial\psi}\mathcal{J}^{[l} \subset \mathcal{J}^{[l}, \qquad (6.9a)$$

$$\mathcal{J}^{(k)} = \mathcal{J}^{[l} \odot \mathcal{J}^{]l}, \qquad (6.9b)$$

$$\exp(in\ \psi) \in \mathcal{J}^{[l} \Longleftrightarrow |n| \in l + 2\mathbb{N}. \qquad (6.9c)$$

We shall also define for $i = 1,2,3$:

$$\mathcal{G}_i^{[l;k} = \mathcal{H}^{(k)} \cap L^2(T_i^*\backslash SU(2); \mathcal{J}^{[l}), \qquad (6.10a)$$

$$\mathcal{G}_i^{]l;k} = \mathcal{H}^{(k)} \cap L^2(T_i^*\backslash SU(2); \mathcal{J}^{]l}). \qquad (6.10b)$$

It is easy to check that if $\mathcal{J}^{[l} \neq \{0\}$, then $\mathcal{J}^{]l}$ is $(l-1)$-dimensional for integer l and infinite dimensional for half-integer l. The irreducible factors of $M_3^{\mu,k}$ operate on $\mathcal{J}^{]l}$ and $\mathcal{J}^{]l}$ when $l = |\mu| + 1$. The restriction of $M_3^{\mu,k}$ on $\mathcal{J}^{[l}$ is the coupling through σ_2 of two contragredient representations of $SL(2,\mathbb{R})$ belonging to its discrete series of UIR's, and it can easily be seen that $M_3^{\mu,k} | \mathcal{J}^{[l}$ is equivalent to $M_3^{-\mu,k} | \mathcal{J}^{[l}$, even if $k = 1$ or 2. This is not the case however for the restriction to $\mathcal{J}^{]l}$, as it is not for V_3^L irreducible.

The corresponding factors of U^L, acting on $\mathcal{G}_3^{[l;k}$ and $\mathcal{G}_3^{]l;k}$, will be denoted by $U_3^{\lambda,[l}$ and $U_3^{\lambda,]l;k,\epsilon}$, the parameter $\epsilon = \mu/|\mu|$ being relevant only if $k = 1$ or 2, and $U_3^{\lambda,]l;1,\epsilon} \sim U_3^{\lambda,]l;2,-\epsilon}$; ϵ may be omitted if $k \neq 1,2$. By cyclic permutations over the indices one can define analogous splittings through the other subgroups, if any.

Notice that both factors are reducible for $k = \frac{1}{2}$, and, like $U^{(\vec{\lambda},1/2)}$, they are multiples of order 2 of their isotypic components. Notice also that for $l = 1$ and $k = 1,2$, $\mathcal{J}^{[l} = \mathcal{J}^{(k)}$. This means that $\mu = 0 = \lambda_1 - \lambda_2$, so that $U^{(\vec{\lambda};1)} \sim U^{(\vec{\lambda};2)}$.

To obtain the j-multiplicity of each factor we introduce the functions $M(j;l)$ and $\rho_i(l,k)$ (the latter being invariant under permutations acting on i and k, so that we put $i = 3$ in what follows):

$$\rho_3(l,k) = \begin{cases} 2 & \text{if } l \in \frac{1}{2}+1+\mathbb{N} \text{ and } k = \frac{1}{2}, \\ 1 & \text{if } l \in 1+\mathbb{N} \quad \text{and } k = 1 \text{ or } 2, \\ 1 & \text{if } l \in 2+\mathbb{N} \quad \text{and } k = 0 \text{ or } 3, \\ 0 & \text{otherwise}, \end{cases} \qquad (6.11)$$

TABLE III. Splitting in induction by steps of $U^{(\lambda,k)}$, $\vec{\lambda}=(\lambda_1,\lambda_2,\lambda)$, $l=1+|\lambda_1-\lambda_2|$, $\epsilon=|\lambda_1-\lambda_2|/(\lambda_1-\lambda_2)$, $\rho_3(l,k)\neq 0$.

Factor	Carrier space	Inducing factor
$U_3^{\lambda,l}$	$\mathcal{G}_3^{l;k}$	$\lambda\otimes(M_3^{(l-1),k}\mid\mathcal{J}^{(l)})$
$U_3^{\lambda,l;k,\epsilon}$	$\mathcal{G}_3^{l;k}$	$\lambda\otimes(M_3^{(l-1),k}\mid\mathcal{J}^{1l})$

$$M(j;l)=\begin{cases} 1+\tfrac{1}{2}(j-l) & \text{if } j\in l-2+2\mathbb{N},\\[4pt] \tfrac{1}{2}(j-l+1) & \text{if } j\in l-1+2\mathbb{N},\\[4pt] 0 & \text{otherwise.}\end{cases} \qquad (6.12)$$

Notice that if $k=1,2$, or 3, then $N(j;k)=M(j;1)$. We then have the following proposition.

Proposition 1: When $\rho_3(l,k)\neq 0$, the multiplicity of D^j in the factors of U^L for $L=(\tfrac{1}{2}(-\lambda+\epsilon l-\epsilon,\ -\lambda-\epsilon l+\epsilon,\ 2\lambda);\ k)$ is $\rho_3(l,k)\,M(j;l)$ for $U_3^{\lambda,l l}$ and the complement to $N(j;k)$ for $U_3^{\lambda,1l;k,\epsilon}$. The latter is unbounded if and only if $k=\tfrac{1}{2}$.

Most of the above results and notation are summarized in Table III.

The further study of V_3^L from the unitarity point of view gives sufficient conditions for U^L to be unitary (within infinitesimal equivalence) according to Fact 6. Thus suppose V_3^L or one of its factors to be unitary. If we replace $i\partial/\partial\psi$ by $-n$ in (6.8) and express that $-(X+iY)(X-iY)$ must be a positive self-adjoint operator we obtain

$$(n\pm 1)^2-\mu^2\geqslant 0 \qquad (6.13)$$

for every n. If $M_3^{\mu,k}$ is irreducible the lowest value taken by $(n\pm 1)^2$ is 1, $\tfrac{1}{4}$, 0 according to whether $k=0$ or 3, $\tfrac{1}{2}$, 1 or 2. If it is reducible and $\mu=-l+1$, the lowest value of $(n\pm 1)^2$ is $\mu^2=(l-1)^2$ in $M^{l l,k}$; $\tfrac{1}{4}$ in $M^{1l,1/2}$; 0 in $M^{1l,k}$ $k=1,2$; and 1 in $M^{1l,k}$, $k=0,3$; so that $M^{13/2,1/2}$, $M^{12,0}$, $M^{12,3}$, and $M^{l l,k}$ are equivalent to unitary representations. In view of the fact that λ_3 must be imaginary to have a unitary V_2^L and using Theorem 3, one obtains for the induced representations:

Theorem 4 (sufficient conditions for unitarity): The following representations of G are unitary (with $m\in\{1,2,3\}$):

(a) $U^{(\vec{\lambda},k)}$ for any k and $\lambda\in i\mathbb{R}^3$,

(b) $U^{(\vec{\lambda},0)}$ and $U^{(\lambda,m)}$ for $\lambda_m\in i\mathbb{R}$ and $\mu_m\in\mathbb{R}$, $0<\mu_m^2<1$,

(c) $U^{(\vec{\lambda},1/2)}$ if there is m such that $\lambda_m\in i\mathbb{R}$,

$\mu_m\in\mathbb{R}$, $0<\mu_m^2<\tfrac{1}{4}$,

(d) $U_m^{\lambda,l l}$ $(l>1)$ if $\lambda_m\in i\mathbb{R}$,

(e) $U_m^{\lambda,12;0}$, $U_m^{\lambda,12,m}$, $U_m^{\lambda,13/2,1/2}$ if $\lambda_m\in i\mathbb{R}$.

E. Case of multiple splitting

The splitting of U^L into the factors of Table III is co-ordinate-dependent since it derives from the induction by steps to Γ_3^*. If more than one of the numbers $\mu_i=\tfrac{1}{2}(\lambda_{i'}-\lambda_{i''})$ are integers or half-integers (in which case all of them are, since $\mu_1+\mu_2+\mu_3=0$) U^L may be reducible in more than one way. So we are going to study the splitting of H^k into irreducible subspaces.

Let us first observe that if V_i^L is not irreducible, in which case $2\mu_i\in Z-\{0\}$, then there is one and only one invariant proper subspace $I\mathcal{J}_i$ of $\mathcal{J}^{(k)}$, isomorphic to either $\mathcal{J}^{1l}$ or $\mathcal{J}^{l l}$, for $l=1+|\mu_i|$ as seen through (6.7) and (6.8), the representation being indecomposable. The corresponding invariant subspace of $H^{(k)}$ (which is fixed once and for all by the parametrization $T_3^* T_2^* T_3^*$) will be denoted by $\mathcal{G}_i^{(k)}$ and it is one of the two sub-spaces defined by (6.10).

To carry on the reduction of U^L we shall begin by proving four lemmas:

Lemma 1: For fixed k and $i\in\{1,2,3\}$ at most two of the three $V_i^{(\vec{\lambda},k)}$ are reducible.

Proof: Let $\mu_1=\tfrac{1}{2}(\lambda_2-\lambda_3)$ and μ_2 μ_3 be defined by circular permutation. Reducibility of $V_i^{(\vec{\lambda},k)}$ implies μ_i half-integer (resp. odd, even) if $k=\tfrac{1}{2}$ (resp. $k=0$ or i; $k\neq 0,i,\tfrac{1}{2}$). This cannot happen for $i=1,2,3$ and fixed k, because $\mu_1+\mu_2+\mu_3=0$. Q.E.D.

Lemma 2: Let $A\oplus A'$ be a decomposition of $\mathbb{C}^{2l+1}$ into two eigenspaces of the generator J_{12} of SU(2) represented by D^l on $\mathbb{C}^{2l+1}$, with $l>1$. Let C be the two-dimensional subspace generated by the eigenvectors ψ_1 and ψ_{-1} of J_{23}. Then $C\cap A\neq\{0\}$ implies $A'=\{0\}$ and conversely.

Proof: Since $l>1$, the five-dimensional SU(2) sub-module $J^{(2)}$ of the enveloping algebra of su(2) does not vanish identically on $\mathbb{C}^{2l+1}$. If $C\cap A\neq\{0\}$, then A contains an eigenvector of $(J_{23})^2-\tfrac{1}{3}l(l+1)$, which is an element of $J^{(2)}$, for the eigenvalue $\tfrac{1}{3}(2l-1)$; since A is invariant under J_{23} it is also an eigenspace of each element of $J^{(2)}$, hence of $[J^{(2)},J^{(2)}]=J^{(3)}+J^{(1)}$, hence A is a proper invariant subspace of $\mathbb{C}^{2l+1}$, hence it is $\mathbb{C}^{2l+1}$ itself because of the irreducibility of D^l. Q.E.D.

Suppose now that V_1^L and V_2^L are reducible. We are interested in the intersection $\mathcal{G}_1^{(k)}\cap\mathcal{G}_2^{(k)}$ and the sum $\mathcal{G}_1^{(k)}+\mathcal{G}_2^{(k)}$. We can always assume, without loss of generality, that $l=l_1\geqslant l_2=l'$ and that $I\mathcal{J}=\mathcal{J}^{l l}$. Lemma 2 still holds if we take a fixed multiple of D^l on a space $F=\mathbb{C}^{2l+1}\otimes E$ instead of D^l on $\mathbb{C}^{2l+1}$. Taking for F the subspace H_l of $L^2(SU(2))$, on which the Casimir takes the value $l(l+1)$, with respect to the *left* regular representation of SU(2) (which serves to indicate the SU(2) multiplicity in U^L), putting $C\subset\mathcal{G}_1^{(k)}$, $A\subset\mathcal{G}_2^{(k)}$, we may apply Lemma 2, knowing that $A'\neq\{0\}$ (because $l'\leqslant l$). Thus we obtain:

Lemma 3: Let l_{12} be the lowest value of j for which $H_j\cap\mathcal{G}_1^{[l;k}\cap\mathcal{G}_2^{(k)}\neq\{0\}$, with $l=l_1\geqslant l_2=l'$; then $l_{12}>l$.

On the other hand, let $\text{Mult}(j;i;k)$ denote the j-multiplicity of $\mathcal{G}_i^{(k)}$, obtained from Table III for the different choices of $I\mathcal{J}_i$ and $N(j;k)$ the one of $H^{(k)}$ as indicated in Table II. The inequality

$$\text{Mult}(j;1;k)+\text{Mult}(j;2;k)\leqslant N(j;k)$$

is always satisfied, except for the case of integer l_1 and l_2, of $I\mathcal{J}_2=\mathcal{J}^{l l}$ and of $j\geqslant l_1+l_2-1$ as seen by direct computation. So we have:

Lemma 4: Let l_{12} be as in Lemma 3; let l_1 and l_2 be integers greater than 1 and $I\mathcal{J}_i=\mathcal{J}^{l l}$ for $i=1,2$; then $\mathcal{G}_1^{(k)}\cap\mathcal{G}_2^{(k)}\neq\{0\}$ and $l_{12}\leqslant l_1+l_2-1$.

One can now observe that the expression of the infinitesimal generators of $sl(3, \mathbb{R})$ depends only on $\tilde{\lambda}$, so that the expression of the Casimir operators of G is the same for $U^{(\tilde{\lambda}, k)}$ and $U^{(\tilde{\lambda}, k')}$ and independent of the splitting (if there is one) of $\mathcal{H}^{(k)}$. One can see without any calculation that they must be represented by scalars since they must be scalars on the subspace of constant functions of $SU(2)$ which is the only eigenspace in $\mathcal{H}$ for the eigenvalue zero of w. Their exact expression has been calculated by Šijački in Ref. 4, the correspondence between his notation and ours being

$$(\psi, \varphi, \vartheta) \to (\alpha, \beta, \gamma),$$

$$\lambda \to (\sigma_1 + i\sigma_2)/3, \quad \mu \to \lambda_1 + i\lambda_2, \quad l \to K_{\min},$$

$$a \to 4I_2 + 1, \quad b \to -24iI_3.$$

Remark: The suspicious reader who will carry out the calculations will observe that the λ's and μ's introduced in this section independently of those introduced previously happen to coincide, for the biggest relief of everyone who has read that far!

The fact that the Casimir operators are expressed by scalars makes possible the use of Theorem 1 which gives an exhaustive and finite list of all possible irreducible factors of U^L. The limitations on the RLW representations enables us to find their multiplicity and the multiplicity of D^j in each factor by mere subtraction, using the results of Table III. In particular one can see that, if $l_1 \geq l_2 > 1$ are two lowest weights which effectively occur, the third one is either $l_3 = l_1 + l_2 - 1 > l_1$, corresponding to the condition

$$(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) > 0 \tag{6.14a}$$

or $l_3 = l_1 - l_2 + 1 < l_1$, corresponding to

$$(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) < 0. \tag{6.14b}$$

It is easy to check that if the two $\mathcal{G}_i^{(k)}$ are of the same kind, then (6.14a) holds and if not it is (6.14b) which is true, so that one obtains:

Lemma 5: When $l = l_1 \geq l' = l_2$, a necessary and sufficient condition that $\mathcal{G} = \mathcal{G}_1^{[l;k} \cap \mathcal{G}_2^{(k)} \neq \{0\}$ is (a) l and l' integers and (b) $\mathcal{G}_2^{(k)} = \mathcal{G}_2^{[l';k}$ [or, equivalently, (6.14a)]. If and only if that happens, then the lowest weight of $\mathcal{G}$ is $l + l' - 1$ and there is a finite dimensional factor in U^L, acting on $\mathcal{H}^{(k)}/(\mathcal{G}_1^{l;k} + \mathcal{G}_2^{l';k})$ with highest weight $l + l' - 3$.

We shall now proceed to the ultimate reduction of U^L and each of its factors. The three different parametrizations (λ_i, μ_i) of λ transform to one another by

$$(\lambda, \mu) \to \tfrac{1}{2}(\lambda, \mu) \cdot \begin{pmatrix} -1 & \pm 3 \\ \pm 1 & -1 \end{pmatrix}$$

so that

$$|\mu_\alpha| = \mathrm{Sup}_i |\mu_i| \Longleftrightarrow |\mu_\alpha| \geq 3 |\lambda_\alpha|.$$

On the other hand, if $|\mu_\alpha| < 3 |\lambda_\alpha|$, then

$$2 \, \mathrm{Sup}_i |\mu_i| = |\mu_\alpha| + 3 |\lambda_\alpha|.$$

From what precedes one obtains (denoting by $\overline{U}$ the direct sum of the irreducible factors of the representation U):

Theorem 5: Suppose that more than one of the numbers $2\mu_i$ obtained from $\tilde{\lambda}$ are nonzero relative integers and let $l_i = 1 + |\mu_i|$; for any couple (λ, l) obtained from $\tilde{\lambda}$ let $l' = \tfrac{1}{2}(l - 1 + 3|\lambda|) + 1$, $l'' = \tfrac{1}{2}|l - 1 - 3|\lambda||+1$, $\lambda' = \tfrac{1}{2}[-\lambda + (l-1)\lambda/|\lambda|]$, $\lambda'' = -\lambda - \lambda'$, and let ϵ denote the sign of the permutation $\tau: (\lambda_1, \lambda_2, \lambda_3) \to (\lambda, \lambda', \lambda'')$ and $(i, i', i'') = \tau(1, 2, 3)$. Let $k(a, b)$ be the function defined by

$$a \in \tfrac{3}{2} + \mathbb{N}, 2 + 2\mathbb{N}, 2 + 2\mathbb{N}, 1 + 2\mathbb{N}, 1 + 2\mathbb{N},$$

$$b \in \tfrac{3}{2} + \mathbb{N}, 2 + 2\mathbb{N}, 1 + 2\mathbb{N}, 2 + 2\mathbb{N}, 1 + 2\mathbb{N},$$

$$k(a, b) = \tfrac{1}{2}, 0, 1, 2, 3, .$$

Then if and only if $l < 3|\lambda| + 1$ and $3l + 3|\lambda| + 1 \in 2\mathbb{N}$ (in which case $l' \geq 3|\lambda'| + 1$), we have the following decompositions for $k' = \tau k(l, l')$:

$$\overline{U}^{(\tilde{\lambda}, k')} = \overline{U}_i^{\lambda, 1; k', \epsilon} + \overline{U}_{i, i'}^{[l_i] l'', \epsilon} + \overline{U}_{i'}^{\lambda', [l''},$$

$$\overline{U}_i^{\lambda, [l} = \overline{U}_{i, i'}^{[l_i] l'', \epsilon} + \overline{U}_{i'}^{\lambda', [l''},$$

$$\overline{U}_{i'}^{\lambda', 1l''; k', \epsilon} = \overline{U}_i^{\lambda, 1l; k', \epsilon} + \overline{U}_{i, i'}^{[l_i] l'', \epsilon}.$$

If and only if $l < 3|\lambda| + 1$ and $k(l, l'') = \tfrac{1}{2}$, then

$$\overline{U}^{(\tilde{\lambda}, 1/2)} = \overline{U}_i^{\lambda, [l} + \overline{U}_{i'}^{\lambda'', [l''} + \overline{U}_{i, i''}^{]l_i] l'', \epsilon}.$$

If and only if $l < 3|\lambda| + 1$ and $k(l'', l) \in \mathbb{N}$, then

$$U_i^{\lambda, 1l; k'', \epsilon} = \overline{U}_{i'', i'}^{[l'''] l'', \epsilon} + \overline{U}_{i, i'}^{]l_i] l'', \epsilon} \quad \text{with } k'' = \tau k(l'', l).$$

All factors appearing in the right-hand side of the decompositions are either irreducible (if $l \in \mathbb{N}$) or multiples of order two of an IR (if $l \in \tfrac{1}{2} + \mathbb{N}$).

Remark 1: The finite-dimensional representations $U_{i, i'}^{]l_i] l'', \epsilon}$ are equivalent to the well-known tensor representations $(l - 2, l'' - 2)$ for $\epsilon = 1$ or $(l'' - 2, l - 2)$ for $\epsilon = -1$.

The j-multiplicity of these representations is obtained by adequate subtractions from Proposition 1 and Table II. The results are shown in Table IV(a)—(e); since for given $\tilde{\lambda}$ there is no multiple splitting for all values of k (for some of them there is no splitting at all, e.g., $l, l', l'' \in \mathbb{N}$ and $k = \tfrac{1}{2}$), only the decompositions due to Theorem 5 figure in these tables. We point out that for half-integer l the representations are reducible and they are multiples of order two of an IR, so that the j-multiplicities of the isotypic components are obtained from Table IV(a) and (b) by taking the half of the values given (all of them are even, of course).

7. SUMMARY AND REMARKS

(a) The exhaustive description of IR's of $\overline{SL(3, \mathbb{R})}$ is given in Theorem 1 (Sec. 4), in terms of the values taken by the Casimir operators and of the lowest $SU(2)$ representation which occurs. In Theorem 3 (Sec. 6) the equivalence criterion between induced representations is given.

(b) The necessary condition for unitarity, in Theorem 2A + 2B + 2C + 2D (Sec. 5), and the sufficient one in Theorem 4, coincide. We are not aware of any previous rigorous proof of the necessary condition. The sufficient condition has in fact already been established by Šijački, although the formulation of his results is not very clear in what concerns the supplementary series; we have in fact proved by the induction-by-steps theo-

TABLE IV. j-multiplicities in multiple splitting.

(a): $l_1 = l$ and $l_2 = l'$ half-integers, $\xi, \xi' \in \{0, 1\}$.

$j - l = 2n + \xi - 2$ $j - l' = 2n' + \xi' - 2$	$U_1^{\lambda, 1l; 1/2, +}$	$U_{1,2}^{[l_1, 1l', +}$	$U_2^{\chi, l l'}$
$\tfrac{1}{2} \leqslant j < l$	$2j + 1$	0	0
$l - 2 \leqslant j < l'$	$j + l + \xi - 1$	$2n$	0
$l' - 2 \leqslant j$	$j + l + \xi - 1$	$l' - l + \xi' - \xi$	$2n'$

(b): $l_1 = l$ and $l_3 = l''$ half-integers, $\xi, \xi'' \in \{0, 1\}$.

$j - l = 2n + \xi - 2$ $j - l'' = 2n'' + \xi'' - 2$	$U_{1,3}^{1l_1 1l'', +}$	$U_1^{\lambda, l l}$	$U_3^{\chi'', l l''}$
$1/2 \leqslant j < l$	$2j + 1$	0	0
$l - 2 \leqslant j > l''$	$j + l + \xi - 1$	$2n$	0
$l'' - 2 \leqslant j$	$l + \xi + l'' + \xi'' - 3$	$2n$	$2n''$

(c): $l_1 = l = 2p + 1$, $l_3 = l'' = 2q + 1$, $l_2 = l' = 2p + 2q + 1$.

$j = 2m + \xi, \ \xi \in \{0, 1\}$	$U_1^{\lambda, 1l; 3, +}$	$U_{1,2}^{[l_1 1l', +}$	$U_{1,3}^{1l_1, 1l'', +}$	$U_2^{\chi, l l'}$	$U_{3,2}^{[l'', 1l', -}$	$U_3^{\chi', 1l''; 1, -}$
$0 \leqslant j < l$	$m + \xi$	0	$m + \xi$	0	0	$m + \xi$
$l - 2 \leqslant j < l''$	p	$m - p + \xi$	p	0	0	$m + \xi$
$l'' - 2 \leqslant j < l'$	p	$m - p + \xi$	$p + q - m - \xi$	0	$m - q + \xi$	q
$l' - 2 \leqslant j$	p	q	0	$m - p - q + \xi$	p	q

(d): $l_1 = l = 2p + 2$, $l_3 = l'' = 2q + 1$, $l_2 = l' = 2p + 2q + 2$.

$j = 2m + \xi, \ \xi \in \{0, 1\}$	$U_1^{\lambda, 1l; 0, +}$	$U_{1,2}^{[l_1, 1l', +}$	$U_{4,3}^{1l_1 1l'', +}$	$U_2^{\chi, l l'}$	$U_{3,2}^{[l'', 1l', -}$	$U_3^{\chi', 1l''; 2, -}$
$0 \leqslant j < l$	$m + 1 - \xi$	0	$m + \xi$	0	0	$m + \xi$
$l - 2 \leqslant j < l''$	$p + 1 - \xi$	$m - p$	$p + \xi$	0	0	$m + \xi$
$l'' - 2 \leqslant j > l'$	$p + 1 - \xi$	$m - p$	$p + q - m$	0	$m - q + \xi$	q
$l' - 2 \leqslant j$	$p + 1 - \xi$	q	0	$m - p - q$	$p + \xi$	q
$0 \leqslant j < l''$	$m + 1 - \xi$	0	$m + \xi$	0	0	$m + \xi$
$l'' - 2 \leqslant j < l$	$m + 1 - \xi$	0	q	0	$m - q + \xi$	q
$l - 2 \leqslant j < l'$	$p + 1 - \xi$	$m - p$	$p + q - m$	0	$m - q + \xi$	q
$l' - 2 \leqslant j$	$p + 1 - \xi$	q	0	$m - p - q$	$p + \xi$	q

(e): $l_1 = l = 2p + 2$, $l_3 = l'' = 2q + 2$, $l_2 = l' = 2p + 2q + 3$.

$j = 2m + \xi, \ \xi \in \{0, 1\}$	$U_1^{\lambda, 1l; 1, +}$	$U_{1,2}^{[l_1, 1l'', +}$	$U_{1,3}^{1l_1 1l'', +}$	$U_2^{\chi, l l'}$	$U_{3,2}^{[l'', 1l', -}$	$U_3^{\chi', 1l''; 3, -}$
$0 \leqslant j < l$	$m + \xi$	0	$m + 1 - \xi$	0	0	$m + \xi$
$l - 2 \leqslant j < l''$	$p + \xi$	$m - p$	$p + 1 - \xi$	0	0	$m + \xi$
$l'' - 2 \leqslant j < l'$	$p + \xi$	$m - p$	$p + q + 1 - m - \xi$	0	$m - q$	$q + \xi$
$l' - 2 \leqslant j$	$p + \xi$	$q + 1 - \xi$	0	$m - p - q + \xi - 1$	$p + 1 - \xi$	$q + \xi$

rem that both $l = 1$ and $l = 0$ occur as lowest weights for this series [case (b) of Theorem 4].

(c) The SU(2) content is studied in Sec. 6, as well as the decomposition of the induced representations into irreducible components. When no RLW occur for given values of the Casimirs, the multiplicity of D^j inside an IR increases (roughly) like j if $j \in \tfrac{1}{2} + \mathbb{N}$, and like $\tfrac{1}{2}j$ if $j \in \mathbb{N}$, as described in Table II. For IR's of the RLW series it increases like $\tfrac{1}{2}(j - l)$ regardless of the parity of $2l$. If more than one RLW IR occurs, the D^j content is given in Table IV(a)—(e); notice that there are factors having bounded or even constant j-multiplicities in this case.

(d) The finite-dimensional representations—corresponding to integer j only (the twofold covering is not isomorphic to any matrix group)—yield, by complexification, the UIR's of SU(3)—see Remark 2 following Theorem 5. On the contrary, the UIR's of the other real form SU(2, 1) do not appear here: Except the trivial IR, the ones treated here yield finite-dimensional representations of SO(2, 1) $\subset$ SU(2, 1), hence nonunitary ones.

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APPENDIX

We give here, without proof, the expression of the compact infinitesimal generators and of the noncompact generator H_{33} in terms of the Euler angles ψ, φ, ϑ:

$$J_2 = \frac{\partial}{\partial \vartheta},$$

$$J_{13} \pm iJ_{23} = \exp(\pm i\vartheta) \left(\pm i \frac{\partial}{\partial \varphi} + \cot g \varphi \frac{\partial}{\partial \vartheta} - \sin\varphi^{-1} \frac{\partial}{\partial \psi} \right),$$

$$H_{33} = - \sin 2\varphi \frac{\partial}{\partial \varphi} - (1 + \lambda)(3 \cos^2\varphi - 1)$$

$$+ \sin^2\varphi \left(\sin 2\psi \frac{\partial}{\partial \psi} + (1 - \mu) \cos 2\psi \right).$$

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Scattering in the depth direction for an anisotropic random medium

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We consider in this paper the asymptotic nature of scattering in the depth direction in an anisotropic random medium like the ocean. We use the coherence equation developed previously. It is shown that as the propagation distance goes to infinity, the integral of the square of the coherence function, $\{\hat{\Gamma}\}$, approaches zero. [The integral of $\{\hat{\Gamma}\}$ itself remains constant as a result of conservation of energy.] It is pointed out, however, that the approach to zero is expected to be very slow once we are well into the multiple-scatter region. For most practical cases, a quasi-asymptotic state is reached for which the integral of the square of $\{\hat{\Gamma}\}$ is approximately a finite constant and the shape of $\{\hat{\Gamma}\}$ no longer changes.

1. INTRODUCTION

Beran and McCoy[1,2] derived the following equation governing the coherence function in an anisotropic, statistically homogeneous random medium,

$$\frac{d}{dz}\{\tilde{\Gamma}(x_{12},\ \zeta_y,\ z)\} = -2\pi\tilde{\sigma}_3(0,\ 0,\ \zeta_y)\{\tilde{\Gamma}(x_{12},\ \zeta_y,\ z)\}$$
$$+ 2\pi\int_{-\infty}^{\infty}\tilde{\sigma}_3(x_{12},\ \zeta_y,\ \zeta_y')\{\tilde{\Gamma}(x_{12},\ \zeta_y',\ z)\}d\zeta_y'.$$

(1)

We have here

$$\{\tilde{\Gamma}(x_{12},\ \zeta_y,\ z)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(iy_{12}\zeta_y)\{\hat{\Gamma}(x_{12},\ y_{12},\ z)\}dy_{12},$$

where

$$\{\hat{\Gamma}(x_{12},\ y_{12},\ z)\} = \{p(x_1,\ y_1,\ z)p^*(x_2,\ y_2,\ z)\},$$

$$x_{12} = x_1 - x_2,$$

$$y_{12} = y_1 - y_2.$$

The pressure is denoted by $p(x, y, z)$, where z is the propagation direction, y is the depth direction, and x is the transverse direction. The brackets indicate an ensemble average and $\{\tilde{\Gamma}(0, \zeta_y, z)\}$ gives the angular spectrum in the depth direction. The coherence function, $\{\hat{\Gamma}(x_{12}, y_{12}, z)\}$ is assumed to depend only on the transverse difference coordinates x_{12} and y_{12} (in addition to z). Thus the initial condition at $z = 0$ must satisfy this requirement. For example, a plane wave, propagating in the z direction, is an appropriate initial condition.

The function $\bar{\sigma}_3(x_{12}, y_{12}, \zeta_y')$ is given by

$$\bar{\sigma}_3(x_{12},\ y_{12},\ \zeta_y') = \left(\frac{2}{\pi}\right)^{1/2}\frac{\bar{k}^3}{8\pi}\int_0^{\infty}\left[\cos\left(\frac{\bar{k}y_{12}^2}{2z_{12}} - \frac{\pi}{4} + \frac{z_{12}\zeta_y'^2}{2\bar{k}}\right)\right.$$
$$\left.\times\left(\frac{1}{|\bar{k}z_{12}|}\right)^{1/2}\left[\int_{-\infty}^{\infty}\sigma(x_{12},\ y_{12}',\ z_{12})\ dy_{12}'\right]\right]dz_{12}$$

(2)

where $\sigma(x_{12},\ y_{12},\ z_{12}) = \{\mu'(x_1,\ y_1,\ z_1)\mu'(x_2,\ y_2,\ z_2)\}$

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is the correlation function associated with variations in the ocean index of refraction. The wave number k is written as

$$k^2 = \bar{k}^2(1 + \mu'),$$

(3)

where $\bar{k}$ is the average wave number.

The function σ is characterized by two length scales, l_H in the horizontal direction and l_V in the vertical direction. The following conditions are assumed to be satisfied:

$$\bar{k}l_H \gg 1, \quad \frac{\bar{k}l_V^2}{l_H} \ll 1.$$

(4)

The function $\tilde{\sigma}_3(x_{12},\ \zeta_y,\ \zeta_y')$ is the Fourier transform of $\bar{\sigma}_3(x_{12},\ y_{12},\ \zeta_y')$.

In Ref. 1 it was conjectured that $\{\tilde{\Gamma}(0, \zeta_y, z)\}$ reached an asymptotic state as $z \to \infty$. (z will be suitably nondimensionalized later in the paper.) Here it is shown that such an asymptotic state cannot exist but instead a quasiasymptotic state is to be expected. We first show, by considering the equation governing $J^2 = \int_{-\infty}^{\infty}\{\tilde{\Gamma}(0, \zeta_y, z)\}^2 d\zeta_y$, that a strict asymptotic solution for $\{\tilde{\Gamma}\}$ cannot exist. We next examine the nature of $d\{\tilde{\Gamma}\}/dz$ and dJ^2/dz and show that a quasiasymptotic state is expected in the multiple scatter region. The existence of this quasiasymptotic state is demonstrated by numerical calculation using typical ocean parameters.

2. NATURE OF THE SOLUTION OF EQ. (1) WHEN $x_{12} = 0$

A. Basic equation

When $x_{12} = 0$ we find from Eq. (1)

$$\frac{d}{dz}\{\tilde{\Gamma}(0,\ \zeta_y,\ z)\} = -2\pi\bar{\sigma}_3(0,\ 0,\ \zeta_y)\{\tilde{\Gamma}(0,\ \zeta_y,\ z)\}$$
$$+ 2\pi\int_{-\infty}^{\infty}\tilde{\sigma}_3(0,\ \zeta_y,\ \zeta_y')\{\tilde{\Gamma}(0,\ \zeta_y',\ z)\}d\zeta_y'.$$

(5)

We can show from Eq. (2) that $\tilde{\sigma}_3(0, \zeta_y, \zeta_y')$ is symmetric in ζ_y and ζ_y' and we note that

$$\bar{\sigma}_3(0,\,0,\,\zeta_y) = \int_{-\infty}^{\infty} \tilde{\sigma}_3(0,\,\zeta_y,\,\zeta_y')\,d\zeta_y'. \tag{6}$$

B. Conservation of intensity

If we integrate both sides of Eq. (1) with respect to $d\zeta_y$, we find

$$\frac{d}{dz} \int_{-\infty}^{\infty} \{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}\,d\zeta_y = 0 \tag{7}$$

and

$$\hat{I} = \int_{-\infty}^{\infty} \{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}\,d\zeta_y, \tag{8}$$

where $\hat{I}$, the intensity, is, as demanded, a constant.

C. Variation of $J^2 = \int_{-\infty}^{\infty} \{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}^2 d\zeta_y$

We next multiply Eq. (5) by $\{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}$ and then integrate both sides with respect to ζ_y. We find, after some manipulation, in which we make use of the symmetry of $\tilde{\sigma}_3(0,\,\zeta_y,\,\zeta_y')$,

$$\frac{d}{dz} \int_{-\infty}^{\infty} \{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}^2 \,d\zeta_y$$

$$= -2\pi \int_{-\infty}^{\infty}\!\!\int [\{\tilde{\Gamma}(0,\,\zeta_y,\,z\} - \{\tilde{\Gamma}(0,\,\zeta_y',\,z)\}]^2$$

$$\times \tilde{\sigma}_3(0,\,\zeta_y,\,\zeta_y')\,d\zeta_y d\zeta_y'. \tag{9}$$

D. Nonexistence of an asymptotic state

If $\{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}$ is to become independent of z as $z \to \infty$, then the integral

$$J^2 \equiv \int_{-\infty}^{\infty} \{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}^2 \,d\zeta_y$$

must, of course, become independent of z. From Eq. (9), we see that J^2 must decrease as z increases. Since it is nonnegative, its limit, as $z \to \infty$, must exist. Now, $(d/dz)J^2$ is strictly negative so long as $\{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}$ is not constant (in ζ_y). Thus, if $\{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}$ has any limit at all as $z \to \infty$, the limit must be a constant (in ζ_y). This constant must be zero, for any other would violate the conservation of energy. Hence we conclude that $\{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}$ does not approach an asymptotic state in the usual sense of the term.

E. Quasiasymptotic state

If we had no further knowledge of the nature of $\tilde{\sigma}_3(0,\,\zeta_y,\,\zeta_y')$, we could not say more about the nature of $\{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}$ as $z \to \infty$. In physical problems, however, we usually have additional knowledge about the nature of this function. We see from Eq. (5) that $\tilde{\sigma}_3(0,\,\zeta_y,\,\zeta_y')$ is a scattering function that determines how much energy is transferred from ζ_y' to ζ_y. From Eq. (2) we find that

$$\tilde{\sigma}_3(0,\,\zeta_y,\,\zeta_y') = \frac{\bar{k}^2}{8\pi}$$

$$\times \int_{-\infty}^{\infty} \exp\!\left(i\frac{s_z}{2\bar{k}}(\zeta_y^2 - \zeta_y'^2)\right) \sigma_2(0,\,s_z)ds_z, \tag{10}$$

where

$$\sigma_2(0,\,s_z) \equiv \int_{-\infty}^{\infty} \sigma(0,\,s_y',\,s_z)ds_y'.$$

In general, $\sigma_2(0,\,s_z)$ is a function that has a maximum characteristic decay length l_H. From Eq. (10), we see that if ζ_y' is of the order $(2\bar{k}/l_H)^{1/2}$ or less energy is transferred from ζ_y' to a range of values ζ_y, where ζ_y is also of order $(2\bar{k}/l_H)^{1/2}$ or less. If, however, ζ_y' is of the order $C(2\bar{k}/l_H)^{1/2}$ where $C \gg 1$, then significant amounts of energy are transferred only to those ζ_y for which $|\zeta_y' - \zeta_y|/|\zeta_y| \ll 1$. This may be seen more clearly if we write

$$\tilde{\sigma}_3(0,\,\zeta_y,\,\zeta_y') = \frac{\bar{k}^2}{8\pi}$$

$$\times \int_{-\infty}^{\infty} \exp\!\left(\frac{is_z}{2\bar{k}}(\zeta_y + \zeta_y')(\zeta_y - \zeta_y')\right) \sigma_2(0,\,s_z)ds_z. \tag{11}$$

If $\zeta_y' \sim O[C(2\bar{k}/l_H)^{1/2}]$, then the value of the exponent will be very large unless $|\zeta_y' - \zeta_y|/|\zeta_y| \ll 1$. If the value of the exponent is very large, the exponential will oscillate rapidly on the scale $s_z = l_H$ and little energy will be transferred from ζ_y' to ζ_y. Hence we may conclude that as the characteristic decay scale of $\{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}$, say, ζ_{yc} grows large [that is, $\gg (2\bar{k}/l_H)^{1/2}$], less and less energy will be scattered from any particular ζ_y and hence a quasiasymptotic state will be reached. We cannot of course prove that this is true for all functions $\sigma_2(0,\,s_z)$. However, we expect that if $\sigma_2(0,\,s_z)$ does not have very unusual properties $\{\tilde{\Gamma}(0,\,\zeta_y,\,z)\}$ will behave in this manner.

We note that the above behavior is consistent with dJ^2/dz (suitably nondimensionalized) becoming small as z becomes large. This may be seen from Eq. (9) for when $|\zeta_y' - \zeta_y|/|\zeta_y| \to 0$, the difference $[\{\tilde{\Gamma}(0,\,\zeta_y,\,z)\} - \{\tilde{\Gamma}(0,\,\zeta_y',\,z)\}]^2$ similarly approaches zero. On the other hand, if $|\zeta_y' - \zeta_y|/|\zeta_y| \sim O(1)$, $\sigma_3(0,\,\zeta_y,\,\zeta_y')$ approaches zero when ζ_y' becomes very large. We therefore expect the integral on the right-hand side of Eq. (9) to approach zero as $z \to \infty$.

In order to test the validity of the above arguments, we have performed a numerical calculation and we present the results in Figs. 1 and 2.[3] The parameters have been chosen to model a real ocean situation and we have chosen

$$\tilde{\sigma}_3(0,\,\zeta_y,\,\zeta_y') = \frac{A}{8\pi}\left(1 + \frac{1}{(2\bar{k}p_M)^2}(\zeta_y^2 - \zeta_y'^2)^2\right)^{-1}, \tag{12}$$

where $p_M = 2\pi/l_H$ and A is a constant which determines the strength of the scattering. A discussion at ocean temperature fluctuations may be found in a report by Moseley and Del Balzo.[4]

As an initial condition we would ideally like to choose a plane wave incident in the z direction. For numerical reasons, however, we could not use a delta function in ζ_y. We used instead the condition

$$\tilde{\Gamma}(0,\,\zeta_y,\,0) = \frac{b\hat{I}\exp(-b^2\zeta_y^2/2)}{\sqrt{2\pi}}. \tag{13}$$

We chose the following numerical values which would be typical of ocean propagation:

$$\bar{k} = .42m \quad (\bar{\nu} = 100 \text{ Hz}),$$

$$p_M = \pi \times 10^{-4}m^{-1} \quad (l_H = \frac{2\pi}{p_M} = 20 \text{ km}),$$

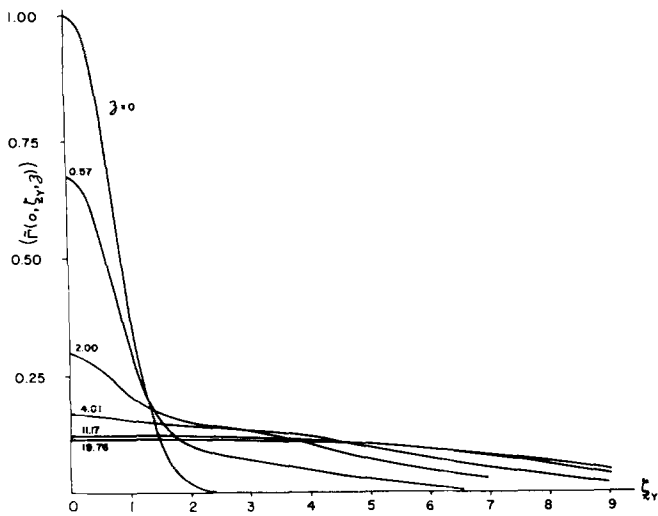FIG. 1. $\{\tilde{\Gamma}(0, \zeta_y, \jmath)\}$ vs. ζ_y for various $\jmath$.

$A = .08,$

$b = 313$ m.

We use the nondimensionalized parameters,

$$\jmath = 2\pi\tilde{\sigma}_3(0, 0, 0)z, \quad \zeta_y = \zeta_y(b/\sqrt{2}).$$

When $\jmath \gg 1$ we consider that we are well into the multiple scatter region. When $\jmath \ll 1$ we are in the single scatter region.

In Fig. 1 we can see the change in $\{\tilde{\Gamma}(0, \zeta_y, \jmath)\}$ as a function of $\jmath$. The change from $\jmath = 0$ to $\jmath = 2$ is very significant. However, beyond $\jmath = 4$ it is rather small.

The same trend is observed if we study dJ^2/dz. Here we nondimensionalize Eq. (9) by dividing both sides by $2\pi J^2\tilde{\sigma}(0, 0, 0)$. This yields

$$\kappa_c = \frac{1}{J^2}\frac{d}{d\jmath}\cdot J^2$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[\{\tilde{\Gamma}(0, \zeta_y, z)\} - \{\tilde{\Gamma}(0, \zeta_y', z)\}]^2}{\int_{-\infty}^{\infty}\{\tilde{\Gamma}(0, \zeta_y, z)\}^2 d\zeta_y}$$

$$\times\left[\frac{\tilde{\sigma}_3(0, \zeta_y, \zeta_y')}{\int_{-\infty}^{\infty}\tilde{\sigma}_3(0, 0, \zeta_y')d\zeta_y'} d\zeta_y d\zeta_y'\right].$$

We see from Fig. 2 that beyond the distance $\jmath \sim 10$ the value of κ_c is less than $\sim .01$. In fact, beyond $\jmath \sim 5$ the value is very small to values when $\jmath < 2$.

We emphasize again that we do not wish to imply by the above analysis and this numerical example that one could not skillfully choose initial radiation patterns, $\{\tilde{\Gamma}(0, \zeta_y, 0)\}$, and functions, $\tilde{\sigma}_3(0, \zeta_y, \zeta_y')$, for which a quasiasymptotic state would not be reached until distances z of order of say, 100. Our purpose, instead, was to demonstrate that for the problem, as it is often formulated, a quasiasymptotic state is to be expected as soon as we are well into the multiscatter region.

FIG. 2. κ_c vs. $\jmath$.
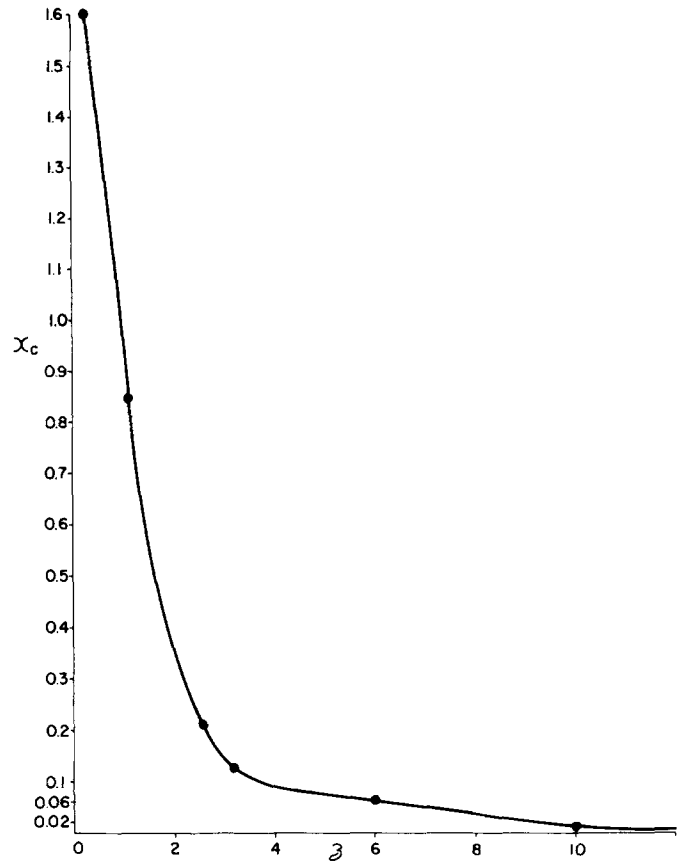
3. SUMMARY

We have shown in this paper that a true asymptotic state for $\{\tilde{\Gamma}(0, \zeta_y, z)\}$ cannot be reached in the limit $\jmath = 2\pi\sigma_3 z \to \infty$. However, for scattering functions such as the one given in Eq. (12), we expect to find a quasiasymptotic state in the multiple scatter region ($\jmath \gg 1$). These functions are characterized by the fact that when $\zeta_y' \gg (2\bar{k}p_M)^{1/2}$ the function $\tilde{\sigma}_3(0, \zeta_y, \zeta_y')$ $\to 0$ unless $|\zeta_y - \zeta_y'|/(2\bar{k}p_M)^{1/2} \ll 1$. The results were demonstrated by a numerical calculation.

ACKNOWLEDGMENT

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[3]The numerical procedure will be presented in detail in a separate paper. Our results were obtained by using an iteration technique in which the ζ_y' dependence in $\tilde{\sigma}_3(0, \zeta_y, \zeta_y')$ was neglected in the initial trial.

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Conditions on the uniqueness of the solution of the elastic unitarity equation

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We report new conditions for the uniqueness of the solution of the unitarity integral equation. These conditions follow from the consideration of the smallest value of the integral.

The elastic unitarity equation, considered as an integral equation for the phase of the amplitude when the modulus is known at all angles, has been investigated by several authors[1-7] in the past. We write the equation as

$$\sin\alpha(z) = \frac{q}{2\pi}\iint_{-1}^{+1} \frac{|f(x)|\,|f(y)|}{|f(z)|}\cos[\alpha(x) - \alpha(y)]\frac{\Theta(K)}{\sqrt{K}}dxdy. \tag{1}$$

Here f is the amplitude the square of which is the differential cross section. $\Theta(K)$ is the step function and $K = 1 - x^2 - y^2 - z^2 + 2xyz$. q is the center of mass wavenumber.

We consider the integral

$$\frac{q}{2\pi}\iint_{-1}^{+1} \frac{|f(x)|\,|f(y)|}{|f(z)|}\frac{\Theta(K)}{\sqrt{K}}dxdy = \sin\mu(z), \tag{2}$$

which is a function of z. We shall call the supremum of $\sin\mu(z)$ for all values of z ($-1 \le z \le +1$) $\sin\mu$. Its smallest value will be called $\sin\nu$. Since all the quantities under the integral are positive and the amplitude is assumed not to vanish anywhere, $\sin\nu$ will exist. It is clear that

$$\sin\alpha(z) \le \sin\mu(z) \le \sin\mu. \tag{3}$$

We consider $\alpha(z)$ in the domain $(0, \pi/2)$ (see for details Ref. 2,3). Since the smallest value of the unitarity integral (1) is obtained when $\cos[\alpha(x) - \alpha(y)]$ is smallest $|\alpha(x) - \alpha(y)|$ must be largest. But the largest value of $\alpha(z)$ is μ. Hence

$$\sin\alpha(z) \ge \frac{q}{2\pi}\iint_{-1}^{+1} \frac{|f(x)|\,|f(y)|}{|f(z)|}\cos\mu\,\frac{\Theta(K)}{\sqrt{K}}dxdy$$
$$= \cos\mu\sin\mu(z) \ge \cos\mu\sin\nu. \tag{4}$$

We shall define this as

$$\sin\alpha(z) \ge \sin\theta_{min} \equiv \cos\mu\sin\nu. \tag{5}$$

Hence the smallest value of the unitarity integral will be obtained when

$$\cos[\alpha(x) - \alpha(y)] = \cos(\mu - \theta_{min}) \tag{6}$$
$$= \cos\mu[(1 - \cos^2\mu\sin^2\nu)^{1/2} + \sin\mu\,\sin\nu] = \cos\mu a_1, \tag{7}$$

where Eq. (5) has been used. Hence

$$\sin\alpha(z) \ge \frac{q}{2\pi}\iint_{-1}^{+1} \frac{|f(x)|\,|f(y)|}{|f(z)|}\cos\mu\,a_1\frac{\Theta(K)}{\sqrt{K}}dxdy \tag{8}$$
$$= a_1\cos\mu\sin\mu(z) \ge a_1\cos\mu\sin\nu.$$

We shall write this as

$$\sin\alpha(z) \ge \sin\theta'_{min} \equiv a_1\cos\mu\sin\nu. \tag{9}$$

It is clear that $\cos(\mu - \theta_{min}) > \cos\mu$ or $a_1 > 1$. Therefore,

$$\theta'_{min} > \theta_{min}. \tag{10}$$

Substituting Eq. (9) into the integral (1), we find

$$\sin\alpha(z) \ge \frac{q}{2\pi}\iint_{-1}^{+1} \frac{|f(x)|\,|f(y)|}{|f(z)|}\cos(\mu - \theta'_{min})\frac{\Theta(K)}{\sqrt{K}}dxdy. \tag{11}$$

Here

$$\cos(\mu - \theta'_{min}) = \cos\mu[(1 - a_1^2\cos^2\mu\sin^2\nu)^{1/2} + a_1\sin\mu\sin\nu] \equiv \cos\mu\,a_2, \tag{12}$$

where use has been made of Eq. (9). Hence

$$\sin\alpha(z) \ge a_2\cos\mu\sin\mu(z) > a_2\cos\mu\sin\nu. \tag{13}$$

We shall write this as

$$\sin\alpha(z) \ge \sin\theta''_{min} = a_2\cos\mu\sin\nu. \tag{14}$$

It is clear that since $\theta'_{min} > \theta_{min}$,

$$\cos(\mu - \theta'_{min}) > \cos(\mu - \theta_{min}), \quad a_2 > a_1. \tag{15}$$

Thus we have the sequence

$$a_1 = (1 - \cos^2\mu\sin^2\nu)^{1/2} + \sin\mu\sin\nu,$$
$$a_2 = (1 - a_1^2\cos^2\mu\sin^2\nu)^{1/2} + a_1\sin\mu\sin\nu,$$
$$\cdots$$
$$a_n = (1 - a_{n-1}^2\cos^2\mu\sin^2\nu)^{1/2} + a_{n-1}\sin\mu\sin\nu, \tag{16}$$
$$\cdots$$

where each a_n is larger than a_{n-1}. If this sequence did not have a limit, then $\sin\alpha(z)$ would not have a lower limit. We shall now prove two results:

(a) $\displaystyle\lim_{n\to\infty} a_n(\mu,\nu) = (1 - 2\sin\mu\sin\nu + \sin^2\nu)^{-1/2}$. (17)

This limit is obtained by setting in the Eq. (16) for a_{n-1}, a_n and solving the equation. In particular for $\mu = \nu$, we have

$$\lim_{n\to\infty} a_n = 1/\cos\mu.$$

(b) If we call $\displaystyle\lim_{n\to\infty} a_n = a$, then

$$\frac{da}{d\nu}\bigg|_{\nu=\mu} = 0. \tag{18}$$

This property can either be obtained by differentiating the Eq. (16) and then going to the limit $a_{n-1} = a_n$, or directly from the Eq. (17). With the relation (17) we

can write

$$\sin\alpha(z) \geqslant a\cos\mu\sin\nu$$

$$= \frac{\cos\mu\sin\nu}{(1 - 2\sin\mu\sin\nu + \sin^2\nu)^{1/2}} \equiv \sin\theta_{Min}. \quad (19)$$

We now consider the uniqueness question of the solutions of the Eq. (1). Assume that there exist two solutions $\alpha(z)$ and $\beta(z)$ with $|f| = |g|$ where α and β are the respective phases of f and g. Following the method of Martin,[3] we can write

$$\mathrm{Im}f(z) - \mathrm{Im}g(z)$$

$$= \frac{q}{2\pi} \int\int\limits_{-1}^{+1} [\mathrm{Im}f(x) - \mathrm{Im}g(x)] \left[\frac{\mathrm{Im}f(y) + \mathrm{Im}g(y)}{\mathrm{Re}f(y) + \mathrm{Re}g(y)} \right.$$

$$\left. - \frac{\mathrm{Im}f(x) + \mathrm{Im}g(x)}{\mathrm{Re}f(x) + \mathrm{Re}g(x)} \right] \times [\mathrm{Re}f(y) + \mathrm{Re}g(y)] \frac{\Theta(K)}{\sqrt{K}} dx dy. \quad (20)$$

The large bracket under the integral can be majorized by taking the largest value of the first term and the smallest value of the second term. The integration of $\mathrm{Re}f(y) + \mathrm{Re}g(y)$ gives the real parts of the S-wave partial waves which are bounded by $\frac{1}{2}$ each so that their sum is bounded by 1:

$$\left| \mathrm{Im}f(z) - \mathrm{Im}g(z) \right| \leqslant (\tan\mu - \tan\theta_{Min}) \max \left| \mathrm{Im}f - \mathrm{Im}g \right|. \quad (21)$$

The uniqueness condition for the solution of the integral equation thus becomes

$$\tan\mu - \tan\theta_{Min} < 1. \quad (22)$$

Using the Eq. (19) in (22), we find the condition

$$\sin\nu > \frac{\sin\mu - \cos\mu}{1 - \sin\mu\cos\mu}. \quad (23)$$

Thus the solution is unique if the lowest value of the integral (2) and its supremum satisfy this inequality.

We notice that the relation (23) is always satisfied if

$$\mu \leqslant 45°.$$

That is when $\mu \leqslant 45°$, for the solution to be unique there is no lower bound required on the integral (2). Hence the modulus of f has complete freedom at the lower end.

The relations (17), (19), and (23) are our main results. As with the existence condition ($\sin\mu < 1$) for the solutions of the unitarity equation, the inequality (23) is a sufficiency condition and may not be necessary. For example, Martin[3] has shown that for up to $\sin\mu = 0.79$ no lower bound is needed, whereas the inequality (23) restricts $\sin\nu$ to $\sin\nu > 0.327$. On the other hand for larger values of $\sin\mu$, that is, for larger than 0.79, the uniqueness has not been proven yet. The inequality (23) should fill in this gap as a sufficiency condition in the region $45° - 90°$ for the uniqueness of the solution. It may be that for up to $\sin\mu = 1$, that is between 0.79 and 1 no restriction is needed for the lowest value of $\sin\mu(z)$, even though we do not know this. However, we feel that with more input for the modulus function one can obtain more definite results.

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Tensor fields invariant under subgroups of the conformal group of space-time[a]

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This work is concerned with the characterization of tensor fields in (compactified) Minkowski space which are invariant under the action of subgroups of the conformal group. The general method for determining all invariant fields under the smooth action of a Lie group G on a manifold M is given, both in global and in local form. The maximal subgroups of the conformal group are divided into conjugacy classes under the Poincaré group and the most general fields of 1-forms, 2-forms, symmetric (0,2) tensors and scalar densities which are invariant under representatives of each class (as well as certain other subgroups) are then determined. The results are then discussed from the viewpoint of physical interpretation (as, e.g., electromagnetic fields, metric tensors, etc.) and applicability; in particular, for studies of spontaneously or otherwise broken conformal invariance.

I. INTRODUCTION

Statement of the problem

In this paper we concern ourselves with the problem of determining the most general types of tensor fields on the space—time manifold which are invariant under subgroups of the conformal group. We shall obtain a complete solution of this problem for the following types of fields and groups: 1-forms, 2- forms, symmetric (0, 2) tensors and scalar densities invariant under the maximal subgroups of the conformal group, as well as under certain other subgroups of interest in physics. Such fields may be identified, for example, as electromagnetic potentials and fields, metric tensors, etc. The particular model for the space—time manifold that is used is the conformally compactified Minkowski space $\bar{M}$, upon which the conformal group acts as a global transformation group. This is a compact, infinitely connected manifold which possesses the same local structure as Minkowski space. That is, it admits a local (infinitesimal) causal orientation and therefore a pseudo-Riemannian metric of signature $(1, 3)$. However $\bar{M}$ does not possess a global causal structure since there exists within this space an infinity of closed, time-like geodesics, and hence it is not quite an adequate model for the space—time of relativistic physics. This difficulty may be satisfactorally resolved[1-3] by replacing $\bar{M}$ by its noncompact, simply connected covering space $\tilde{M}$ and, correspondingly, the conformal group with its universal covering group. Since such a replacement has no effect upon the invariance properties of

tensor fields, except insofar as relating the fields defined on different sheets of $\tilde{M}$, we shall make no use of it.

1. Background

The conformal group as a local transformation group in Minkowski space has long been known in physics as the invariance group for Maxwell's equations,[4,5] and more generally, for a wide variety of other field equations[6-9] (particularly those describing massless particles). Since it contains the invariance group of special relativity; that is, the Poincaré or inhomogeneous Lorentz group, it has often been suggested that the conformal group might have an equally basic interpretation as fundamental symmetry group for the geometry of space—time and the equations defined in it. In particular, considerable study has been made of the implications of conformal invariance in quantum field theory,[10,11] primarily motivated by attempts to interpret the approximate scaling invariance found experimentally in high energy scattering of elementary particles.[12] Apart from this, conformal invariance of field equations is a property shared by most models for the unified gauge field theories of weak, electromagnetic, and strong interactions,[13,14] due to the fact that the fields involved (prior to renormalization) are all massless. In this context, particle masses are introduced through spontaneous breaking of the gauge symmetry which also leads to breaking of the conformal invariance at the level of solutions to the field equations.[15-18] Solutions which minimize the total energy represent the classical analog of the quantum ground state, and the largest subgroup of the invariance group of the field equations which also leaves invariant these solutions will play the role of the fundamental symmetry group for the physical system. If for reasons of stability, either due to the type of interaction involved[19] or to the topological properties of space—time implied by the solutions,[20-22] this ground state possesses less symmetry than the dynamical equations, we necessarily have a spontaneously broken symmetry. At the level of space—time

symmetries it is therefore quite basic to ask which solutions to conformally invariant equations are themselves invariant under large subgroups of the conformal group. Certain particular cases of this question have been studied in the literature. For example, the "one instanton" solution for the SU(2) gauge theory is known to be completely characterized by its property of invariance under the O(5) subgroup of the conformal group O(5, 1) for Euclidean space—time.[13] Similarly, for the $\lambda \phi^4$ (scalar density) field theory, solutions invariant under O(3, 2) have been studied as a model for introducing a fundamental scale of length, thereby spontaneously breaking the O(4, 2) invariance of the field equations.[17]

In general relativity, conformal transformations also play a central role, several of the known cosmological models having the property of conformal flatness; that is, differing from a flat space by a conformal change of metric. The symmetry groups of some of these models [e. g. , O(4, 1) or O(3, 2) for the de Sitter spaces] moreover, are maximal subgroups of the conformal group. Alternative models for explaining cosmological data, such as the chronometric theory[1] of Segal, formulated in the space $\tilde{M}$, have also put particular emphasis on the underlying conformal invariance properties of space—time. The conformal group has furthermore been studied as a gauge group for field theories formulated in a non-Riemannian geometry, in which it plays a role analogous to that of the Lorentz group in general relativity.[23]

The subalgebras of the conformal Lie algebra have been studied recently as part of a general program for subalgebra structure analysis.[24] All the maximal subalgebras have been identified, up to conjugacy under the group,[25] and their subalgebra structures, in turn, have been determined completely.[26–27] These, and other subalgebra analyses have been applied in particular to the classification of symmetry breaking interactions in the Scrödinger equation[28, 29] as well as to other problems of interest in physics. The electromagnetic fields invariant under certain subgroups of the Poincaré group have been studied systematically with methods similar to those developed in the present work.[30–33]

2. Outline of development

The present work is divided into several distinct parts and may be read in a variety of ways, depending upon the interests of the reader. Sections II 3—II 5 deal with the problem of characterizing the most general tensor field on a manifold, invariant under a transformation group. The discussion is presented first in coordinate-free terms, both for global and local invariance (Sec. II 3). The relevant equations are then given in a coordinate representation [Eqs. (4.2), (4.3), and (4.6)]. Finally (Sec. II 5) a formulation in terms of fibre bundles is given which makes precise certain notions used in Sec. II 4. The following Secs. III 6—III 8

introduce all the definitions and notations needed with regard to the conformal group, realized successively as: a local transformation group in Minkowski space; a global transformation group in conformally compactified Minkowski space; the group $O(4,2)/Z_2$, acting in a six-dimensional real space; and the group $SU(2,2)/\bar{Z}_4$ acting in a four complex dimensional space. In Secs. IV 9—IV 11 the maximal subgroups of the conformal group are divided into conjugacy classes under the Poincaré group. The reason for doing this is the following. When obtaining the most general invariant field under a given group, we may naturally generate an infinity of other fields by applying Poincaré transformations, and each of these fields will be invariant under the group obtained from the original invariance group through conjugation by the corresponding transformation. However, such fields will not be essentially different from one another in terms of interpretation since, by virtue of the duality between point and coordinate transformations, they may be interpreted as the *same* field viewed in different Lorentz frames. Therefore, it is sufficient to determine the invariant fields corresponding to a single representative of each conjugacy class of groups under the Poincaré group. Such an argument holds, of course, for any invariant quantity under a given group, making the conjugacy classes under the Poincaré group the only relevant characterization from the point of view of relativistic invariance. An essential distinction must be made, however, between the *restricted* Poincaré group, preserving orientation and causal sequence, and the *general* Poincaré group, containing the transformation P (space inversion), T (time inversion), and PT. Equivalence of reference frames under the latter represents an additional *physical* assumption, beyond that of relativistic invariance, whose validity depends upon the nature of the dynamical equations involved. With regard to the general problem of determining classes of subgroups, conjugated under a particular subgroup, a formulation is given in Sec. IV 11 in terms of double cosets which leads to two possible approaches to such an analysis. Both these methods are illustrated for the particular cases of maximal subgroups of the conformal group and the results are summarized in Table I. In Secs. V 12—V 14, the methods of Secs. II 3—II 5 are applied with respect to these maximal subgroups (as well as certain nonmaximal ones) so as to obtain the most general invariant fields of 1- forms, 2- forms, (0, 2) symmetric tensors, and scalar densities. The cases for which nonzero invariant fields exist are also indicated in Table I. Finally, the results are discussed in Sec. V 15 from the viewpoint of physical interpretation and a summary is given in which possible applications and extensions are suggested.

General references: For differential geometric definitions and notations, as used in Secs. II 3—II 5, we recommend the standard texts of Refs. 34—37 to the reader. For further background and references on the conformal group, see Refs. 1, and 38—40. For a summary regarding nonlinear group action on manifolds, see Ref. 41 and for more detailed mathematical background, Refs. 42—44.

TABLE I. Maximal Subalgebras of c(3,1).

Algebra $\mathscr{H}$ (dim $\mathscr{H}$) $[n,n_-n_0]$	Normalizer $N_G(\mathscr{H})$ in O(4,2)	Invariant Subspace V_i	Transition Element $g_i : V_i \to V_i$	Orbit in C° under $N_G(\mathscr{H}_i)$	Basis for $\mathscr{H}_i$	1 form A	2 form F	(0,2) tensor G	density ϕ
sim(3,1) (11) [0 0 1]	SIM(3,1)$\times$Z$_2$	$V_1\{(0,0,0,0,-x,x)\}$	I	$\eta^\mu=0,\ \eta^4+\eta^5=0$	$\{L_i,K_i,D,P_\mu\}$				
		$V_2=\{(0,0,0,0,x,x)\}$	$\exp\left(\frac{\pi}{2}(P_3-C_3)\right)$	$\eta^4+\eta^5\neq0$	$\{L_i,K_i,D,C_\mu\}$				
		$V_3=\{x,0,0,-x,0,0\}$	$\exp\left(\frac{\pi}{4}(P_0+C_0)\right)$ $\times\exp\left(\frac{\pi}{4}(P_3-C_3)\right)$	$\eta^0\neq0,\ \eta^4+\eta^5=0$	$\{L_3,K_3,L_2+K_1,L_1-K_2,$ $P_1,P_2,P_0-P_3,C_1,C_2,C_0-C_3\}$				
o(4,1) (10) [1 0 0]	O(4,1)$\times$Z$_2$	$V_1\{(x,0,0,0,0,0)\}$	I	$\eta_1^2+\eta_2^2+\eta_3^2$ $+\eta_4^2-\eta_5^2=0$	$\{D,L_i,P_i,C_i\}$			$\checkmark$	$\checkmark$
		$V_2=\{(0,0,0,0,$ $x\sinh\lambda,x\cosh\lambda)\}$	$e^{\lambda D}$ $\times\exp\left(\frac{\pi}{4}(P_0+C_0)\right)$	$\eta_1^2+\eta_2^2+\eta_3^2$ $+\eta_4^2-\eta_5^2=e^{2\lambda}$	$\{L_i,K_i,e^\lambda P_\mu-e^{-\lambda}C_\mu\}$			$\checkmark$	$\checkmark$
o(3,2) (10) [0 1 0]	O(3,2)$\times$Z$_2$	$V_1=\{(0,0,0,x,0,0)\}$	I	$\eta_0^2-\eta_1^2-\eta_2^2$ $-\eta_4^2+\eta_5^2=0$	$\{K_1,K_2,L_3,D,P_0,$ $P_1,P_2,C_0,C_1,C_2\}$			$\checkmark$	$\checkmark$
		$V_2=\{(0,0,0,0,$ $x\cosh\lambda,x\sinh\lambda)\}$	$e^{\lambda D}$ $\times\exp\left(-\frac{\pi}{4}(P_3-C_3)\right)$	$\eta_0^2-\eta_1^2-\eta_2^2-\eta_4^2$ $+\eta_5^2=e^{2\lambda}$	$\{L_i,K_i,e^\lambda P_\mu+e^{-\lambda}C_\mu\}$			$\checkmark$	$\checkmark$
opt(3,1) (10) [0 0 2]	OPT(3,1)$\times$Z$_2$	$V_1=\{(x,0,0,$ $-x,-y,y)\}$	I	$\eta^4+\eta^5=\eta^0+\eta^3$ $=\eta^1=\eta^2=0$	$\{D,L_3,K_3,L_1+K_1,$ $L_1-K_2,P_\mu,C_0-C_3\}$				
		$V_2=\{(x,0,0,-x,y,y)\}$	$\exp\left(\frac{\pi}{2}(P_3-C_3)\right)$	$\eta^4+\eta^5\neq0$ or $\eta^0+\eta^3\neq0$	$\{D,L_3,K_3,L_2+K_1,$ $L_1-K_2,C_\mu,P_0-P_3\}$				
su(2,1) $\oplus$ u(1) (9) [0 1 0]	$\dfrac{S(U(2,1)\times U(1))}{\overline{Z}_2}$ $\times Z_2$	$V_1=\{(0,0,0,\omega)\}$	I	C^5	$\{L_3,D-K_3,P_0+C_0,P_0+P_3,$ $C_0-C_3,K_1+P_2+C_2,$ $K_2-P_1-C_1,L_1-P_1+C_1,$ $2L_3-P_2+C_2\}$ $\oplus\{2L_3+P_0-P_3+C_0-C_3\}$				
o(2) $\oplus$ o(4) (7) [2 0 0]	O(2)$\times$O(4)	$(V_1=\{(x,0,0,0,0,y)\})$ $\lambda=0$	(I) $\lambda=0$	$(\eta_0^2+\eta_5^2$ $=\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2=1)$					
		$V_1(\lambda)$ $=\{(x,0,0,0,x\sinh\lambda,$ $x\cosh\lambda)\}$	$e^{\lambda D}$	$\eta_0^2+\eta_5^2$ $=\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2=e^{2\lambda}$	$\{L_i,e^\lambda P_i-e^{-\lambda}C_i\}$ $\oplus\{e^\lambda P_0+e^{-\lambda}C_0\}$	$\checkmark$		$\checkmark$	$\checkmark$
o(2) $\oplus$ o(2,2) (7) [0 2 0]	O(2)$\times$O(2,2)	$V_1=\{(0,x,y,0,0,0)\}$	I	$\eta_1^2+\eta_2^2$ $=\eta_0^2-\eta_3^2-\eta_4^2+\eta_5^2=0$	$\{K_3,D,P_0,C_0,P_3,C_3\}$ $\oplus\{L_3\}$	$\checkmark$		$\checkmark$	$\checkmark$
		$V_2=\{(0,0,0,y,$ $x\cosh\lambda,x\sinh\lambda)\}$	$e^{\lambda D}\exp\left(\frac{\pi}{2}L_1\right)$ $\times\exp\left(-\frac{\pi}{4}(P_1-C_1)\right)$	$\eta_1^2+\eta_2^2$ $=\eta_0^2-\eta_3^2-\eta_4^2+\eta_5^2=e^{2\lambda}$	$\{K_1,K_2,L_3,e^\lambda P_0+e^{-\lambda}C_0,$ $e^\lambda P_1+e^{-\lambda}C_1,e^\lambda P_2+e^{-\lambda}C_2\}$ $\oplus e^\lambda P_3-e^{-\lambda}C_3\}$	$\checkmark$		$\checkmark$	$\checkmark$
o(3) $\oplus$ o(2,1) (6) [0 3 0]	O(3)$\times$O(2,1)	$V_1=\{(0,x,y,z,0,0)\}$	I	$\eta_1^2+\eta_2^2+\eta_3^2$ $=\eta_0^2-\eta_4^2+\eta_5^2=0$	$\{L_1,L_2,L_3\}\oplus\{D,P_0,C_0\}$		$\checkmark$	$\checkmark$	$\checkmark$
		$V_2=\{(0,x,y,0,$ $z\cosh\lambda,z\sinh\lambda)\}$	$e^{\lambda D}$ $\times\exp\left(-\frac{\pi}{4}(P_3-C_3)\right)$	$\eta_1^2+\eta_2^2+\eta_3^2$ $=\eta_0^2-\eta_4^2+\eta_5^2=e^{2\lambda}$	$\{K_3,e^\lambda P_0+e^{-\lambda}C_0,e^\lambda P_3$ $+e^{-\lambda}C_3\}\oplus\{L_3,e^\lambda P_1-e^{-\lambda}C_1,$ $e^\lambda P_2-e^{-\lambda}C_2\}$		$\checkmark$	$\checkmark$	$\checkmark$

TABLE I. Maximal Subalgebras of c(3,1).

Algebra $\mathcal{H}$ (dim $\mathcal{H}$) $[n, n, n_0]$	Normalizer $N_G(\mathcal{H})$ in O(4,2)	Invariant Subspace V_i	Transition Element $g_i : V_i \rightarrow V_i$	Orbit in C under $N_G(\mathcal{H}_1)$	Basis for $\mathcal{H}_i$	1-form A	2-form F	(0,2) tensor G	density ϕ
o(2,1) $\oplus$ o(2,1) (6) [1 2 0]	O(2,1)×O(2,1)×Z_2 (see text)	$V_1 = \{(x,y,z,0,0,0)\}$	I	$\eta_0 = \eta_1 = \eta_2 = 0, \ \eta_3 \neq 0$ or $\eta_3 = \eta_4 = \eta_5 = 0, \ \eta_0 \neq 0$	$\{K_1, K_2, L_3\} \oplus \{D, P_3, C_3\}$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
		$V_2 = \{(0,y,z,0, \ x \sinh\lambda, x \cosh\lambda)\}$	$e^{\lambda D}$ $\times\exp\left(\dfrac{\pi}{4}(P_0 + C_0)\right)$	$\eta_3^2 + \eta_4^2 - \eta_5^2 = \pm e^{2\lambda},$ $\eta_0^2 - \eta_1^2 - \eta_2^2 = \pm e^{2\lambda}$	$\{2K_2 - P_2 - C_2, 2K_3 - 2D + P_0 - C_0$ $-P_3 - C_1, 2L_1 - P_2 + C_2\}$ $\oplus \{2K_1 + P_1 + C_1, 2K_3 - 2D - P_0$ $+ C_0 + P_1 + C_1, 2L_2 - P_1 + C_1\}$	$\checkmark$	$\checkmark$		$\checkmark$
		$V_3 = \left\{\left(\dfrac{x}{\sqrt{2}}, y, 0, \dfrac{z}{\sqrt{2}}, \dfrac{z}{\sqrt{2}}, \dfrac{x}{\sqrt{2}}\right)\right\}$	$\exp\left(-\dfrac{\pi}{8}(P_3 - C_3)\right)$ $\times\exp\left(\dfrac{\pi}{2}L_1\right)$ $\times\exp\left(-\dfrac{\pi}{8}(P_0 + C_0)\right)$	$\eta_3^2 + \eta_4^2 - \eta_5^2 = 0, \ \eta_3 \neq 0;$ $\eta_0^2 - \eta_1^2 - \eta_2^2 = 0, \ \eta_0 \neq 0$	$\{K_3, e^\lambda P_0 - e^{-\lambda} C_0,$ $e^\lambda P_3 - e^{-\lambda} C_3\}$ $\oplus \{L_3, e^\lambda P_1 + e^{-\lambda} C_1,$ $e^\lambda P_2 + e^{-\lambda} C_2\}$	$\checkmark$	$\checkmark$		$\checkmark$

3. Coordinate independent formulation

We address ourselves now to the following problem. Given a C^∞ manifold M of dimension n and a transformation group G acting on M by C^∞ diffeomorphisms (i. e., we have a homomorphism $f : G \rightarrow \text{Diff}(M)$ into the group of C^∞ diffeomorphisms of M), determine the most general $\psi \in D^{(r,s)}(M)$ (element of the module of s-covariant r-contravariant tensor fields on M) such that $f_g^*(\psi) = \psi$ for all $g \in G$ [where f_g^* is the mapping $D^{(r,s)}(M) \rightarrow D^{(r,s)}(M)$ induced by the differential of f_g]. In fact, it is sufficient to consider tensor fields of definite symmetry under permutation of indices, since the submodules of a given permutation symmetry are invariant under $\text{Diff}(M)$. We may also extend the class of invariant fields considered to include tensor densities. The requirement of invariance may be weakened to that of invariance under groups of local diffeomorphisms defined only in a neighborhood of each point. If the group G is a Lie group, this becomes equivalent to the condition that the Lie derivative of ψ with respect to the vector fields X induced by the one-parameter subgroups of G should vanish,

$$(\mathcal{L}_X \psi)_p = \lim_{t \to 0} \left(\frac{[\psi - f_{g(t)}^*(\psi)]}{t}\right)_p = 0, \qquad (3.1)$$

where $X(p)$ is the tangent vector at $p \in M$ to the curve $f_{g(t)}(p)$ generated by the one-parameter subgroup $g(t)$.

Now let M_0 be the orbit of a point p_0 under the group action and let G_0 be the isotropy subgroup of G at p_0 [i. e., $G_0 = \{g \mid f_g(p_0) = p_0\}$]. Then for $g_0 \in G_0$, $f_{g_0}^*$ maps the tensor space $D^{(r,s)}(p_0)$ into itself linearly, and defines a representation of G_0 on this space which is a (reduced) tensor product of the linear isotropy representation r times with itself and s times with its conjugate representation. Clearly, for the field to be invariant under G, it must satisfy in particular, at

$$f_{g_0}^*(\psi)(p_0) = \psi(p_0), \qquad \forall g_0 \in G_0 \qquad (3.2)$$

which represents a set of linear algebraic equations for each g_0. Now, given a tensor $\psi(p_0)$ at p_0 which satisfies (3.2), we may generate an invariant tensor field on M_0 by defining

$$\psi(f_g(p_0)) \equiv f_g^*(\psi(p_0)). \qquad (3.3)$$

Because of (3.2), this mapping depends only on the left G_0 coset to which g belongs and hence $\psi(p)$, $p \in M_0$ really only depends on the point p and not the particular element g in the coset which maps p_0 to p. Conversely, any invariant tensor field on M_0 is uniquely determined by its value at p_0 through Eq. (3.3). The particular point p_0 chosen is evidently immaterial due to the transitivity of the group action on the orbit, the isotropy group of any other point on a given orbit being conjugate to G_0 and the corresponding linear isotropy representation equivalent. The above remarks may be summarized as follows:

There is a one-to-one correspondence between the irreducible tensor fields ψ of type (r, s) on M_0, invariant under G, and the GL(n)-irreducible tensors ψ_0 of the same permutation symmetry type as ψ, invariant under the linear isotropy tensor representation at a fixed $p_0 \in M_0$ realized in the $D^{(r,s)}(p_0)$ tensor space; the correspondence being given by Eqs. (3.2), (3.3) with $\psi(p_0)$ identified as ψ_0.

Suppose now that G is a Lie group and that the mapping $G \times M \rightarrow M$ defined by $(g, p) \rightarrow f_g(p)$ is C^∞. We shall assume, moreover that M has an everywhere dense submanifold M', which is the un__ n of a finite number of submanifolds $\{M_i\}$ each of which is an open stratum[41] (that is, each M_i is the union of all orbits with conjugate isotropy subgroups). We shall refer to

such M_i's as the regular strata (and the orbits they contain as regular orbits) and all others as singular strata (and singular orbits). It will be assumed that either M_i consists of a finite number of orbits or that the space of regular orbits $G \backslash M_i$ is itself a differentiable manifold under the differentiable structure inherited from M_i and G. In the former case, the identification of invariant fields is complete when they are known on each orbit, while in the latter we may use the manifold structure of $G \backslash M_i$ to characterize the invariant fields throughout each M_i. A local coordinate system for $G \backslash M_i$ is equivalent to a functionally complete set of scalar fields defined on the orbit of an open set $U_0 \subset M_i$, invariant under the group G. To see this, let k be the dimension of $G \backslash M_i$ and let

$$\beta : U \to \overline{U} \qquad (GU_0 \equiv U \subset G \backslash M_i, \quad \overline{U} \subset \mathbb{R}^k) \tag{3.4}$$

be such a local coordinate system (U_0, $\overline{U}$, and U being open neighborhoods). Then

$$\overline{\beta}(p) \equiv \beta(Gp) \tag{3.5}$$

defines a k-tuple $\{\overline{\beta}^a\}$ ($a = 1, \ldots, k$) of functionally independent G-invariant scalar fields. Furthermore, if

$$\overline{\eta} : GU_0 \to \mathbb{R} \tag{3.6}$$

is a G-invariant scalar field on GU_0 (considered now as an open set in M_i), define the map:

$$\eta : U \to \mathbb{R} \tag{3.7}$$

(considering U as on open set in $G \backslash M_i$) by

$$\eta(Gp) \equiv \overline{\eta}(p). \tag{3.8}$$

Then $\eta \equiv \eta_0 \circ \beta$, where $\eta_0 \equiv \eta \circ \beta^{-1}$ is a mapping from $\overline{U}$ into $\mathbb{R}$, and hence η is functionally dependent on the $\{\overline{\beta}^a\}$.

4. Coordinate systems

Given a k-tuple $\{\overline{\beta}^a\}$ of G-invariant functionally independent scalar fields defined on $GU_0 \subset M_i$, a coordinate system for M_i in this open set may be defined as follows. Let S be a submanifold of M_i which locally intersects each orbit in GU_0 exactly once and moreover such that the isotropy subgroup G_0 for each point $p_{0s} \in S \cap GU_0$ is locally the same. (For further discussion concerning the existence of such an S, see the next section.) Now the space of cosets $K = G/G_0$ may be identified with each orbit in GU_0 in a unique way by the correspondence

$$g p_{0s} \longleftrightarrow g G_0.$$

A coordinate system for K thus provides a coordinate system for the orbit of any point p_{0s}. The orbit is necessarily of dimension $n - k$ and therefore such a coordinate system is a mapping

$$\alpha_A : K_A \to \mathbb{R}^{n-k},$$

where $\{K_A\}$ is an open covering of K. This covering may be used to defined a covering for $GU_0 \subset M_i$ by open neighborhoods U_A,

$$U_A \equiv \bigcup_{p_{0s}} K_A(p_{0s}).$$

Then a natural coordinate system for M_i in GU_0 is given by the mapping $\gamma : GU_0 \to \mathbb{R}^n$ defined as

$$\gamma(p = k p_{0s}) = \{\alpha_A^a(k), \overline{\beta}^b(p)\}_{\substack{a=1,\ldots,n-k \\ b=1,\ldots,k}}, \quad p \in U_A,$$

where $k \in K_A \subset K$. Thus, the first $n - k$ coordinates identify the point in the coset corresponding to the given point on the orbit $G_{p_{0s}}$ while the last k coordinates identify the orbit upon which p_{0s} lies. If the invariant scalar fields $\{\overline{\beta}^b\}$ are defined and functionally independent throughout M_i, this may be used to defined a coordinate system throughout M_i.

In general, let $\{x^i(p)\}$ denote the coordinates of a point $p \in M_i$ in any coordinate system and let $x_g^i(p) \equiv x^i(f_g(p))$ denote the coordinates of the image of this under the mapping f_g for a given $g \in G$. The Jacobian of this map, referred to these coordinates, is represented by the $n \times n$ matrix ${}^g J(p)$ with element

$${}^g J_j^i(p) \equiv \frac{\partial x_g^i(p)}{\partial x^j}. \tag{4.1}$$

Let the components of the tensor ψ relative to the coordinate frame be denoted $\psi_{j_1 \ldots j_s}^{i_1 \ldots i_r}$. Then the conditions for invariance (3.2) and (3.3) become

$$\psi_{j_1 \ldots j_s}^{i_1 \ldots i_r}(p_0) = \sum_{\substack{k_1 \ldots k_r \\ l_1 \ldots l_s}} {}^{g_0} J_{k_1}^{i_1} \ldots {}^{g_0} J_{k_r}^{i_r} \, {}^{g_0} \overline{J}_{j_1}^{l_1} \ldots {}^{g_0} \overline{J}_{j_s}^{l_s} \, \psi_{l_1 \ldots l_s}^{k_1 \ldots k_r}(p_0)$$

$$(\forall \, g_0 \in G_0) \tag{4.2}$$

and

$$\psi_{j_1 \ldots j_s}^{i_1 \ldots i_r}(f_g(p_0)) = \sum_{\substack{k_1 \ldots k_r \\ l_1 \ldots l_s}} {}^g J_{k_1}^{i_1} \ldots {}^g J_{k_r}^{i_r} \, {}^g \overline{J}_{j_1}^{l_1} \ldots {}^g \overline{J}_{j_s}^{l_s} \, \psi_{l_1 \ldots l_s}^{k_1 \ldots k_r}(p_0)$$

$$(\forall \, g \in G) \tag{4.3}$$

(where ${}^g \overline{J}$ represents the inverse of the Jacobian matrix). Thus, the most general invariant field ψ is obtained by solving (4.2) for the independent components of $\psi(p_0)$, allowing these to be arbitrary functions of the scalars $\overline{\beta}^a(p_0)$ characterizing the orbits and then applying (4.3) to generate the field throughout the orbit. The above expressions become considerably simpler within the $\{\alpha^a, \overline{\beta}^b\}$ coordinate system, since the Jacobian matrix then takes on the block form

$$J = \begin{bmatrix} A_{n-k} & 0 \\ \hline 0 & \mathbf{1}_k \end{bmatrix}. \tag{4.4}$$

If invariant tensor densities are being considered,

rather than tensors, Eqs. (4.2) and (4.3) are modified by multiplying the right-hand side by a suitable power of $|\det(J)|$, depending upon the density weight. In particular, for a scalar density ϕ of weight δ, we have

$$\phi(p_0) = |\det{}^{g_0}J|^\delta \, \phi(p_0), \quad g_0 \in G_0 \tag{4.2'}$$

and

$$\phi(p) = |\det{}^{g}\bar{J}|^\delta \, \phi(p_0), \quad g \in G, \quad g p_0 = p. \tag{4.3'}$$

Finally, let $g(t)$ be a one-parameter Lie group which induces the vector field X, denoted in the coordinate system $\{x^i\}$ by

$$X = \xi^i \frac{\partial}{\partial x^i}. \tag{4.5}$$

The requirement (3.1) for local invariance of the field ψ is expressed in terms of components as

$$(L_X \psi)^{i_1 \cdots i_r}_{j_1 \cdots j_s} = \xi^k \psi^{i_1 \cdots i_r}_{j_1 \cdots j_s, k} + \sum_{p=1} \xi^k_{,j_p} \psi^{i_1 \cdots i_r}_{j_1 \cdots j_{p-1}, k, j_{p+1} \cdots j_s}$$
$$- \sum_{p=1}^{r} \xi^{i_p}_{,k} \psi^{i_1 \cdots i_{p-1}, k, i_{p+1} \cdots i_r}_{j_1 \cdots j_s}$$
$$= 0. \tag{4.6}$$

Again, for tensor densities, a suitable multiple of $\xi^k_{,k} \psi^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ must be added to the expression for the Lie derivative.

5. Fibre bundle formulation

In terms of fibre bundles, the preceding structures may be described as follows. Let M_i denote the ith stratum. Then the action of G on M_i defines an equivalence relation turning M_i into a fibre bundle over the base manifold $G \backslash M_i$ with group G, the fibres being the orbits, and the fibre-type, the coset space $K = G/G_0$. An open subset of the submanifold S intersecting each orbit once represents a local cross section of this bundle which is invariant under G_0. Such cross sections may be shown to exist, in particular, if G_0 is compact,[41–43] and under weaker assumptions as well, such as the existence of a G_0-invariant local metric. For the cases treated in Sec. V either $G \backslash M_i$ is discrete or a smooth local section exists. A coordinate system for $G \backslash M_i$ defined on any covering by open sets $\{V_A\}$ together with a G_0 invariant local section S_A over each V_A and a coordinate system in K gives rise to a coordinate system for M_i through the identification of the identity coset with the intersection of S_A with each fibre over V_A (the open covering being $\{K_B S_A(V_A)\}$, where $\{K_B\}$ is the covering by open sets of K).

An invariant tensor field ψ is a cross section of the bundle $T^{(r,s)}(M_i)/G$ over $G \backslash M_i$ as base manifold, $T^{(r,s)}(M_i)$ being the bundle of (r, s)-tensors on M_i, and the G-action on $T^{(r,s)}(M_i)$ defined by ϕ^*_g. Each point in $T^{(r,s)}(M_i)/G$ over a point $x \in G \backslash M_i$ corresponds to a G-invariant tensor field defined on the orbit x, and these are completely characterized by the Eqs. (3.2) and (3.3). Thus, the problem of finding the most general tensor field ψ of type (r, s), invariant under G is reduced to that of characterizing the sections of the bundle $T^{(r,s)}(M_i)/G$. Locally this may be done by specifying the coordinates of a point $x \in G \backslash M_i$ and the components of the (r, s)-tensor [satisfying the linear

isotropy condition (3.2)] which lies over x. A coordinate system for $G \backslash M_i$ is a C^∞, 1-1 onto mapping between open sets $\{V_A\}$ covering $G \backslash M_i$ and $\{\bar{U}_A\}$ in $\mathbb{R}^k$. This may also be interpreted as a set of local cross sections over $\{V_A\}$ of the bundle $G \backslash M_i \times \mathbb{R}^k$, each defining a set of k functionally independent, local G-invariant scalar fields in M_i. That they are functionally independent follows from the fact that the Jacobians of the C^∞ maps $V_A \rightarrow \mathbb{R}^k$ are nondegenerate; that k is moreover the maximum number is evident from the fact that the Jacobian's rank cannot exceed the dimension of $G \backslash M_i$.

III. THE CONFORMAL GROUP OF MINKOWSKI SPACE

6. The local transformation group

Let $M(p, q)$ be a pseudo-Euclidean space with metric g_M of signature (p, q). That is, within any rectilinear coordinate system $\gamma_g : M(p, q) \rightarrow \mathbb{R}^{p+q}$, g_M is identified with a nondegenerate bilinear form

$$g : \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \rightarrow \mathbb{R}.$$

Unless stated otherwise, we shall always identify g_M with the diagonal matrix

$$\mathrm{diag}(\underbrace{1, \ldots, 1}_{q}, \underbrace{-1, \ldots, -1}_{p}),$$

using the same symbol for both. By $O(p, q)$, we shall mean the orthogonal group corresponding to this particular g_M. A C^∞ transformation $T : U_T \rightarrow M(p, q)$ [U_T being an open set in $M(p, q)$] is called a local conformal transformation if, for any $\mathbf{x} \in U_T$, the Jacobian matrix J of the transformation within the coordinate system γ_g is of the form

$$J(\mathbf{x}) = f(\mathbf{x}) R(\mathbf{x}), \tag{6.1}$$

where $R(\mathbf{x}) \in O(p, q)$ and $f(\mathbf{x}) \in \mathbb{R}^+$ (multiplicative group of positive real numbers). This is equivalent to the requirement that

$$T^* g_M(\mathbf{x}) = \lambda(\mathbf{x}) g_M(\mathbf{x}), \quad \lambda(\mathbf{x}) \in \mathbb{R}^+, \tag{6.2}$$

where g_M in this equation simply denotes the metric tensor, and not any particular coordinate representation of it. If the open set U_T is all of $M(p, q)$, then T is a global conformal transformation. The definition (6.2) applies, moreover, both in local and in global form if $M(p, q)$ is replaced by an arbitrary (pseudo)-Riemannian manifold. It is a well-known result that for $p + q \geq 3$, the conformal transformations of $M(p, q)$ are generated by translations, pseudo-orthogonal transformations, dilatations and the inversion $\mathbf{x} \rightarrow \mathbf{x}/g_M(\mathbf{x}, \mathbf{x})$. These generate a local Lie group $C(p, q)$ of transformations on $M(p, q)$ [local, because the open sets U_T for certain $T \in C(p, q)$ do not cover the whole space].

For $(p, q) = (3, 1)$, $M(p, q)$ becomes Minkowski space (denoted hereafter as M), for which Klein[45] proved that $C(3, 1)$ is isomorphic to the projective orthogonal group $PO(4, 2) \sim O(4, 2)/Z_2$, where Z_2 denotes the centre $\{1, -1\}$ of $O(4, 2)$. Klein's proof is based on the sphere geometry of Lie which allows the local action of $C(3, 1)$ on M to be extended to a global action on another manifold,

having the same local properties as M; namely, the conformally compactified Minkowski space $\bar{M}$, which we shall define in the next section.

The one parameter subgroups of $C(3,1)$ induce a set of vector fields on M which close under commutation to form a 15-dimensional Lie algebra $(4,1)$. A basis for $c(3,1)$ which is convenient for distinguishing the Lorentz subalgebra consists of the infinitesimal homogeneous Lorentz transformations $M_{\mu\nu}$, translations P_μ, dilatations D and special conformal transformation C_μ $(\mu = 0, 1, 2, 3)$. Within a Cartesian coordinate system these are of the form:

$$M_{\mu\nu} = -x_\mu \partial_\nu + x_\nu \partial_\mu,$$

$$P_\mu = -\partial_\mu, \quad D = x^\mu \partial_\mu, \tag{6.3}$$

$$C_\mu = x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu$$

[summation convention used throughout, and raising or lowering of indices done with the Minkowski metric $g_M = \text{diag}(1, -1, -1, -1)$]. Now, denoting an arbitrary vector field $X \in C(3,1)$ with components given by

$$X = \tfrac{1}{2}\omega^{\mu\nu}M_{\mu\nu} + \rho D + a^\mu P_\mu + c^\mu C_\mu \tag{6.4}$$

corresponding to the infinitesimal transformation

$$x^\mu \to x^\mu - \omega^\mu_\nu x^\nu + a^\mu - c^\mu x^2 + 2c \cdot xx^\mu - \lambda x^\mu \tag{6.5}$$

$(\omega^{\mu\nu}, \rho, a^\mu, c^\mu$ being real constants), we have

$$\mathcal{L}_X g_M = 2(\lambda - 2c \cdot x)g_M \tag{6.6}$$

showing explicitly that these vector fields represent infinitesimal conformal transformations. The nonzero commutators within this basis are:

$$[M_{\mu\nu}, M_{\sigma\tau}] = g_{\mu\sigma}M_{\nu\tau} + g_{\nu\tau}M_{\mu\sigma} - g_{\mu\tau}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\tau},$$

$$[M_{\mu\nu}, P_\sigma] = g_{\mu\sigma}P_\nu - g_{\nu\sigma}P_\mu,$$

$$[D, P_\mu] = -P_\mu,$$

$$[D, C_\mu] = C_\mu, \tag{6.7}$$

$$[M_{\mu\nu}, C_\sigma] = g_{\mu\sigma}C_\nu - g_{\nu\sigma}C_\mu,$$

$$[P_\mu, C_\nu] = 2g_{\mu\nu}D + 2M_{\mu\nu}.$$

At the algebraic level, the $o(4,2)$ structure may be seen by defining a new basis $J_{ab} = -J_{ba}$ $(a, b = 0, 1, 2, 3, 4, 5)$ with

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{45} = D, \quad J_{4\mu} = \tfrac{1}{2}(P_\mu - C_\mu), \quad J_{5\mu} = \tfrac{1}{2}(P_\mu + C_\mu) \tag{6.8}$$

in terms of which the commutation relations become

$$[J_{ab}, J_{cd}] = g_{ac}J_{bd} + g_{bd}J_{ac} - g_{ad}J_{bc} - g_{bc}J_{ad}, \tag{6.9}$$

where

$$g_{ab} = \text{diag}(+1, -1, -1, -1, -1, +1). \tag{6.10}$$

Another basis frequently used is obtained by separating $M_{\mu\nu}$ into rotations L_i and boosts K_i $(i = 1, 2, 3)$

$$L_i \equiv -\tfrac{1}{2}\epsilon_{ijk}M_{jk}, \quad K_i \equiv M_{0i}. \tag{6.11}$$

7. Compactified Minkowski space

In order to realize $C(3,1)$ as a global transformation group, we proceed in a standard way,[1] replacing M by another manifold $\bar{M}$ defined to be the projective cone of

$\mathbb{R}^6$,

$$\bar{M} \equiv C^5/\sim,$$

where C^5 is the cone of nonzero null vectors for the quadratic form associated with the matrix (g_{ab}); that is the set of points $\eta = (\eta^0, \eta^1, \eta^2, \eta^3, \eta^3, \eta^4, \eta^5) \in \mathbb{R}^6$ for which

$$\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 + \eta_5^2 = 0 \tag{7.1}$$

[where $(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5) \equiv (\eta^0, -\eta^1, -\eta^2, -\eta^3, -\eta^4, \eta^5)$]. The equivalence relation $\sim$ is defined by

$$\eta \sim \eta' \Longleftrightarrow \eta' = \lambda\eta, \quad \lambda \neq 0. \tag{7.2}$$

This gives an identification between points in $\bar{M}$ and rays on the cone C^5.

The action of $O(4,2)$ on $\bar{M}$ is the natural one induced by the linear action of $O(4,2)$ on $\mathbb{R}^6$,

$$g : [\eta] \to [g\eta],$$

where $[\eta]$ is the class of points on C^5 equivalent to η. This action is transitive but not effective, since the center $\mathbb{Z}_2 = \{1, -1\}$ acts as the identity at all points of $\bar{M}$. The projective orthogonal group $PO(4,2) = O(4,2)/\mathbb{Z}_2$ acts effectively and the isotropy group at any point in $\bar{M}$ is easily seen to be an 11-dimensional Lie group which is isomorphic to the similitude group $SIM(3,1)$; that is, the semidirect product of the inhomogeneous Lorentz group $P(3,1)$ (Poincaré group) with the group $\mathbb{R}^+$ of dilatations. Choosing the point $[(0,0,0,0,-1,1)]$, the isotropy group may be characterized as the set of all matrices which can be written as a product

$$g(a, \lambda, L) \equiv \exp(a^\mu P_\mu)\exp(\lambda D)g_L, \tag{7.3}$$

where

$$\exp(a^\mu P_\mu) = \begin{bmatrix} \mathbb{1}_4 & a & a \\ \hline a^T g_M & 1 + \tfrac{1}{2}a^2 & \tfrac{1}{2}a^2 \\ -a^T g_M & -\tfrac{1}{2}a^2 & 1 - \tfrac{1}{2}a^2 \end{bmatrix}, \tag{7.4a}$$

$$\exp(\lambda D) = \begin{bmatrix} \mathbb{1}_4 & 0 & 0 \\ \hline 0 & \cosh\lambda & \sinh\lambda \\ 0 & \sinh\lambda & \cosh\lambda \end{bmatrix}, \tag{7.4b}$$

and

$$g_L = \begin{bmatrix} L & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{7.4c}$$

where $L \in O(3,1)$, $a \in \mathbb{R}^4$ with components a^μ and $a \cdot b \equiv a^T g_M b$ for any $a, b \in \mathbb{R}^4$ considered as column vectors. This may readily be verified to define a six-dimensional representation of the similitude group. We may thus identify $\bar{M}$ with the homogeneous space $PO(4,2)/SIM(3,1)$.

From (7.3) and (7.4) we see that $P(3,1)$ acts on the cone C^5 as follows:

$$\exp(a^\mu P_\mu)g_L \begin{bmatrix} \eta \\ \eta^4 \\ \eta^5 \end{bmatrix} \mapsto \begin{bmatrix} L\eta + (\eta^4 + \eta^5)a \\ \eta^4 + a \cdot L\eta + \tfrac{1}{2}a^2(\eta^4 + \eta^5) \\ \eta^5 - a \cdot L\eta - \tfrac{1}{2}a^2(\eta^4 + \eta^5) \end{bmatrix} \tag{7.5}$$

[where $\eta = (\eta^\mu) \in \mathbb{R}^4$].

The orbit of the point $(0, 0, 0, 0, 1, 1)$ in C^5 under $P(3, 1)$ is thus the four-dimensional submanifold of points with coordinates

$$\begin{bmatrix} 2a \\ 1 + a^2 \\ 1 - a^2 \end{bmatrix}.$$

This allows us to identify the orbit with Minkowski space M through the injection mapping

$$j : x \mapsto \begin{bmatrix} x \\ \frac{1}{2}(1 + x^2) \\ \frac{1}{2}(1 - x^2) \end{bmatrix}. \tag{7.6}$$

The image $j[M]$ is then the set of all (η, η^4, η^5) for which $\eta^4 + \eta^5 = 1$. The injection of M into $\bar{M}$ is given by

$$\bar{j} : x \mapsto [j(x)]$$

and the image consists of the submanifold of rays $[\eta]$ for which $\eta^4 + \eta^5 \neq 0$. The local inverse $\bar{j}^{-1}[\eta] \to M$ defines a coordinate system corresponding to the Cartesian coordinates $\{x^\mu\}$ of M,

$$x^\mu \equiv \frac{\eta^\mu}{\eta^4 + \eta^5}. \tag{7.7}$$

The entire manifold $\bar{M}$ may be seen to be diffeomorphic to the projective space $[S^1 \times S^3]/\mathbb{Z}_2$ by introducing the projective coordinates

$$u^a \equiv \frac{\eta^a}{[\eta_0^2 + \eta_5^2]^{1/2}} \quad (a = 0, \ldots, 5) \tag{7.8}$$

which maps $\bar{M}$ diffeomorphically into $S^1 \times S^3/\mathbb{Z}_2$ since

$$u_0^2 + u_5^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1 \tag{7.9}$$

and the points $\{u^a\}$ and $\{- u^a\}$ must be identified to make the correspondence one—one. The set of points of $\bar{M}$ which are not the image of any point in M are just those for which $u^4 + u^5 = 0$, so that

$$u_0^2 - u_1^2 - u_2^2 - u_3^2 = 0. \tag{7.10}$$

This may be identified as a light-cone "at infinity" (i. e., $x^\mu \to \infty$).

The action of SIM(3, 1) defined by the mapping j is exactly that of the similitude group on M; namely

$$g(a, \lambda, L) : x \mapsto \exp(- \lambda) Lx + a. \tag{7.11}$$

The Abelian subgroup of PO(4, 2) consisting of elements of the form

$$g(b) \equiv \begin{bmatrix} I & \vdots & -b & b \\ \hline - b^T g_M & \vdots & 1 + \frac{1}{2}b^2 & -\frac{1}{2}b^2 \\ - b^T g_M & \vdots & \frac{1}{2}b^2 & 1 - \frac{1}{2}b^2 \end{bmatrix} \tag{7.12}$$

moreover, defines the "special conformal transformations"

$$g(b) : x \mapsto \frac{x - x^2 b}{1 - 2b \cdot x + b^2 x^2}. \tag{7.12'}$$

These latter are clearly not defined globally on M.

The isotropy subgroups of O(4, 2) acting on C^5 are all

conjugate to $P(3, 1)$ and therefore we have the identification

$$C^5 \sim O(4, 2)/P(3, 1). \tag{7.13}$$

Finally, we should like to distinguish between the different connected components of the conformal group in a manner analogous to that done for the Lorentz group[46]; namely, according to which, among certain discrete transformations, each component contains. It is known that O(4, 2) as a manifold has four connected components [as does each O(p, q) group for which $p, q > 0$], and that these may be distinguished according to the values of two multiplicative discrete characters,[26] namely the determinant $d \ (= \pm 1)$ and the spinor norm $n \ (= \pm 1)$. The latter may be defined as

$$\text{sgn det} \begin{bmatrix} G_{00} & G_{05} \\ G_{50} & G_{55} \end{bmatrix}$$

for any O(4, 2) element (G_{ab}). We thus have

$$\mathbb{Z}_4 = O(4, 2)/SO_0(4, 2), \tag{7.14}$$

where the four cosets in $\mathbb{Z}_4$ may be identified with the following four elements of $C(3, 1) \sim O(4, 2)/\mathbb{Z}_2$,

$$I = \epsilon \, \text{diag}(1, 1, 1, 1, 1, 1) \qquad (n = +1, d = +1),$$

$$P = \epsilon \, \text{diag}(1, -1, -1, -1, 1, 1) \qquad (n = +1, d = -1),$$

$$T = \epsilon \, \text{diag}(-1, 1, 1, 1, 1, 1,) \qquad (n = -1, d = -1),$$

$$\tag{7.15}$$

$$PT = \epsilon \, \text{diag}(-1, -1, -1, -1, 1, 1) \qquad (n = -1, d = \pm 1)$$

$$(\epsilon = \pm 1).$$

[Note that the center $\mathbb{Z}_2$ of O(4, 2) is not contained in $\mathbb{Z}_4$, but is contained in SO$_0$(4, 2).] Using notation which is standard for the Lorentz group, we may identify the components of $C(3, 1)$ according to which components of the Lorentz group[46] they contain:

$$C_+^\dagger(3, 1) \sim SO_0(4, 2)/\mathbb{Z}_2 \supset L_+^\dagger \supset I,$$

$$C_+^\dagger(3, 1) \supset L_+^\dagger \supset PT,$$

$$C_-^\dagger(3, 1) \supset L_-^\dagger \supset P, \tag{7.16}$$

$$C_-^\dagger(3, 1) \supset L_-^\dagger \supset T.$$

Correspondingly, we have the following five types of conformal groups:

(i) $C(3, 1) \sim O(4, 2)/\mathbb{Z}_2$ (general conformal group)

(ii) $C_+^\dagger(3, 1) \sim SO_0(4, 2)/\mathbb{Z}_2$ (restricted conformal group)

(iii) $C_+(3, 1) \sim SO(4, 2)/\mathbb{Z}_2 \sim C_+^\dagger \cup C_+^\dagger$ (proper conformal group)

(iv) $C^\dagger(3, 1) \sim C_+^\dagger \cup C_-^\dagger$ (orthochronous conformal group)

(v) $C_0(3, 1) \sim C_+^\dagger \cup C_-^\dagger$ (orthochorous conformal group)

8. The conformal group and SU (2,2)

In the following, we shall need an explicit construction[47] of the isomorphism

$$SU(2, 2)/\bar{\mathbb{Z}}_2 \sim SO_0(4, 2), \tag{8.1}$$

where $\bar{\mathbb{Z}}_2 = \{\pm 1\}$, or, equivalently

$$SU(2, 2)/\bar{\mathbb{Z}}_4 \sim SO_0(4, 2)/\mathbb{Z}_2 \sim C_+^\dagger(3, 1), \tag{8.2}$$

where $\bar{\mathbb{Z}}_4 = \{\pm 1, \pm i 1\}$ is the discrete center of SU(2, 2).

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Let $h = h^\dagger$ denote a Hermitian form of signature $(2, 2)$ on $\mathbb{C}^4$. Then SU(2, 2) is the group of complex 4×4 matrices U such that

$$U^\dagger h U = h, \quad \det U = 1. \tag{8.3}$$

h can always be chosen such that $h^2 = 1$ [and consequently, $h \in SU(2, 2)$] and we choose

$$h = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}. \tag{8.4}$$

Let $\rho(U)$ be the six complex dimensional, linear representation of SU(2, 2) on the space $\wedge^2 \mathbb{C}^4$ of bivectors (i. e., antisymmetric tensor product of $\mathbb{C}^4$ with itself), defined by

$$\rho(U): v_1 \wedge v_2 \mapsto U v_1 \wedge U v_2, \quad v_1, v_2 \in \mathbb{C}^4. \tag{8.5}$$

Now let us define the following symmetric form on $\wedge^2 \mathbb{C}^4$,

$$(v_1 \wedge v_2, v_3 \wedge v_4) = \det(v_1, v_2, v_3, v_4) \tag{8.6}$$

(which extends, through bilinearity, to any pair of bivectors). Since for each $A \in GL(4, \mathbb{C})$ we have

$$\det(A v_1, A v_2, A v_3, A v_4) = \det A \cdot \det(v_1, v_2, v_3, v_4)$$

and since $\det U = 1$ for $U \in SU(2, 2)$, the symmetric form is invariant under $\rho(U)$,

$$(\rho(U): v_1 \wedge v_2, \rho(U): v_3 \wedge v_4) = (v_1 \wedge v_2, v_3 \wedge v_4) \tag{8.7}$$

and this invariance property defines a homomorphism of SU(2, 2) into $SO_0(6, \mathbb{C})$ [which incidently can be trivially extended to a homomorphism of SL(4, $\mathbb{C}$) into SO(6, $\mathbb{C}$)]. Moreover, the representation ρ is virtually real; that is, the complex vector space $\wedge^2 \mathbb{C}^4$, considered as a twelve-dimensional real space, contains a six-dimensional linear subspace invariant under $\rho(U)$ and such that the restriction of (8.6) is a real bilinear form of signature (4, 2). This gives rise to the homomorphism of SU(2, 2) onto $SO_0(4, 2)$.

To exhibit this decomposition explicity, it is convenient to represent $\wedge^2 \mathbb{C}^4$ by the space τ of all antisymmetric 4×4 matrices $T = -T^T$. The representation ρ acts on T as

$$\rho(U) T = U T U^T \tag{8.8}$$

and the symmetric form becomes

$$(T, T') = \tfrac{1}{2}\epsilon_{ijkl} T_{ij} T'_{kl} = -\operatorname{tr}(* T T'), \tag{8.9}$$

where ϵ_{ijkl} is the Levi—Civita symbol and $*T$ denotes the (affine) dual of T,

$$(*T)_{ij} = \tfrac{1}{2}\epsilon_{ijkl} T_{kl}. \tag{8.10}$$

We note that

$$*(\rho(U) T) = \rho(U^{-1T}) * T \tag{8.11}$$

and

$$** T = T. \tag{8.12}$$

Consider now the involutive, antilinear transformation D on L defined by

$$DT = \rho(h) * \overline{T} \tag{8.13}$$

(where $\overline{T}$ denotes the complex conjugate of T). The fact that D is involutive $(D^2 = 1)$ follows from (8.11) and (8.12) and the relations

$$h^\dagger = \overline{h}^T = h = h^{-1}. \tag{8.14}$$

The set of matrices invariant under D,

$$\tau_D = \{T \in \tau \mid DT = T\} \tag{8.15}$$

is a real vector space of dimension six which is invariant under $\rho(U)$ since

$$\rho(U) \circ D = D \circ \rho(U). \tag{8.16}$$

Within τ_D, we have

$$* T = h \overline{T} h, \quad \forall\, T \in \tau_D \tag{8.17}$$

and therefore the restriction of the form (8.6) to τ_D becomes

$$(T, T') = -\operatorname{tr}(h \overline{T} h T'). \tag{8.18}$$

By direct computation, one verifies that τ_D consists of all matrices of the form

$$T = \begin{bmatrix} 0 & p & q & r \\ -p & 0 & -\overline{r} & \overline{q} \\ -q & \overline{r} & 0 & \overline{p} \\ -r & -\overline{q} & -\overline{p} & 0 \end{bmatrix}, \quad p, q, r, \in \mathbb{C}. \tag{8.19}$$

Therefore, we may coordinatize τ_D by points in $\mathbb{R}^6$,

$$\Gamma(\eta) = \tfrac{1}{2} \begin{bmatrix} 0 & \eta^0 - i\eta^5 & -\eta^2 - i\eta^1 & \eta^4 + i\eta^3 \\ -\eta^0 + i\eta^5 & 0 & -\eta^4 + i\eta^3 & -\eta^2 + i\eta^1 \\ \eta^2 + i\eta^1 & \eta^4 - i\eta^3 & 0 & \eta^0 + i\eta^5 \\ -\eta^4 - i\eta^3 & \eta^2 - i\eta^1 & -\eta^0 - i\eta^5 & 0 \end{bmatrix}, \tag{8.20}$$

where

$$\eta = (\eta^0, \eta^1, \eta^1, \eta^2, \eta^3, \eta^4, \eta^5) \in \mathbb{R}^6.$$

We then have

$$(\Gamma(\eta), \Gamma(\eta)) = \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 + \eta_5^2 \equiv Q(\eta) \tag{8.21}$$

which thus defines a quadratic form in $\mathbb{R}^6$ of signature (4, 2). The homomorphism

$$\phi: SU(2, 2) \to SO_0(4, 2) \text{ is now defined by}$$

$$U \Gamma(\eta) U^T = \Gamma(\phi(U)\eta) \tag{8.22}$$

and it is easily verified that $\ker \phi = \{\pm I\}$ establishing the isomorphism (8.1). The particular choice of coordinatization in (8.20) is arbitrary up to an O(4, 2) transformation and has been chosen so as to give a simple form to the translations, homogeneous Lorentz transformations, dilatations, and special conformal transformations in the SU(2, 2) representation with Hermitian form

$$\tilde{h} = \begin{bmatrix} 0 & -i\mathbb{1} \\ i\mathbb{1} & 0 \end{bmatrix} = S h S^\dagger, \tag{8.23}$$

where

$$S \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbb{1} & i\mathbb{1} \\ i\mathbb{1} & \mathbb{1} \end{bmatrix} = (S^\dagger)^{-1}. \tag{8.24}$$

Translations:

$$\tilde{g}_T = \begin{bmatrix} \mathbb{1} & k \\ 0 & \mathbb{1} \end{bmatrix}, \tag{8.25}$$

where

$$k = \begin{bmatrix} a^0 + a^3 & a^1 - ia^2 \\ a^1 + ia^2 & a^0 - a^3 \end{bmatrix} \in \mathbb{H}(2) \quad \text{(Hermitian 2×2 matrix)}.$$

Special conformal transformations:

$$\tilde{g}_c = \begin{bmatrix} 1 & 0 \\ \tilde{k} & 1 \end{bmatrix}, \tag{8.26}$$

where

$$\tilde{k} = \begin{bmatrix} -b^0 + b^3 & b^1 - ib^2 \\ b^1 + ib^2 & -b^0 - b^3 \end{bmatrix} \in \mathbb{H}(2). \tag{8.27}$$

Homogeneous Lorentz transformations:

$$\tilde{g}_L = \begin{bmatrix} a & 0 \\ 0 & a^{\dagger-1} \end{bmatrix}, \quad a \in \mathrm{SL}(2, \mathbb{C}). \tag{8.28}$$

Dilatations:

$$\tilde{g}_0 = \begin{bmatrix} \exp(-\lambda/2)\,1 & 0 \\ 0 & \exp(+\lambda/2)\,1 \end{bmatrix}, \quad \lambda \in \mathbb{R}. \tag{8.29}$$

The corresponding matrices in the basis with diagonal invariant Hermitian form h is given, of course, by conjugation with the matrix S. Another useful realization of the action of $C_+^\uparrow(3, 1)$ directly on Minkowski space may be defined using the representation (8.23)—(8.29). Identifying the point of M with Cartesian coordinates $\{x^\mu\}$ by the Hermitian matrix

$$x \equiv \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix} \in \mathbb{H}(2) \tag{8.30}$$

we obtain a realization of $\mathrm{SU}(2,2)/\mathbb{Z}_4 \sim C_+^\uparrow(3.1)$ as a local transformation group acting effectively on $M \sim \mathbb{H}(2)$ by[8]:

$$\tilde{g} : x \mapsto x' \equiv (ax + b)(cx + d)^{-1}, \tag{8.31}$$

where

$$\tilde{g} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SU}(2,2) \tag{8.32}$$

$$\tilde{g}^{\dagger} \tilde{h} \tilde{g} = \tilde{h}, \quad a, b, c, d, \in \mathrm{GL}(2, \mathbb{C}). \tag{8.33}$$

Moreover, the action may be extended to a global one,[1] by replacing $\mathbb{H}(2)$ by a space diffeomorphic to $\bar{M}$, namely the group $U(2)$. The diffeomorphic injection l [cf. Eq. (7.6)] of $\mathbb{H}(2)$ into $U(2)$ is given by the Cayley transform

$$l : x \mapsto U \equiv \frac{1 + ix}{1 - ix} \in U(2), \quad x \in \mathbb{H}(2) \tag{8.34}$$

with local inverse

$$l^{-1} : U \mapsto x \equiv i\left[\frac{1 - U}{1 + U}\right] \in \mathbb{H}(2), \quad U \in U(2), \tag{8.34'}$$

The "light-cone at infinity" is given precisely by the subspace of U(2) for which the denominator in (8.34′) is singular, that is; where $\det(1 + U)$ vanishes. The global action of $\mathrm{SU}(2,2)$ on $U(2)$ corresponding to the local action (8.32) on $\mathbb{H}(2)$ is :

$$g : U \mapsto (AU + B)(CU + D)^{-1}, \tag{8.35}$$

where

$$A = \tfrac{1}{2}[a + d + i(b - c)],$$
$$B = \tfrac{1}{2}[d - a + i(b + c)],$$
$$C = \tfrac{1}{2}[d - a - i(b + c)],$$
$$D = \tfrac{1}{2}[d + a - i(b - c)]. \tag{8.36}$$

The $\mathbb{C}^4$ nonsingular matrices defined by

$$\bar{g} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = t^{\dagger} S^{\dagger} \tilde{g} S t \tag{8.37}$$

with

$$t \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} (1 - i) & 0 \\ 0 & (1 + i) \end{bmatrix} \tag{8.38}$$

define a representation of $\mathrm{SU}(2,2)$ which preserves the diagonal form h whenever $\tilde{g}$ preserves $\tilde{h}$. Moreover, these two conditions define constraints for the matrices $\{A, B, C, D\}$ and $\{a, b, c, d\}$ which guarantee that (8.35) and (8.31) actually define transformations on $\mathbb{H}(2)$ and $U(2)$ respectively.

As a final remark concerning the isomorphism (8.1), note that with the choice (8.4) for the Hermitian form h, the matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{8.39}$$

although not an element of $\mathrm{SU}(2,2)$, nevertheless defines an automorphism by conjugation since

$$J^{-1} h J = -h. \tag{8.40}$$

Moreover, the homomorphism (8.22) can be extended to the two-component group obtained by multiplying $\mathrm{SU}(2,2)$ elements by $\{I, J\}$ (within this matrix representation). Applying the homomorphism to J, we find

$$\phi(J) : \eta \mapsto (\eta^0, \eta^1, \eta^2, \eta^3, -\eta^4, -\eta^5). \tag{8.41}$$

That is,

$$\pm \phi(J) = PT \tag{8.42}$$

and the two component group is homomorphic to $\mathrm{SO}(4,2)$.

IV. MAXIMAL SUBGROUPS OF THE CONFORMAL GROUP

9. Remark on maximal subgroups, subalgebras, and normalizers

In the following, the notion of a "maximal" Lie subgroup of $C(3, 1)$ is defined purely at the level of the Lie algebra. Thus a subgroup of $C(3, 1)$ will be referred to as "algebraically maximal" if its Lie algebra is a maximal subalgebra of $c(3,1)$. Evidently, this will allow the identification of several algebraically maximal subgroups corresponding to the same algebra. As will become apparent from the discussion of Sec. IV 11, the relevant group, from the point of view of conjugacy classes, is the normalizer [in $C(3, 1)$] of any given group or algebra. Thus, two groups with equal (or conjugate) normalizers will have a bijective correspondence between their conjugacy classes under any given subgroup. Therefore, we shall define a "normally maximal" subgroup as any algebraically maximal subgroup whose normalizer equals the normalizer of its algebra. (Note

that the normalizer of any algebraically maximal sub-
group is always contained in the normalizer of its Lie
algebra and hence also that of any normally maximal
subgroup with the same algebra.) It is easy to verify
in particular that the identity component of any alge-
braically maximal subgroup is also normally maximal.
Furthermore, the normalizer of any maximal sub-
algebra is either a maximal subgroup or the entire
group, the latter being impossible for simple groups
such as $C(3, 1)$.

10. The maximal subalgebras of the conformal algebra

A complete classification of the subalgebras of $c(3, 1)$
into conjugacy classes under $C(3, 1)$ is in progress[25-27];
in particular, all the maximal subalgebras of $c(3, 1)$ are
known. We describe here briefly how the classification
of the maximal subalgebras was made in order to char-
acterize these for use in the following sections. The
method given is applicable to any semisimple Lie
algebra.

(i) We choose a specific representation of $c(3, 1)$;
namely, the one provided by the $o(4, 2)$ Lie algebra
[Eqs. (6.7), (6.9)], defined on $\mathbb{R}^6$ with respect to the
symmetric, bilinear form associated with the matrix
g_{ab} [Eq. (6.10)]. In this representation, a subalgebra
H of $c(3, 1) \sim o(4, 2)$ is called *reducible* if it leaves some
proper linear subspace $V \subset \mathbb{R}^6$ invariant, and irreduci-
ble otherwise.

(ii) Due to the isomorphism of Eq. (7.17), the $C(3, 1)$
conjugacy classes

$$[H]_c \equiv \{g^{-1}Hg, \ g \in C(3, 1)\} \qquad (10.1)$$

are identical to the $O(4, 2)$ classes. For reducible sub-
algebras, these may be associated with $O(4, 2)$ equiva-
lence classes of vector spaces,

$$[V]_c = \{gV, \ g \in O(4, 2)\} \qquad (10.2)$$

by the surjective mapping

$$gV \mapsto gHg^{-1}. \qquad (10.3)$$

In general, the correspondence is not one—one; indeed,
the kernel of the mapping is identifiable with $N(H)/H$,
where $H \subset O(4, 2)$ is the subgroup consisting of all ele-
ments leaving V invariant and $N(H)$ is the normalizer
of H [in $O(4, 2)$]. If H is a *maximal* subalgebra of $o(4, 2)$
however, then $N(H)$ is a maximal subgroup of $O(4, 2)$
and H a normally maximal one, both having H as Lie
algebra. For these cases $N_C(H)/H$ is always a finite
group and easy to determine by inspection.

The classification of the subspaces V is done by
direct application of a theorem of Witt,[48] which for the
present case states that any two subspaces V and V'
of $\mathbb{R}^6$ are related by an $O(4, 2)$ transformation if and
only if they have the same *signature* under restriction
of the quadratic form Q [Eq. (8.21)] to V and V'. Since
the induced quadratic forms on the subspaces are no
longer necessarily nondegenerate, they must be charac-
terized by three integers $[n_+ n_- n_0]$ denoting, respective-
ly, the number of positive, negative, and null eigen-
vectors. The conjugacy classes of reducible subalgebras
may now be completely identified by the signature of the
space they leave invariant. (Of course, since the

classes with signatures $[n_+ n_- n_0]$ and $[2 - n_+ - n_0,$
$4 - n_- - n_0, n_0]$ are identical, we limit the range to
$n_- + n_+ + n_0 \leqslant 3, n_- < 3.$) It follows from general theorems[26]
that the following classes consist of maximal subalge-
bras: [100], [010], [001], [200], [020], [002], [120],
and [210]. All other classes of reducibly embedded sub-
algebras within this representation are nonmaximal
(e. g., any subalgebra $H \in [101]$ is properly contained
in a subalgebra $H \in [001]$ because the invariance of a
subspace of signature [101] implies the invariance of
its isotropic subspace of signature [001]).

(iii) An irreducible subalgebra of a semisimple Lie
algebra can either be semisimple or the direct sum of
a semisimple algebra with a one-dimensional compact
Lie algebra.[26] The algebra $c(3, 1)$ has precisely one such
class [for the $o(4, 2)$ representation], consisting of sub-
algebras isomorphic to $su(2, 1) \oplus u(1)$. When conjugacy
under $C^1_+(3, 1)$ or $C'(3, 1)$ is considered however, these
split into two inequivalent classes. Furthermore, these
are reducible in the defining representation of $su(2, 2)$
on the space $\mathbb{C}^4$ with the invariant Hermitian form h of
Sec. III 8. The remarks of paragraphs (i) and (ii) are
equally applicable in this representation, with the cor-
responding signatures for subspaces $V \subset \mathbb{C}^4$ relating
to the induced form obtained by the restriction of h to
V. The two different $SU(2, 2)$ [or $C^1_+(3, 1)$] conjugacy
classes of $su(2, 1) \oplus u(1)$ subalgebras can be character-
ized by the signatures [100] and [010] of the subspaces
left invariant by the representatives of the two classes.
If H belongs to one class, the subalgebra $J^{-1}HJ$ [see
Eqs. (8.39)—(8.42)] belongs to the other one. We have
seen that this transformation induces a transformation
of $SO(4, 2)$; consequently, depending upon whether we
classify under $C_+(3, 1)$ or under $C^1_+(3, 1)$ we have,
respectively, either one or two classes of $su(2) \oplus u(1)$
subalgebras.

11. Conjugacy classes as orbits in homogeneous spaces

(a) General method

Before treating the maximal subgroups of $C(3, 1)$ in
particular, we shall formulate the general problem of
determining conjugacy classes for subgroups of a given
group G. Let K and H be two subgroups of G and let

$$[H]_G = \{g^{-1}Hg, \ g \in G\} \qquad (11.1)$$

denote the set of subgroups of G conjugate to H. We
should like to divide $[H]_G$ into conjugacy classes under
K,

$$[g^{-1}Hg]_K = \{k^{-1}g^{-1}Hgk, \ k \in K\}. \qquad (11.2)$$

If we define the action of K in $[H]_G$ by

$$k : g^{-1}Hg \mapsto k^{-1}gHgk, \qquad (11.3)$$

then each class $[g^{-1}Hg]_K$ is a K-orbit, and $[H]_G$ is the
(disjoint) union of all the orbits. Let $N_G(H)$ denote the
normalizer of H in G. The mapping $\phi : [H]_G \to N_G(H) \backslash G$

$$\phi : g^{-1}Hg \mapsto N_G(H)g \qquad (11.4)$$

is bijective and, under this mapping, the natural action
ψ_G of G on $[H]_G$,

$$\psi_{\bar{g}} : g^{-1}Hg \mapsto \bar{g}^{-1}g^{-1}Hg\bar{g}, \ \bar{g} \in G, \qquad (11.5)$$

and Θ_G on $N_G(H)\backslash G$,

$$\Theta_{\bar{g}}: N_G(H)g \mapsto N_G(H)g\bar{g}, \quad \bar{g} \in G \qquad (11.6)$$

are equivariant.

Through the mapping ϕ, the problem of determining the classes $[g^{-1}Hg]_K$ becomes equivalent to that of identifying the double cosets $N_G(H)gK$. The latter can be seen *either* as the set of K-orbits in $N_G(H)\backslash G$ *or* as $N_G(H)$ orbits in G/K, and this dual interpretation gives two equivalent methods of solution. Of course, the right and left cosets are interchangeable; however, the nature of the orbits under K and H in the respective homogeneous spaces $N_G(H)\backslash G$ and G/K may be entirely different, the one being more readily identifiable than the other. We should like furthermore to remark that if in the above, the group G is a Lie group and $\not{H}$ is the Lie algebra for some subgroup H, the entire analysis carries through *mutatis mutandis* if we make the replacements $H \rightarrow \not{H}$, $N_G(H) \rightarrow N_G(\not{H})$.

For the reasons outlined in the Introduction, we should now like to characterize the maximal subgroups of the conformal group by their conjugacy classes under the Poincaré group $P(3,1)$; that is, for our case, $G = O(4,2)$, $K = P(3,1)$ and H is any normally maximal subgroup of $C(3,1)$. [The central elements of $O(4,2)$ do not, of course, affect the classification.] We thus have

$$N_G(H) = N_G(\not{H}) \qquad (11.7)$$

and the particular choice of H for a given $\not{H}$ is immaterial since the conjugacy classes $[\not{H}]_{P(3,1)}$ and $[H]_{P(3,1)}$ will be isomorphic under the correspondence

$$g^{-1}Hg \longleftrightarrow g^{-1}H'g \longleftrightarrow g^{-1}\not{H}g$$

for any two normally maximal groups H, H' with the same algebra $\not{H}$.

The two methods provided by the two interpretations of the double cosets are both worthwhile considering, since the first $[P(3,1)$ orbits in $O(4,2)/N_G(H)]$ is directly related to the contents of the preceding section (i.e., identification of maximal subalgebras) while the second $[N_G(H)$ orbits in $O(4,2)/P(3,1) \sim C^5]$ is directly related to the contents of Sec. V. The computations are straightforward, and the results are summarized in Table I. Readers mainly interested in the applications to invariant fields are invited to omit the details below.

(b) P(3,1) orbits in O(4,2)/N$_G$ (H)

For each reducible maximal subalgebra $\not{H}$, we have seen in Sec. IV 10 that the space $O(4,2)/N_G(H) \sim [\not{H}]_C$ can be realized as the set of all vector subspaces V of $\mathbb{R}^6$ which are related by $O(4,2)$ to a given $\not{H}$-invariant subspace. Then each $P(3,1)$-orbit consists of an equivalence class of vector spaces $[V]_{P(3,1)}$ related by $P(3,1)$ transformations,

$$[V]_{P(3,1)} = \{pV, p \in P(3,1)\} \qquad (11.8)$$

which, in turn, corresponds to a conjugacy class $[\not{H}]_{P(3,1)}$ of algebras leaving the spaces in $[V]_{P(3,1)}$ invariant. For subalgebras which are reducible within the su(2,2) realization on $\mathbb{C}^4$, precisely the same

method may be applied with regard to the corresponding invariant subspaces $V \subset \mathbb{C}^4$. The procedure for identifying the classes $[V]_{P(3,1)}$ is as follows:

(α) Choose a particular space V of given signature which is invariant under the algebra $\not{H}$ and identify it by the $O(4,2)$ [or $SU(2,2)$] signature.

(β) Identify a basis for V in terms of $\mathbb{R}^6$ (or $\mathbb{C}^4$) coordinates and apply the most general $P(3,1)$ transformation to it, as given by Eqs. (7.4) [or (8.25), (8.28)].

(γ) By suitable choice of the $P(3,1)$ transformation, reduce the basis elements to a set of standard forms which may no longer be related to each other by a $P(3,1)$ transformation.

(δ) Identify for each orbit a representative set of basis elements; one for each standard form.

(ϵ) Identify a set of elements $g \in O(4,2)$, one for each standard basis, which maps the basis for V in its standard form onto the other standard bases representing different conjugacy classes.

We now treat the maximal algebras of $c(3,1)$ by this procedure. In each case, rather than referring to the algebras, we shall identify the normally maximal group H, defined as the largest subgroup (mod $\mathbb{Z}_2$) of $O(4,2)$ [or $SU(2,2)$] to leave invariant the same space V which is invariant under its Lie algebra $\not{H}$. For the cases (i)—(vi) the group H equals its normalizer (mod $\mathbb{Z}_2$). For case (vii), there are two elements in $N(\not{H})/H$ and this will be discussed separately.

(i) *The group* $O(4,1)$: This group is identified by an invariant subspace of signature $[100]$, i.e., it is spanned by one positive vector $\mathbf{T} \in \mathbb{R}^6$, which we normalize to length $Q(\mathbf{T}) = 1$. Applying translations, we can immediately reduce $\mathbf{T}$ to the form $\mathbf{T}(\lambda)$ $= (0,0,0,0,\sinh\lambda,\cosh\lambda)$ if $T^4 + T^5 \neq 0$. If $T^4 + T^5 = 0$, then by translation and homogeneous Lorentz transformations we reduce $\mathbf{T}$ to $\mathbf{T}^{(1)} = (1,0,0,0,0,0)$. Thus, we obtain two types of $P(3,1)$ orbits of $[100]$ signature spaces: a one parameter family, represented by the spaces with basis elements $\exp(\lambda D)\mathbf{T}^{(2)}$, with $\mathbf{T}^{(2)}$ $= (0,0,0,0,0,1)$, and an isolated one, represented by the basis element $\mathbf{T}^{(1)}$. Taking the latter as the standard basis for the defining representation of $O(4,1)$, an $O(4,2)$ element which maps this onto the corresponding basis in the second family with parameter λ, is $g(\lambda)$ $= \exp[\lambda D]\exp[\pi/4(P_0 + C_0)]$. A basis for the $o(4,1)$ algebras leaving $\mathbf{T}^{(1)}$ and $\exp(\lambda D)\mathbf{T}^{(2)}$ invariant is given by

$$\{D, L_i, P_i, C_i\} \quad \text{and} \quad \{L_i, K_i, \exp(\lambda)P_\mu - \exp(-\lambda)C_\mu\},$$

respectively.

(ii) *The group* $O(3,2)$: This corresponds to an invariant subspace of signature $[010]$, spanned by one negative length vector $\mathbf{S}$, which analogously to the above, may be reduced by $P(3,1)$ transformations to one of the two forms $\mathbf{S}^{(1)} = (0,0,0,1,0,0)$ or $\exp(\lambda D)\mathbf{S}^{(2)}$, where $\mathbf{S}^{(2)} = (0,0,0,0,1,0)$. The corresponding Lie algebras and the transition elements in $O(4,2)$ leading from one conjugacy class to the others are given in Table I (as they are for all the cases below).

(iii) *The group* SIM(3,1): The invariant subspace is of signature [001] and is spanned by an isotropic vector $\mathbf{L} \in \mathbb{R}^6$ [i. e., satisfying $Q(\mathbf{L}) = 0$]. Applying $P(3,1)$ transformations this may be reduced to the form $\mathbf{L}^{(1)} = (0,0,0,0,1,1)$ if $L^4 + L^5 \neq 0$ or to one of the forms $\mathbf{L}^{(2)} = (1,0,0,-1,0,0)$ or $\mathbf{L}^{(3)} = (0,0,0,0,1,-1)$ if $L^4 + L^5 = 0$.

(iv) *The group* OPT(3,1): The invariant subspace is of signature [002], i. e., it is spanned by two orthogonal vectors $\mathbf{X}$ and $\mathbf{Y}$ [$g(\mathbf{X},\mathbf{X}) = g(\mathbf{Y},\mathbf{Y}) = g(\mathbf{X},\mathbf{X}) = 0$]. Vector $\mathbf{X}$ can be transformed to one of the three standard forms $\mathbf{L}^{(1)}$, $\mathbf{L}^{(2)}$, $\mathbf{L}^{(3)}$ obtained above in the SIM(3,1) case and vector $\mathbf{Y}$ may then be standardized by transformations in the subgroups of $P(3,1)$ which leave invariant the space spanned by $\mathbf{X}$. Doing this, and taking suitable linear combinations among the pairs of basis vectors to simplify their form, we obtain two orbits of spaces, inequivalent under $P(3,1)$, represented by the following pair of standardized basis vectors:

$$\{\mathbf{L}, \mathbf{L}\}^{(1)} = \{1,0,0,-1,0,0,\}, \; (0,0,0,0,-1,1)\}$$

and

$$\{\mathbf{L}, \mathbf{L}\}^{(2)} = \{(1,0,0,-1,0,0,\}, \; (0,0,0,0,1,1,)\}.$$

(v) *The groups* O(4)×O(2) *and* O(2,2)×O(2): These two cases, corresponding, respectively to invariant subspaces of signature [200] and [020] are treated analogously to the above; that is, one of the basis vectors is put in a previously established standard form of type $\mathbf{S}$ or type $\mathbf{T}$, and the other is standardized using the stabilizer of the corresponding one-dimensional space in $P(3,1)$. The results are summarized in Table I.

(vi) *The group* O(3)×O(2,1): The signature of the invariant subspace is [030], i. e., it is spanned by three orthogonal negative length vectors $\mathbf{S}^{(1)}$, $\mathbf{S}^{(2)}$, and $\mathbf{S}^{(3)}$. Two of them may be put into one of the two standard forms identifiable from Table I for the group O(2,2) ×O(2). Considering each separately, let us first choose $\{\mathbf{S}^{(1)}, \mathbf{S}^{(2)}\} = \{(0,1,0,0,0,0), \; (0,0,1,0,0,0)\}$. Then $\mathbf{S}^{(3)} = (S^0, 0,0, S^3, S^4, S^5)$, with $S_0^2 - S_3^2 - S_4^2 + S_5^2 = -1$. If $S^4 + S^5 \neq 0$, we can translate $S^{(3)}$ into $(0,0,0,0,\cosh\lambda, \sinh\lambda)$ without altering $\{\mathbf{S}^{(1)}, \mathbf{S}^{(2)}\}$. If $S^4 + S^5 = 0$, we can transform $\mathbf{S}^{(3)}$ by a homogeneous Lorentz transformation and a translation into $(0,0,0,1,0,0)$ without affecting $\{\mathbf{S}^{(1)}, \mathbf{S}^{(2)}\}$. Now, take the other case $\{\mathbf{S}^{(1)}, \mathbf{S}^{(2)}\} = \{(0,0,0,1,0,0), \; (0,0,0,0,\cosh\lambda, \sinh\lambda)\}$. Then $\mathbf{S}^{(3)} = (S^0, S^1, S^2, 0, S^4 \sinh\lambda, S^4 \cosh\lambda)$. By an O(2,1) transformation $\mathbf{S}^{(3)}$ can be taken into $(0,0,\cosh\mu, 0, \sinh\mu \sinh\lambda, \sinh\mu \cosh\lambda)$. The triplet $\{\mathbf{S}^{(1)}, \mathbf{S}^{(2)}, \mathbf{S}^{(3)}\}$ now determines a hyperplane in the 2345 subspace. A vector orthogonal to this hyperplane within the 2345 subspace determines the hyperplane completely and can be written as $\mathbf{T}_\perp = (0,0,\sinh\mu, 0,\cosh\mu \sinh\lambda, \cosh\mu \cosh\lambda)$. By a translation $(a,1)$ with $a = (0, -\tanh\mu \exp(-\lambda), 0, 0)$ we can take $T_\perp$ into $(0,0,0,0,\sinh\lambda, \cosh\lambda)$. The orthogonal hyperplane then goes into $\{\mathbf{S}^{(1)}, \mathbf{S}^{(2)}, \mathbf{S}^{(3)}\} = \{(0,0,0,1,0,0), \; (0,0,0,0,\cosh\lambda, \sinh\lambda), \; (0,0,0,1,0,0)\}$. Thus the classes for this case are characterized by this one parameter family of orbits together with the single

orbit represented by the standardized set of basis elements $\{(0,1,0,0,0,0), \; (0,0,1,0,0,0), \; (0,0,0,1,0,0)\}$.

(vii) *The group* O(2,1)×O(2,1): For this single case, the normalizer $N(H)$ is not equal to the group H and hence, according to the discussion of Sec. IV 10, the mapping (10.3) does not define a one—one correspondence between the set of equivalence classes $\{[V]_{P(3,1)}\}$ of vector spaces invariant under the algebra $o(2,1) \oplus o(2,1) \equiv H$ and the set of conjugacy classes $\{[H]_{P(3,1)}\}$. However, $N(H)/H$ has only two elements and hence there will be at most two classes $[V]_{P(3,1)}$ corresponding to the same class $[H]_{P(3,1)}$.

To see this, we note first that the O(4,2) conjugacy class of groups O(2,1)×O(2,1) (and their Lie algebras) is defined by the O(4,2) class of subspaces $V \subset \mathbb{R}^6$ with signature [120]. We may take, as the defining representation, the group (or algebra) leaving invariant the subspace V_1 spanned by the vectors $\{(0,1,0,0,0,0,), \; (0,0,1,0,0,0), \; (1,0,0,0,0,0)\}$. Then the normalizer of the group H (or algebra H) leaving this space invariant is

$$N(H) = H \times_s \mathbf{Z}_2',$$

where $\mathbf{Z}_2' = \{1, 1'\}$ with

$$\mathbf{1}' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The transformation $1'$ takes the space V_1 into its orthogonal complement $V_1^\perp$ and consequently two equivalence classes $[V_1^\perp]_{P(3,1)}$ and $[V_1]_{P(3,1)}$ correspond to the same conjugacy class $[H]_{P(3,1)}$. In general, any other subspace V_i of signature [120] will be mapped either into itself or its orthogonal compliment $V_i^\perp$ by elements of the normalizer of the group leaving V_i invariant, and hence will correspond to the same conjugacy class of groups (or algebras). Turning to the detailed classification of these, the [020] subspace of any such space can be chosen in one of the forms given in Table I for the invariant spaces under O(2,2)×O(2). Consider first the case when $\{\mathbf{S}^{(1)}, \mathbf{S}^{(2)}\} = \{(0,1,0,0,0,0), \; (0,0,1,0,0,0)\}$. Then the third basis vector, orthogonal to these first two, has the form $\mathbf{T} = (T^0, 0, 0, T^3, T^4, T^5)$ with normalization $T_0^2 - T_3^2 - T_4^2 + T_5^2 = 1$. If $T^4 + T^5 = 0$, we can transform the basis, using translations and homogeneous Lorentz transformation which leave $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ fixed to precisely the form given above defining the space V_1. If $T^4 + T^5 \neq 0$, we can apply a translation to transform the basis to the form $\{\mathbf{S}^{(1)}, \mathbf{S}^{(2)}, \mathbf{T}\} = \{(0,1,0,0,0,0), \; (0,0,1,0,0,0), \; (0,0,0,0,\sinh\lambda, \cosh\lambda)\}$ defining a one-parameter family of spaces $V_2(\lambda)$. Consider now the other possibility; namely, $\{\mathbf{S}^{(1)}, \mathbf{S}^{(2)}\} = \{(0,0,0,1,0,0), \; (0,0,0,0,\cosh\lambda, \sinh\lambda)\}$. We then have $\mathbf{T} = (T^0, T^1, T^2, 0, T^4 \sinh\lambda, T^4 \cosh\lambda)$, with $T_0^2 - T_1^2 - T_2^2 = 1 - T_4^2 \equiv K$. If $K < 0$ or $K > 0$, we can transform the basis, in a manner analogous to case (vi) into one of the standard bases defining V_1, $V_2(\lambda)$ or their orthogonal complements $V_1^\perp$, $V_2^\perp(\lambda)$. If $K = 0$ with $T^0 = T^1 = T^2 = 0$, we already have the space $V_1^\perp$. However, the

case $K = 0$, $T^0 \neq 0$ is different. We then have, after a suitable rotation $\mathbf{T} = (T^0, -T^0, 0, 0, \sinh\lambda, \cosh\lambda)$. Applying a translation with $a = (a^0, a^1, 0, 0)$ we obtain: $\mathbf{S}^{(1)}$
$= (0, 0, 0, 1, 0, 0)$, $\mathbf{S}^{(2)} = (a^0 e^\lambda, a^1 e^\lambda, 0, 0, \cosh\lambda$
$+ [(a_0^2 - a_1^2)/2] e^\lambda, \sinh\lambda - [(a_0^2 - a_1^2/2] e^\lambda)$, and $\mathbf{T}$
$= (T^0 + a^0 e^\lambda, -T^0 + a^1 e^\lambda, 0, 0, (a_0 + a_1) T_1 + \sinh\lambda$
$+ [a_0^2 - a_1^2/2] e^\lambda, -(a^0 + a^1) T_0 + \cosh\lambda - [(a_0^2 - a_1^2)/2] e^\lambda)$.
We now replace $\mathbf{S}^{(2)}$ and $\mathbf{T}$ by $\overline{\mathbf{T}} = \cosh b \mathbf{T} + \sinh b \mathbf{S}^{(2)}$
and $\overline{\mathbf{S}} = \sinh b \mathbf{T} + \cosh b \mathbf{S}^{(2)}$. Choosing $e^b = e^{-\lambda}/\sqrt{2}$,
$T^0 = e^{-\lambda}/2$, $a^0 - a^1 = e^{-2\lambda}/2$, $a^0 + a^1 = 1$, we obtain a space spanned by $\{(1, 0, 0, 0, 0, 1), (0, 1, 0, 0, 1, 0), (0, 0, 0, 1, 0, 0)\}$. (Note that T^0 may be chosen to have any nonzero value by applying a Lorentz boost in the 1- direction to the original expression for $\mathbf{T}$.). To identify the conjugacy class, we apply another rotation in the 1-3 plane to yield a space V_3 spanned by basis vectors $\{(1, 0, 0, 0, 0, 1), (0, 1, 0, 0, 0, 0), (0, 0, 0, 1, 1, 0)\}$. We note that, unlike the space V_1 and $V(\lambda)$, the space V_3 may be mapped into its orthogonal complement $V_3^\perp$ by an element of the (general) Poincaré group, consisting of a product of spatial rotations and a time inversion. Hence, we have five types of equivalence classes of vector spaces; $[V_1]_{P(3,1)}, [V_1^\perp]_{P(3,1)}, [V_2(\lambda)]_{P(3,1)}, [V_2^\perp(\lambda)]_{P(3,1)}, [V_3]_{P(3,1)}$, but only three conjugacy classes of groups (or algebras) leaving invariant spaces in the V_1 (and $V_1^\perp$), V_2 (and $V_2^\perp$), and $V_3 \sim V_3^\perp$ classes.

(viii) *The group* $S[U(2,1) \times U(1)]$: As discussed in Sec. IV 10, there is only one $C(3,1)$ [or $C_+(3,1)$] conjugacy class of subalgebras of $SU(2,2)$ leaving, within the defining representation on $\mathbb{C}^4$, a one-complex dimensional, nonisotropic subspace invariant, whereas under $SU(2,2)$, $C_+'(3,1)$, or $C'(3,1)$, the conjugacy classes split into two, characterized by $SU(2,2)$ signature $[100]$ and $[010]$. Similarly there is just one conjugacy class of this algebra under the general Poincaré group $P(3,1)$, but two, characterized by the signature, under the restricted $[P_+'(3,1)]$ or orthochronous $[P'(3,1)]$ Poincaré groups. To see this, we may proceed as follows. Working in the basis with off-diagonal form $\tilde{h}$ [Eqs. (8.25)–(8.29)] we may transform an arbitrary, nonisotropic vector in $\mathbb{C}^4$ into one of the forms: $(0, (a + i) q, 0, q)$ or $(0, (a - i) q, 0, q)$, with $q \in \mathbb{C}$, $a \in \mathbb{R}$, through a suitably chosen homogeneous Lorentz transformation. Next, by multiplication with a normalization factor and application of a translation in the $(0, 3)$ plane, we may transform this to the form $(0, i, 0, 1)$ or $(0, 1, 0, i)$ depending upon whether the signature is $+$ or $-$. No further transformation in the identity component of $P(3, 1)$ can relate the two spaces which these vectors span. However the transformation J (i.e., PT) [Sec. III 8, Eq. (8.39)], which in the off-diagonal basis takes the form

$$\tilde{J} = \begin{bmatrix} -i\mathbb{1} & 0 \\ 0 & i\mathbb{1} \end{bmatrix}, \tag{11.9}$$

maps these two spaces into each other. Thus, we have either one or two conjugacy classes of the group $S(U(2, 1) \times U(1))$ under the groups $P(3, 1)$ $[P_+(3, 1)]$ and $P_+'(3, 1)$ [or $P'(3, 1)]$, respectively. Within the diagonal representation with Hermitian form h [Eq. (8.4)], the bases for the two subspaces of positive and negative signature have the form $(0, 1, 0, 0)$ and $(0, 0, 0, 1)$, respectively.

(c) $N_G(H)$-Orbits in $O(4,2)/P(3,1)$

The homogeneous space $O(4, 2)/P(3, 1)$ has been identified in Sec. III 7 with the five-dimensional cone C^5 in $\mathbb{R}^6$ defined by Eq. (7.1), with the identity coset identified with the point $(0, 0, 0, 0, -1, 1)$. The conjugacy classes therefore, under $P(3, 1)$, for any subalgebra $g^{-1}Hg$ of $o(4, 2)$, (or subgroup $g^{-1}Hg$) are identified with the orbits in C^5 under the group $N_G(H)$ [or $N_G(H)$], the orbit of the identity coset being identified with the class of H (or H) itself. Since the action of any subgroup of $O(4, 2)$ on C^5 is just given by the linear transformation on $\mathbb{R}^6$, these orbits are easily identified. We therefore simply list below the equations defining the different types of orbits on C^5 corresponding to each maximal subalgebra, together with a geometric characterization of the type of space they describe. In order to recover a representative of the class to which these orbits correspond, we merely conjugate the original algebra H, which is on the identity orbit, with any group element $g \in O(4, 2)$ which maps the point $(0, 0, 0, 0, -1, 1)$ into the orbit considered. But such a group element is given precisely by the (inverse of the) mapping which takes the representative V left invariant under H, into another V', invariant under the conjugate algebra $H' = g^{-1}Hg$. This allows a direct translation of the results as obtained in the present section into those obtained in the previous one. Since these are essentially identical, we only treat three representative examples below in order to illustrate the method.

(i) *The group* $O(4, 1)$: Taking, as the original definition of H, the subgroup of $O(4, 2)$ leaving invariant the space spanned by $(1, 0, 0, 0, 0, 0)$, we obtain the following two types of orbits on C^5:

(α) $\eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 - \eta_5^2 = 0$,

(β) $\eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 - \eta_5^2 = e^{2\lambda}$. $\qquad$ (11.10)

The first, defining a four-dimensional cone C^4, is the orbit of the identity, while the second, corresponding to a one-parameter family of single-sheeted hyperboloids $H^4(\lambda)$ is obtained from the identity by applying the mapping $\exp[-(\pi/4)(P_0 + C_0)] \exp(-\lambda D)$.

(ii) *The group* $SIM(3, 1)$: We define H as the subgroup leaving invariant the space of null vectors $(0, 0, 0, 0, -x, x)$. This gives three orbits in C^5:

(α) $\eta_\mu = 0$, $\quad \eta^4 + \eta^5 = 0$,

(β) $\eta^4 + \eta^5 \neq 0$, $\qquad\qquad$ (11.11)

(γ) $\eta_\mu \eta^\mu = 0$, $\quad \eta^4 + \eta^5 = 0$, $\quad \eta^0 \neq 0$.

The identity orbit (α) defines a line L^1. Orbit (β) defines a five-dimensional subspace of C^5, which geometrically may be identified as a product of a four-dimensional paraboloid P^4 with a line $\overline{L}'$ minus a point. A group element mapping the identity coset into this orbit is given by $\exp[-(\pi/2)(P_3 - C_3)]$. The third orbit ($\gamma$) defines the product $C^3 \times L^1$ of a cone with a line and may be related to the identity by the mapping $\exp[-(\pi/4)(P_3 - C_3)] \exp[-(\pi/4)(P_0 + C_0)]$. The orbits for the remaining cases of reducible subgroups of $O(4, 2)$ are equally simple to identify, and the corresponding results are listed in Table I.

(iii) *The group* $H = S(U(2,1) \times U(1))$: This case is a little more involved, because the subgroup is simply defined only within the $SU(2,2)$ realization. Therefore, to determine its action on C^5, we must make use of the homomorphism ϕ given by Eq. (8.22). We first identify three subgroups of $S(U(2,1) \times U(1))$ that are useful for the orbital analysis. Working in the basis with diagonal form h, let G_0 be the isotropy subgroup of the origin in $\bar{M} \sim U(2)$. Using the realization of Sec. III 8, Eqs. (8.31) or (8.35) for the action on this space, we find G_0 to consist of all elements g_0 of the form

$$g_0 = u_0 g_1,$$ (11.12)

where

$$u_0 = \begin{bmatrix} \exp(-i\varphi) & & & \\ & \pm\exp(i\varphi) & & \\ & & \exp(-i\varphi) & \\ & & & \pm\exp(i\varphi) \end{bmatrix}$$ (11.13)

and

$$g_1 = \begin{bmatrix} \cosh\lambda - \exp(\lambda)\dfrac{|z|^2}{2} - iQ & -iz & Q+i & \sinh\lambda - \exp(\lambda)\dfrac{|z|^2}{2} & 0 \\ -i\exp(\lambda)\bar{z} & 1 & \exp(\lambda)\bar{z} & 0 \\ \hline Q-i & \sinh\lambda + \exp(\gamma)\dfrac{|z|^2}{2} & z & \cosh\lambda + \exp(\lambda)\dfrac{|z|^2}{2} + iQ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$ (11.14)

$$0 \leq \phi < 2\pi, \quad \lambda, Q \in \mathbb{R}, \quad z \equiv x + iy \in \mathbb{C}.$$

Now, we identify the maximal compact subgroup $K = S(U(2) \times U(1) \times U(1))$ whose elements we parametrize in the form

$$k = \bar{k}u_0,$$ (11.15)

where

$$\bar{k} = \begin{bmatrix} \exp(-i\xi/2)\,v & 0 \\ \hline 0 & \exp(i\xi/2) \end{bmatrix}, \quad 0 \leq \xi < \pi, \quad v \in SU(2).$$ (11.16)

Finally, let U_0 be the $U(1) \times \mathbb{Z}_2$ subgroup consisting of elements of the form u_0 and note that this is precisely the intersection of K with G_0. The set $\bar{K}$ of elements of the form $\bar{k}$ is not a group (due to the limits on the range of the angle ξ), however, it may be seen as locally isomorphic to the group $U(2)$ under the mapping

$$\bar{k} \longleftrightarrow \exp(-i\xi)\,v \in U(2).$$ (11.17)

This group $U(2)$ acts on the space $\bar{M}$ transitively and freely [Eq. (8.35), Sec. III 8]. It therefore follows that each element $g \in S(U(2,1) \times U(1))$ may be decomposed *uniquely* into a product

$$g = \bar{k}u_0 g_1, \quad \bar{k} \in \bar{K}, \quad u_0 \in U_0, \quad g_1 \in G_1,$$ (11.18)

where G_1 is the noncompact group consisting of elements of the form (11.14). Thus the orbit in $\bar{M}$ under $S(U(2,1) \times U(1))$ covers the entire space and the space $\bar{K}$ may be identified with the coset $S(U(2,1) \times U(1))/G_0$. To determine the orbits in C^5, we make use of the homomorphism ϕ given by Eq. (8.22). Parametrizing the $SU(2)$ element v as

$$v = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}, \quad a^2 + b^2 + c^2 + d^2 = 1,$$ (11.19)

we obtain

$$\phi(\bar{k}) = \begin{bmatrix} \cos\xi & 0 & 0 & 0 & 0 & -\sin\xi \\ 0 & a & b & -c & d & 0 \\ 0 & -b & a & d & c & 0 \\ 0 & c & -d & a & b & 0 \\ 0 & -d & -c & -b & a & 0 \\ \sin\xi & 0 & 0 & 0 & 0 & \cos\xi \end{bmatrix},$$ (11.20)

$$\phi(u_0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 & 0 & 0 \\ 0 & \sin\phi & \cos\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$ (11.21)

$$\phi(g_0) = \begin{bmatrix} \cosh\lambda + \exp(\lambda)\,\dfrac{|z|^2}{2} & -y & -x & \sinh\lambda + \exp(\lambda)\,\dfrac{|z|^2}{2} & Q & -Q \\[2mm] y\exp(\lambda) & 1 & 0 & y\exp(\lambda) & -x\exp(\lambda) & x\exp(\lambda) \\[2mm] -x\exp(\lambda) & 0 & 1 & -x\exp(\lambda) & y\exp(\lambda) & -y\exp(\lambda) \\[2mm] \sinh\lambda - \exp(\lambda)\,\dfrac{|z|^2}{2} & y & x & \cosh\lambda - \exp(\lambda)\,\dfrac{|z|^2}{2} & -Q & Q \\[2mm] Q & x & -y & \left(\cosh\lambda - \exp(\lambda)\,\dfrac{|z|^2}{2}\right) & -\left(\sinh\lambda - \exp(\lambda)\,\dfrac{|z|^2}{2}\right) \\[4mm] Q & x & -y & -\left(\sinh\lambda + \exp(\lambda)\,\dfrac{|z|^2}{2}\right) & \left(\sinh\lambda + \exp(\lambda)\,\dfrac{|z|^2}{2}\right) \end{bmatrix}.$$

$$\tag{11.22}$$

Applying $\phi(g_0)$ to the point $(0,0,0,0,-1,1)$, we obtain $\left(0,0,0,0,-\exp(\lambda),\exp(\lambda)\right)$. Next applying the transformations $\phi(\overline{k}) \in K$, we obtain all points $(\eta^0,\eta^1,\eta^2,\eta^3,\eta^4,\eta^5)$ with

$$\eta_0^2 + \eta_5^2 = \exp(2\lambda), \quad \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = \exp(2\lambda) \tag{11.23}$$

(that is, submanifolds diffeomorphic to $S^1 \times S^3$). Since λ may take any value, this covers the entire cone C^5. Therefore, there is only one conjugacy class of $S(U(2,1)\times U(1)) \subset SU(2,2)$ under the group $P(3,1)$.

V. INVARIANT FIELDS

12. Local invariance and Lie derivatives

Although the invariant fields to be discussed in Sec. V 13 have been obtained by use of the global method outlined in Sec. II, we give here the differential equations [cf. Eq. (4.6)] describing local invariance of fields under infinitesimal conformal transformations. Using the notations of Sec. III 6 Eqs. (6.3), (6.4) to express the most general field X generating local conformal transformations the condition, in Cartesian coordinates, for invariance of a (symmetric or antisymmetric) tensor of type (0,2) with components $F_{\mu\nu}$ is

$$(x_\alpha\,\omega^{\alpha\beta}\partial_\beta + a^\alpha\,\partial_\alpha)F_{\mu\nu} + (\omega_\mu^\alpha F_{\alpha\nu} + \omega_\nu^\alpha F_{\mu\alpha})$$
$$- \lambda(2 + x^\alpha\partial_\alpha)F_{\mu\nu} - (x^2 c^\alpha\partial_\alpha - 2c^\alpha x_\alpha[2 + x^\beta\partial_\beta])F_{\mu\nu}$$
$$- 2(c^\alpha x_\mu - c_\mu x^\alpha)F_{\alpha\nu} - 2(c^\alpha x_\nu - c_\nu x^\alpha)F_{\mu\alpha} = 0.$$

$$\tag{12.1}$$

For a covariant vector (1-form) with components A_μ, the condition for invariance becomes

$$(x_\alpha\,\omega^{\alpha\beta}\partial_\beta + a^\alpha\,\partial_\alpha)A_\mu + \omega_\mu^\alpha A_\alpha - \lambda(1 + x^\alpha\partial_\alpha)A_\mu$$
$$- (x^2 c^\alpha\partial_\alpha - 2c^\alpha x_\alpha[1 + x^\beta\partial_\beta])A_\mu - 2x_\mu c^\alpha A_\alpha + 2c_\mu x^\alpha A_\alpha = 0.$$

$$\tag{12.2}$$

Finally, for a scalar density ϕ with scaling dimension d (density weight $-d/4$), the invariance condition is

$$(x_\alpha\,\omega^{\alpha\beta}\partial_\beta + a^\alpha\,\partial_\alpha)\,\phi - \lambda(x^\alpha\partial_\alpha - d)\,\phi$$
$$- (x^2 c^\alpha\partial_\alpha - 2c^\alpha x_\alpha[x^\beta\partial_\beta - d])\,\phi = 0. \tag{12.3}$$

For the case of antisymmetric $F_{\mu\nu}$ (2-forms), we should like to make a decomposition of Eq. (12.1) into space and time parts, with a separation of what would be the electric and magnetic components of $F_{\mu\nu}$, if this were interpreted as an electromagnetic field tensor. The notations we use are chosen to correspond with those of Refs. 30—32, where invariance of the electromagnetic field under subgroups of the Poincaré group

was studied. We thus define the three-dimensional vectors $\vec{E}$, $\vec{B}$, $\vec{\phi}$, and $\vec{\theta}$ with components

$$E^i = F^{0i}, \quad B^i = \tfrac{1}{2}\epsilon^{ijk}F_{jk}, \tag{12.4}$$

$$\phi^i = \omega^{0i}, \quad \theta^i = \tfrac{1}{2}\epsilon^{ijk}\omega_{jk}. \tag{12.5}$$

To simplify the expressions below, we also define

$$\tilde{\omega}^{\mu\nu} \equiv \omega^{\mu\nu} + 2(c^\mu x^\nu - c^\nu x^\mu), \tag{12.6a}$$

$$\tilde{\lambda} \equiv \lambda - 2c^\alpha x_\alpha, \tag{12.6b}$$

and correspondingly

$$\vec{\tilde{\phi}} \equiv \vec{\phi} + 2[c^0\vec{x} - x^0\vec{c}], \tag{12.7a}$$

$$\vec{\tilde{\theta}} \equiv \vec{\theta} + 4\vec{c}\times\vec{x}, \tag{12.7b}$$

and the differential operator

$$\tilde{D} \equiv a^\alpha\partial_\alpha + \omega^{\alpha\beta}x_\alpha\partial_\beta - \lambda x^\alpha\partial_\alpha + c^\alpha[2x_\alpha x^\beta - \delta_\alpha^\beta x^2]\partial_\beta \tag{12.8}$$

[which is just the negative of the vector field X of Eq. (6.4) defining the infinitesimal transformation]. In this notation Eq. (12.1) decomposes into the two vector equations:

$$\tilde{D}\vec{E} + \vec{\tilde{\theta}}\times\vec{E} + \vec{\tilde{\phi}}\times\vec{B} - 2\tilde{\lambda}\vec{E} = 0, \tag{12.9a}$$

$$\tilde{D}\vec{B} + \vec{\tilde{\theta}}\times\vec{B} - \vec{\tilde{\phi}}\times\vec{E} - 2\tilde{\lambda}\vec{B} = 0, \tag{12.9b}$$

while Eq. (12.2) decomposes into the scalar and vector equations

$$\tilde{D}A^0 - \vec{\tilde{\phi}}\cdot\vec{A} - \tilde{\lambda}A^0 = 0 \tag{12.10a}$$

and

$$\tilde{D}\vec{A} + \vec{\tilde{\theta}}\times\vec{A} - A^0\vec{\tilde{\phi}} - \tilde{\lambda}\vec{A} = 0, \tag{12.10b}$$

where $A_\mu \equiv \{A^0, -\vec{A}\}$ may, for instance, be interpreted as the electromagnetic 4-potential.

Now, depending on the dimensionality of the particular subalgebra of $c(3,1)$ that is considered, the resolution of these differential equations may be relatively simple or rather complicated. In any case, for the examples treated in the following section we have found

them useful in corroborating the results obtained by the global method and in some cases complementing it. For example, it is evident from these equations that no tensor fields exist (other than the trivial constant scalar) which are invariant under the groups SIM(3, 1) or OPT(3, 1), since the algebra of each contains all the translation and dilatation generators. Local invariance under the translations implies that the fields must be constant, and the dilatation invariance implies furthermore that this constant is zero (except for a scalar field). This observation leads rather simply to the result, whereas trying to solve the linear isotropy conditions at a point on a regular orbit is somewhat more involved.

An example for which Eqs. (12.9) and (12.10) lead to nonvanishing invariant fields relatively simply is the group SO(3)×SO(2, 1) which leaves invariant the space $V_1 = \{(0, x, y, z, 0, 0)\}$ (see Table I). The Lie algebra is defined by the basis $(L_1, L_2, L_3, D, P_0, C_0)$, which corresponds to setting all parameters in Eqs. (12.9) and (12.10) equal to zero except for $\vec{\theta}$, a^0, c^0 and λ. This yields the following equations for $\vec{E}$, $\vec{B}$, $\vec{A}$, and A^0.

$(\vec{\theta})$ $\quad (\vec{\theta} \cdot \vec{x} \times \vec{\nabla})\vec{E} - \vec{\theta} \times \vec{E} = 0, \quad (\vec{\theta} \cdot \vec{x} \times \vec{\nabla})A^0 = 0,$

$\quad (\vec{\theta} \cdot \vec{x} \times \vec{\nabla})\vec{B} - \vec{\theta} \times \vec{B} = 0, \quad (\vec{\theta} \cdot \vec{x} \times \vec{\nabla})\vec{A} - \vec{\theta} \times \vec{A} = 0,$

$$(12.11a)$$

(a^0) $\quad \partial_0 \vec{A} = \partial_0 \vec{E} = \partial_0 \vec{B} = 0, \quad \partial_0 A^0 = 0,$ $\qquad (12.11b)$

(λ) $\quad (\vec{x} \cdot \vec{\nabla})\vec{E} + 2\vec{E} = 0, \quad (\vec{x} \cdot \vec{\nabla})\vec{A} + \vec{A} = 0,$

$\quad (\vec{x} \cdot \vec{\nabla})\vec{B} + 2\vec{B} = 0, \quad (\vec{x} \cdot \vec{\nabla})A^0 + A^0 = 0,$ $\qquad (12.11c)$

(c^0) $\quad 2x_0(\vec{x} \cdot \vec{\nabla})\vec{E} + 2|\vec{x}|^2 \partial_0 \vec{E} + 2\vec{x} \times \vec{B} + 4x_0 \vec{E} = 0,$

$\quad 2x_0(\vec{x} \cdot \vec{\nabla})\vec{B} + 2|\vec{x}|^2 \partial_0 \vec{B} + 2\vec{x} \times \vec{E} + 4x_0 \vec{B} = 0,$

$\quad 2x_0(\vec{x} \cdot \vec{\nabla})A^0 + 2|\vec{x}|^2 \partial_0 \vec{A} - 2\vec{x} \times \vec{A} + 2x_0 A^0 = 0,$

$\quad 2x_0(\vec{x} \cdot \vec{\nabla})\vec{A} + 2|\vec{x}|^2 \partial_0 \vec{A} - 2A^0 \vec{x} + 2x_0 \vec{A} = 0.$

Equations (12.11b) imply that the fields are independent of time and (112.11a) (rotational invariance) then imply they are of the form:

$\vec{E} = e(|\vec{x}|)\vec{x}, \quad \vec{B} = b(|\vec{x}|)\vec{x},$

$\vec{A} = a(\vec{x})\vec{x}, \quad A^0 = f(|\vec{x}|),$ $\qquad (12.12)$

for some scalar functions $e(|\vec{x}|)$, $b(|\vec{x}|)$, $a(|\vec{x}|)$, and $f(|\vec{x}|)$. Solving Eqs. (12.11c) shows that these functions have the form:

$e(|\vec{x}|) = \dfrac{N}{|\vec{x}|^3}, \quad b(|\vec{x}|) = \dfrac{M}{|\vec{x}|^3},$

$a(\vec{x}) = \dfrac{K}{|\vec{x}|^2}, \quad f(|\vec{x}|) = \dfrac{K'}{|\vec{x}|},$ $\qquad (12.13)$

where K, K', M, and N are arbitrary constants. Finally, Eqs. (12.11d) are automatically satisfied by the fields $\vec{E}$ and $\vec{B}$ given by Eqs. (12.12) and (12.13) but cannot be satisfied for nonzero $\vec{A}$, A^0. Thus we find the following solutions for the invariant fields $\vec{E}$ and $\vec{B}$,

$\vec{E} = N \dfrac{\vec{x}}{|\vec{x}|^3}, \quad \vec{B} = M \dfrac{\vec{x}}{|\vec{x}|^3},$ $\qquad (12.14)$

while no invariant fields of the type $\vec{A}$ or A^0 exist. These

results will be re-established with the global method in the following section, where the form of the invariant symmetric tensor and scalar density fields will also be given.

Before proceeding to the detailed examples treated there, we should like to make a remark that bears upon the results obtained. In the case of a *closed*, invariant 2-form F (e.g., the electromagnetic field), one might ask whether, at least locally, the problem may not be reduced to that of finding the most general invariant local 1-form field A (e.g., the vector potential) and taking its exterior derivative. (Since we are referring here to local quantities only, the distinction between closed and exact forms arising from a nontrivial cohomology group will be disregarded.) Suppose that F is such a closed form and that it is invariant under the local transformation group generated by the set of vector fields $\{X_i\}_{i=1,\ldots,r}$. We thus have

$dF = 0,$ $\qquad (12.15)$

$L_{X_i} F = 0.$ $\qquad (12.16)$

Since F is closed, there exists a local 1-form A, such that

$F = dA.$ $\qquad (12.17)$

Since Lie derivatives commute with exterior derivatives, Eqs. (12.15)–(12.17) imply

$d(L_{X_i} A) = 0$ $\qquad (12.18)$

and hence the $L_{X_i} A$ are closed forms, but not necessarily zero. Again, this implies that there exists a set of local scalar fields σ_i such that

$L_{X_i} A = d\sigma_i.$ $\qquad (12.19)$

Adding a closed 1-form $d\lambda$ to A (i.e., making a local change of gauge) does not alter the relation (12.17), therefore we may inquire whether it is possible to do so in such a way that the resulting 1-form has vanishing Lie derivatives with respect to the X_i. We define

$\bar{A} \equiv A + d\lambda;$

$\bar{A}$ will satisfy

$L_{X_i} \bar{A} = 0$

provided that

$d(X_i \lambda - \sigma_i) = 0,$ $\qquad (12.20)$

that is, provided

$X_i \lambda = \sigma_i + C_i, \quad i = 1, \ldots, r,$ $\qquad (12.21)$

where the C_i are arbitrary constants. Thus, the existence of a locally invariant 1-form $\bar{A}$, whose exterior derivative is F reduces to the existence of a solution for Eqs. (12.21). This is a set of r linear, first-order partial differential equations in n (= 4) dimension. If $r = 1$, a solution will always exist, provided the fields considered satisfy sufficient smoothness requirements. In general, the number of independent equations in (12.21) may be fewer than r, due to the Lie algebra commutation relations between the fields X_i, but they need not necessarily be compatible.

Thus, it may easily occur that even though the number of independent equations is less than the dimension of the space, no solution exists.

This implies that there may be invariant fields F which, may not be derived from an invariant field A, but only one satisfying the weaker condition (12.19) of invariance up to a gauge transformation. Indeed, this is precisely what occurs in the example treated above (SO(3)×SO(2,1)), where *no* invariant 1-form exists and will be seen to reoccur in several of the examples following.

13. Global method and results

We shall now apply the method described in Sec. II to determine the most general fields of 1-forms, 2-forms, symmetric $(0,2)$ tensors and scalar densities invariant under the subgroups of $C(3,1)$ whose algebras are listed in Table I. Since several groups may correspond to the same Lie algebra, we shall select among these the ones which lead to the fewest vanishing fields. For example in the case of the $o(3) \oplus o(2,1)$ algebra, as we have seen, the group SO(3)×SO(2,1) leaves invariant certain 2-form fields; but the group O(3)×O(2,1) does not. For most cases, however, the particular choice is immaterial, since the same types of fields arise for each.

To summarize the method, we proceed as follows. First, a convenient choice of coordinate system is made for the space $\bar{M}$. This will always be chosen as the natural coordinates describing the regular orbits of the group on the cone C^5, since these project onto the regular orbits in $\bar{M}$. Secondly, a convenient point p_0 is chosen on each orbit M_0 and the isotropy subgroup at this point is identified. Next, the linear isotropy conditions [Sec. II, Eq. (4.2)] are solved to determine the independent nonvanishing components at p_0. Finally, the group transformation [Sec. II, Eq. (4.3)] is applied to obtain the field at any point on the orbit of each p_0 and the independent parameters are taken as functions of the scalar invariants defining the orbits (if any such exist). To simplify the calculations differential forms are used throughout, thereby permitting the use of six-dimensional coordinates, in which the group action is always linear, by identification of the orbits as lower dimensional submanifolds of $\mathbb{R}^6$ with suitable projections made onto the cotangent spaces of these submanifolds.

(a) The group O(2) × O(4)

We shall treat this example in somewhat greater detail than the others in order to illustrate the method. As we have seen in Sec. IV, a one-parameter family of subgroups of this type exists and each member can be characterized by the fact that it leaves a space of the form $\{(x, 0, 0, 0, y\sinh\lambda, y\cosh\lambda)\}$ invariant. For the group corresponding to a given fixed value of the parameter λ, the regular orbits in C^5, which in this case cover it entirely, are given by the equations

$$\eta_0^2 + \eta_5'^2 = \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4'^2 = K^2 \quad (K > 0), \quad (13.1)$$

where

$$\eta'^4 = \cosh\lambda\eta^4 + \sinh\lambda\eta^5,$$

$$\eta'^5 = \sinh\lambda\eta^4 + \cosh\lambda\eta^5,$$

and K is a constant. Since these orbits are all projectively equivalent they may be identified with one and the same orbit in $\bar{M}$ (which is, in fact, the entire space). We therefore introduce coordinates for $\bar{M}$ by the following six-dimensional parametrization

$$u^\mu \equiv \frac{\eta^\mu}{(\eta_0^2 + \eta_5'^2)^{1/2}}, \quad u^{4/5} = \frac{\eta'^{4/5}}{(\eta_0^2 + \eta_5'^2)^{1/2}} \quad (13.2)$$

in terms of which, we have [Sec. III, Eq. (7.7)]

$$x^\mu = \frac{\eta^\mu}{\eta^4 + \eta^5} = \frac{au^\mu}{u^4 + u^5}, \quad a \equiv e^\lambda \quad (13.3)$$

with

$$u_0^2 + u_5^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1. \quad (13.4)$$

Equation (13.2) may be interpreted as defining a mapping from $C^5 \to \mathbb{R}^6$ (the latter space having coordinates $\{u^a\}$) whose image is the submanifold $S^1 \times S^3$ defined by (13.4). Each ray in C^5 maps onto a pair of antipodal points, allowing the identification $\bar{M} \sim (S^1 \times S^3)/\mathbb{Z}_2$ as discussed in Sec. III. The mapping of each orbit (13.1) of C^5 thus covers the space $\bar{M}$ twice. Indeed, inverting Eq. (13.3), taking into account (13.4), we obtain

$$u^\mu = \epsilon \frac{x^\mu}{\tau}, \quad (13.5)$$

$$u^4 = \epsilon \frac{(a^2 + \mathbf{x}^2)}{2a\tau}, \quad u^5 = \epsilon \frac{(a^2 - \mathbf{x}^2)}{2a\tau}, \quad (13.6)$$

where

$$\tau = \left(x_0^2 + \frac{a^2}{4}(1 - \mathbf{x}^2/a^2)^2 \right)^{1/2} \quad (13.7)$$

and

$$\epsilon = \pm 1$$

($\mathbf{x}^2$ denotes the square of the 4-vector $\mathbf{x}$).

We may furthermore introduce the natural curvilinear coordinates for $S^1 \times S^3$ which can also serve as local coordinates for the projective space if we limit the angular ranges to cover half the space and identify the antipodes. Thus, define:

$$u^0 = \cos\psi, \quad u^1 = \sin\alpha \sin\theta \cos\phi, \quad u^2 = \sin\alpha \sin\theta \sin\phi,$$

$$u^5 = \sin\psi, \quad u^3 = \sin\alpha \cos\theta, \quad u^4 = \cos\alpha, \quad (13.8)$$

$$0 \leq \psi < 2\pi, \quad 0 \leq \alpha \leq \pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi. \quad (13.9)$$

Since the group action is linear in terms of the coordinates $\{u^a\}$ and hence the Jacobian matrix of the transformation is identical to the group element itself, it is convenient to utilize the basis $\{du^a\}$ for the cotangent space in $\mathbb{R}^6$. We can then express 1-forms, 2-forms and symmetric $(0,2)$ tensors as

$$A = A_a du^a \quad \text{(1-form)},$$

$$F = F_{ab} du^a \wedge du^b \quad \text{(2-form)}, \quad (13.10)$$

$$G = G_{ab} du^a du^b \quad \text{[symmetric $(0,2)$ tensor]}.$$

The cotangent space for $S^1 \times S^3$ is then given by the conditions

$$u^0 du^0 + u^5 du^5 = 0,$$

$$u^1 du^1 + u^2 du^2 + u^3 du^3 + u^4 du^4 = 0. \tag{13.11}$$

We now choose the point $p_0 \in \overline{M}$ corresponding to the point in $S^1 \times S^3$ with coordinates

$$(u^0, u^1, u^2, u^3, u^4, u^5)_{p_0} = (0, 0, 0, 0, 1, 1). \tag{13.12}$$

The isotropy subgroup at p_0 is the group O(3) consisting of elements of the form

$$g_0 = \begin{bmatrix} 1 & 0 & 0 \\ \hline 0 & R & 0 \\ \hline 0 & 0 & \mathbb{1}_2 \end{bmatrix}, \tag{13.13}$$

where $\mathbb{1}_2$ is the 2×2 identity matrix and R is any O(3) matrix. The linear isotropy conditions (4.2) of Sec. II imply that at this point, we have:

$$A(p_0) = K \, du^0,$$

$$F(p_0) = 0, \tag{13.14}$$

$$G(p_0) = C (du_0^2) + D (du_1^2 + du_2^2 + du_3^2),$$

for any $A, F,$ and G invariant under the group. Since $\overline{M}$ consists of just one orbit under O(2)×O(4), there are no nontrivial scalar invariants, and hence the parameters $K, C,$ and D must be constants.

The invariant fields at an arbitrary point can now be obtained by applying formulas (4.3). To do this, we parametrize a general element of O(2)×O(4) in terms of the coordinates by

$$\begin{bmatrix} u^0 \\ \vec{u} \\ u^4 \\ u^5 \end{bmatrix} = \begin{bmatrix} \cos\psi & \vec{0}^T & 0 & \sin\psi \\ \vec{0} & M & \vec{l} & \vec{0} \\ 0 & \vec{h}^T & k & 0 \\ -\sin\psi & \vec{0}^T & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} 0 \\ \vec{0} \\ 1 \\ 1 \end{bmatrix}, \tag{13.15a}$$

where

$$\vec{u} \equiv \{u^i\}_{i=1,2,3}, \quad \text{and} \quad \begin{bmatrix} M & \vec{l} \\ \vec{h}^T & k \end{bmatrix} \in O(4). \tag{13.15b}$$

We thus have

$$u^0 = \cos\psi, \quad u^5 = \sin\psi, \quad \vec{u} = \vec{l}, \quad u^4 = k. \tag{13.16}$$

Applying formula (4.3) for this transformation and making use of the conditions (13.11), we obtain the following expression for the fields at point p with coordinates (u^a):

$$A = K(u^5 du^0 - u^0 du^5),$$

$$F = 0, \tag{13.17}$$

$$G = C (du_0^2 + du_5^2) + D (du_1^2 + du_2^2 + du_3^2 + du_4^2).$$

The simple form (13.17) for the invariant fields is a consequence of the linear action of the group expressed in these coordinates. In spherical coordinates for $S^1 \times S^3$ or Cartesian coordinates for M, they take the form:

$$A = K d\psi$$

$$= \frac{K}{2a\tau^2} [(a^2 + x_0^2 + |\vec{x}|^2) dx_0 - 2x_0 \vec{x} \cdot d\vec{x}], \tag{13.18a}$$

$$F = 0, \tag{13.18b}$$

$$G = Cd\psi^2 + D(d\alpha^2 + \sin^2\alpha \, d\theta^2 + \sin^2\alpha \, \sin^2\theta \, d\phi^2]$$

$$= \frac{[C+D]}{4a^2\tau^4} [(a^2 + x_0^2 + |\vec{x}|^2) dx_0 - 2x_0 \vec{x} \cdot d\vec{x}]^2$$

$$- \frac{D}{\tau^2} dx_\mu \, dx^\mu. \tag{13.18c}$$

Although the expression for the tensor G is quite cumbersome in Cartesian coordinates, we may note that if the arbitrary constants satisfy $C = -D$, it simplifies to a form which may be interpreted as a conformally flat metric, namely

$$G = \frac{C}{\tau^2} dx_\mu \, dx^\mu = \frac{C}{\tau^2} g_M. \tag{13.19}$$

To obtain the most general invariant scalar density of scaling dimension d (density weight $\delta = -d/4$), we apply formulas (4.2'), (4.3') of Sec. II. To satisfy the linear isotropy condition (4.2'), we must have $\det(^\varepsilon{}_0 J(p_0)) = \pm 1$ for all g_0 in the isotropy group at p_0. In order to treat the problem in a linear way we could determine the invariant 4-forms in $\mathbf{R}^6$ and project onto the four-dimensional subspace, analogously to the fields $A, F,$ and G, thus defining invariant volume forms, which (up to a sign) are equivalent to densities. Instead, we shall indicate how the calculation is done directly in a four-dimensional coordinate system, for which the O(2)×O(4) action is nonlinear. In terms of Cartesian coordinates for M, the action of an arbitrary group element, parametrized as in Eq. (13.15) is

$$x^0 \to \frac{a\cos\psi x_0 + \frac{1}{2}(a^2 - \mathbf{x}^2)\sin\psi}{-\sin\psi x_0 + \vec{h} \cdot \vec{x} + \frac{1}{2}a[k(a^2 + \mathbf{x}^2) + \cos\psi(a^2 - \mathbf{x}^2)]},$$

$$\vec{x} \to \frac{aM\vec{x} + \frac{1}{2}(a^2 + \mathbf{x}^2)\vec{l}}{-\sin\psi x_0 + \vec{h} \cdot \vec{x} + \frac{1}{2}a[k(a^2 + \mathbf{x}^2) + \cos\psi(a^2 - \mathbf{x}^2)]}. \tag{13.20}$$

At the isotropy point p_0, we have $x^\mu \equiv 0$, and hence

$$\left. \frac{\partial x^\mu(g)}{\partial x^\nu} \right|_{x^\nu=0} = (^\varepsilon J_\nu^\mu(p_0))$$

$$= \frac{2}{(k+\cos\psi)^2} \begin{bmatrix} 1 + k\cos\psi & -\sin\psi \vec{h}^T \\ \hline \sin\psi \vec{l} & (k+\cos\psi)M - \vec{l}\vec{h}^T \end{bmatrix}. \tag{13.21}$$

In view of (13.15'), we have

$$\vec{h}^T = -k^{-1} \vec{l}^T M$$

and hence

$$(^\varepsilon J_\nu^\mu(p_0))$$

$$= \frac{2}{(k+\cos\psi)^2}$$

$$\times \begin{bmatrix} 1 + k\cos\psi & \frac{\sin\psi}{k}\vec{l}^T \\ \hline \sin\psi \vec{l} & (k+\cos\psi)\mathbb{1}_3 + \frac{\vec{l}\vec{l}^T}{k} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \hline 0 & M \end{bmatrix}. \tag{13.22}$$

We thus have [using the further relations implied by (13.15)]

$$\det{}^\varepsilon J(p_0) = \pm \left(\frac{2}{k+\cos\psi}\right)^4 = \pm \left(\frac{2\tau}{a}\right)^4 . \qquad (13.23)$$

For elements of the form (13.13) this equals ± 1 and hence the linear isotropy condition (4.2) is satisfied. Letting the value of the scalar density ϕ at p_0 be $L(a/2)^d$, Eq. (4.2) gives it at any point p with coordinates (x^μ) as

$$\phi = L\tau^d . \qquad (13.24)$$

We have used here the general procedure to derive (13.24). However, if a nondegenerate invariant tensor such as G is known, its determinant may be calculated and used to determine the form (13.23) for $\det{}^\varepsilon J$, since

$$\det G(p) = [\det{}^\varepsilon J(p_0)]^{-2} \det G(p_0) . \qquad (13.25)$$

This gives a much simpler method for arriving at the expression (13.24) for the invariant scalar density.

The procedure for each of the other groups treated is identical to the above, and therefore we present the results in summary form only.

(b) The group O(2) × O(2,2)

(i) We first consider the conjugacy class representative defined as leaving invariant a space of type $\{(0,x,y,0,0,0)\}$.

Orbit M_0: The regular orbits on the cone C^5 are diffeomorphic to the space $S^1 \times H_0^3$ [circle times three-dimensional hyperboloid with signature $(2,2)$]. The orbit in $\bar{M}$, therefore, is the projective space $(S^1 \times H_0^3)/\mathbb{Z}_2$. The four-dimensional surface, embedded in a six-dimensional space is defined by $(v_0, v_1, v_2, v_3, v_4, v_5)$ with

$$v_1^2 + v_2^2 = 1, \quad v_0^2 - v_3^2 - v_4^2 + v_5^2 = 1 . \qquad (13.26)$$

The projection onto the cotangent space is thus given by

$$v_1 dv_1 + v_2 dv_2 = v_0 dv_0 - v_3 dv_3 - v_4 dv_4 + v_5 dv_5 = 0 . \qquad (13.27)$$

Coordinates: The projective coordinates $\{x^\mu\}$ are related to the six-dimensional ones $\{v^a\}$ by

$$x^\mu = \frac{v^\mu}{v^4 + v^5} . \qquad (13.28)$$

Inverting (13.28) using (13.26), we have

$$v^\mu = \frac{\varepsilon x^\mu}{[x_1^2 + x_2^2]^{1/2}} \quad (\varepsilon = \pm 1),$$
$$v^{4/5} = \varepsilon \frac{(1 \pm \mathbf{x}^2)}{2[x_1^2 + x_2^2]^{1/2}} . \qquad (13.29)$$

We may furthermore introduce the following curvilinear coordinates for $S^1 \times H_0^3$:

$$v^1 = \cos\psi, \qquad v^0 = \cosh b \cos\alpha,$$
$$v^2 = \sin\psi, \qquad v^5 = \cosh b \sin\alpha,$$
$$v^3 = \sinh b \cos\beta, \quad 0 \le b < \infty, \quad 0 \le \beta < 2\pi,$$
$$v^4 = \sinh b \sin\beta, \quad 0 \le \alpha < 2\pi, \quad 0 \le \psi < 2\pi. \qquad (13.30)$$

The isotropy subgroup G_0 for any point on the regular orbit in $\bar{M}$ is O(2,1). In fact the intersection of the regular orbit (of which there is only one) with Minkowski space M is the entire space minus the singular surface defined by

$$x_1 = x_2 = 0. \qquad (13.31)$$

Away from the singular surface the most general O(2)×O(2,2) invariant fields now have the form:

$$A = K(v_1 dv_2 - v_2 dv_1)$$
$$= K d\psi \qquad (13.32a)$$
$$= \frac{K}{(x_1^2 + x_2^2)}(x_1 dx_2 - x_2 dx_1),$$

$$F = 0, \qquad (13.32b)$$

$$G = C(dv_1^2 + dv_2^2) + D(dv_0^2 - dv_3^2 - dv_4^2 + dv_5^2)$$
$$= C d\psi^2 + D(\cosh^2 b \, d\alpha^2 - \sinh^2\alpha \, d\beta^2 - db^2)$$
$$= C\left(\frac{dx_1^2 + dx_2^2}{x_1^2 + x_2^2} - \frac{(x_1 dx_1 + x_2 dx_2)^2}{(x_1^2 + x_2^2)^2}\right)$$
$$+ D\left(\frac{dx_0^2 - dx_3^2}{x_1^2 + x_2^2} - \frac{(x_1 dx_1 + x_2 dx_2)^2}{(x_1^2 + x_2^2)^2}\right) , \qquad (13.32c)$$

$$\phi = L(x_1^2 + x_2^2)^{d/2} , \qquad (13.32d)$$

where K, C, D, and L are all constants (since there are no nontrivial invariant scalar fields). Note furthermore that if $C = -D$, we have

$$G = \frac{D}{(x_1^2 + x_2^2)} g_M . \qquad (13.33)$$

(ii) The representative leaving invariant a space of the form: $\{(0, 0, 0, y, x \cosh\lambda, x \sinh\lambda)\}$:

Orbit M_0: $(S^1 \times H_0^3)/\mathbb{Z}_2$

$$\bar{v}_3^2 + \bar{v}_4^2 = \bar{v}_0^2 - \bar{v}_1^2 - \bar{v}_2^2 + \bar{v}_5^2 = 1, \qquad (13.34)$$

Isotropy subgroup G_0: O(2,1),

Coordinates:

$$x^\mu = \frac{a\bar{v}^\mu}{\bar{v}^4 + \bar{v}^5} \qquad (a \equiv e^\lambda), \qquad (13.35)$$

$$\bar{v}^\mu = \frac{\varepsilon x^\mu}{\sigma}, \quad \bar{v}^{4/5} = \frac{\varepsilon[a^2 \pm \mathbf{x}^2]}{2a\sigma} , \qquad (13.36)$$

where

$$\sigma \equiv \left[x_3^2 + \frac{a^2}{4}\left(1 + \frac{\mathbf{x}^2}{a^2}\right)^2\right]^{1/2} ,$$

$$\bar{v}^0 = \cosh b \cos\alpha, \quad \bar{v}^3 = \cos\psi,$$
$$\bar{v}^1 = \sinh b \cos\beta, \quad \bar{v}^4 = \sin\psi, \qquad (13.37)$$
$$\bar{v}^2 = \sinh b \sin\beta, \quad \bar{v}^5 = \cosh b \sin\alpha,$$
$$0 \le b < \infty, \quad 0 \le \beta < 2\pi,$$
$$0 \le \alpha < 2\pi, \quad 0 \le \psi < 2\pi.$$

Singular surface:

$$x_3 = 0, \quad x_1^2 + x_2^2 = a^2 + x_0^2.$$

(In 3-space, a circle in the $x_1 x_2$ plane with radius increasing with the velocity of light.)

Invariant fields:

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$$A = K(\overline{v}_3 d\overline{v}_4 - \overline{v}_4 d\overline{v}_3) = K d\psi$$

$$= \frac{K}{2a\sigma^2}[2x_3(x_0 dx_0 - 2x_1 dx_1 - 2x_2 dx_2)$$

$$- (a^2 + \mathbf{x}^2 + 2x_3^2) dx_3], \tag{13.38a}$$

$$F = 0, \tag{13.38b}$$

$$G = C(d\overline{v}_3^2 + d\overline{v}_4^2) + D(d\overline{v}_0^2 - d\overline{v}_1^2 - d\overline{v}_2^2 + d\overline{v}_5^2)$$

$$= C\,d\psi^2 + D(\cosh^2 b d\alpha^2 - \sinh^2 \alpha d\beta^2 - db^2), \tag{13.38c}$$

$$\phi = L\sigma^d. \tag{13.38d}$$

If $\quad C = -D, \quad G = \dfrac{D}{\sigma^2}\,g_M. \tag{13.39}$

(c) The group SO(3) × SO(2,1)

(i) The representative leaving invariant a space of the type $\{(0, x, y, z, 0, 0)\}$

Orbit M_0: $(S^2 \times H^2)/Z_2$ (in $\overline{M}$)

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega_0^2 - \omega_4^2 + \omega_5^2 = 1. \tag{13.40}$$

Isotropy subgroup G_0: SO(2)×SO(1,1).

Coordinates:

$$x^\mu = \frac{\omega^\mu}{\omega^4 + \omega^5}, \tag{13.41}$$

$$\omega^\mu = \frac{\epsilon x^\mu}{|\vec{x}|}, \quad \omega^{4/5} = \frac{\epsilon(1 \pm \mathbf{x}^2)}{2|\vec{x}|}, \tag{13.42}$$

where:

$$|\vec{x}| \equiv (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

$$\omega^0 = \cosh b \sin\alpha, \quad \omega^3 = \cos\theta,$$

$$\omega^1 = \sin\theta \cos\phi, \quad \omega^4 = \sinh b,$$

$$\omega^2 = \sin\theta \cos\phi, \quad \omega^5 = \cosh b \cos\alpha, \tag{13.43}$$

$$0 \leqslant \phi < 2\pi, \quad -\infty < b < \infty,$$

$$0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \alpha < 2\pi.$$

Singular surface:

$$x_1 = x_2 = x_3 = 0, \tag{13.44}$$

Invariant fields: $A = 0$ \tag{13.45a}

$$F = \frac{M}{2}\epsilon_{ijk}\omega^i d\omega^j \wedge d\omega^k + \frac{N}{2}\epsilon_{abc}\omega^a d\omega^b \wedge d\omega^c$$

$$= M \sin\theta\, d\theta \wedge d\phi + N \cosh b d\alpha \wedge db$$

$$= \frac{M}{2|\vec{x}|^3}\epsilon_{ijk}x^i dx^j \wedge dx^k - \frac{N}{|\vec{x}|^3} dx_0 \vec{x} \cdot d\vec{x}, \tag{13.45b}$$

$$\{ijk\} = 1, 2, 3, \quad \{a, b, c\} = 0, 4, 5,$$

$$G = C(d\omega_1^2 + d\omega_2^2 + d\omega_3^2) + D(d\omega_0^2 - d\omega_4^2 + d\omega_5^2)$$

$$= C(d\theta^2 + \sin^2\theta\, d\phi^2) - D(db^2 - \cosh^2 b d\alpha^2)$$

$$= C\left(\frac{d\vec{x}^2}{|\vec{x}|^2} - \frac{(\vec{x} \cdot d\vec{x})^2}{|\vec{x}|^4}\right) + D\left(\frac{dx_0^2}{|\vec{x}|^2} - \frac{(\vec{x} \cdot d\vec{x})^2}{|\vec{x}|^4}\right), \tag{13.45c}$$

$$\phi = L|\vec{x}|^d \tag{13.45d}$$

If $\quad C = -D, \quad G = \dfrac{D}{|\vec{x}|^2}\,g_M. \tag{13.46}$

Again, M, N, L, C, D are all constants since there are no nontrivial invariant scalar fields. The expression for F must be understood as applying to one half of the space $S^2 \times H^2$ only [as expressed in the coordinates $\omega^a(\theta, \phi, \alpha, b)$] since it is not invariant under the Z_2 mapping $\{\omega^a\} \to \{-\omega^a\}$. If we require invariance, moreover, under O(3)×SO(2,1) or SO(3)×O(2,1), this implies the vanishing of the constant M or N, respectively.

(ii) The representative leaving invariant a space of the type $\{0, x, y, 0, z \cosh\lambda, z \sinh\lambda\}$

Orbit M_0: $(S^2 \times H^2)/Z_2$ (in $\overline{M}$)

$$\overline{\omega}_1^2 + \overline{\omega}_2^2 + \overline{\omega}_4^2 = \overline{\omega}_0^2 - \overline{\omega}_3^2 + \overline{\omega}_5^2 = 1, \tag{13.47}$$

Isotropy subgroup G_0: SO(2)×SO(1, 1).

Coordinates:

$$x^\mu = \frac{a\overline{\omega}^\mu}{\overline{\omega}^4 + \overline{\omega}^5}, \tag{13.48}$$

$$\overline{\omega}^\mu = \frac{\epsilon x^\mu}{\rho}, \quad \overline{\omega}^{4/5} = \frac{\epsilon(a^2 \pm \mathbf{x}^2)}{2a\rho} \quad (a = e^\lambda), \tag{13.49}$$

where

$$\rho = \left[x_1^2 + x_2^2 + \frac{a^2}{4}\left(1 + \frac{\mathbf{x}^2}{a^2}\right)^2\right]^{1/2}.$$

$$\overline{\omega}^0 = \cosh b \sin\alpha, \quad \overline{\omega}^3 = \sinh b,$$

$$\overline{\omega}^1 = \sin\theta \cos\phi, \quad \overline{\omega}^4 = \cos\theta,$$

$$\overline{\omega}^2 = \sin\theta \sin\phi, \quad \overline{\omega}^5 = \cosh b \cos\alpha, \tag{13.50}$$

$$0 \leqslant \phi < 2\pi, \quad -\infty < b < \infty,$$

$$0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \alpha < 2\pi.$$

Singular surface:

$$x_1 = x_2 = 0, \quad x_3^2 = x_0^2 + a^2, \tag{13.51}$$

Invariant fields: $A = 0$ \tag{13.52a}

$$F = \frac{M}{2}\epsilon_{ijk}\overline{\omega}^i d\overline{\omega}^j \wedge d\overline{\omega}^k + \frac{N}{2}\epsilon_{abc}\overline{\omega}^a d\overline{\omega}^b \wedge d\overline{\omega}^c$$

$$= M \sin\theta\, d\theta \wedge d\phi + N \cosh b d\alpha \wedge db$$

$$= \frac{-M}{a\rho^3}[(x_0 dx_0 - x_3 dx_3) \wedge (x_1 dx_2 - x_2 dx_1)$$

$$- [x_1^2 + x_2^2 + \tfrac{1}{2}(a^2 + \mathbf{x}^2)]dx_1 \wedge dx_2]$$

$$+ \frac{N}{a\rho^3}[(x_1 dx_1 + x_2 dx_2) \wedge (x_3 dx_0 - x_0 dx_3)$$

$$+ [x_1^2 + x_2^2 + \tfrac{1}{2}(a^2 + \mathbf{x}^2)]dx_0 \wedge dx_3], $$

$$\{ijk\} = 1, 2, 4, \quad \{abc\} = 0, 3, 5 \tag{13.52b}$$

$$G = C(d\overline{\omega}_1^2 + d\overline{\omega}_2^2 + d\overline{\omega}_4^2) + D(d\overline{\omega}_0^2 - d\overline{\omega}_3^2 + d\overline{\omega}_5^2)$$

$$= C(d\theta^2 + \sin^2\theta\, d\phi^2) - D(db^2 - \cosh^2 b d\alpha^2) \tag{13.52c}$$

$$\phi = L\rho^d. \tag{13.52d}$$

If

$$C = -D, \quad G = \frac{D}{\rho^2}g_M. \tag{13.53}$$

The comments above for case (c) (i) apply equally here.

(d) The group SO(2,1) × SO(2,1)

(i) The representative leaving invariant a space of the type $\{(x,y,z,0,0,0)\}$:

Orbits M_0: Two regular orbits in $\bar{M}$ (one regular stratum), each diffeomorphic to $(H_{\pm}^2 \times H^2)/\mathbb{Z}_2 \sim H_{+}^2 \times H^2$ (where H_{+}^2 is the upper sheet of the two-sheeted hyperboloid $H_{\pm}^2$):

$$p_0^2 - p_1^2 - p_2^2 = p_3^2 + p_4^2 - p_5^2 = \epsilon_0 = \pm 1, \tag{13.54}$$

$$p^\mu = \epsilon \frac{x^\mu}{\lambda}, \quad p^{4/5} = \epsilon \frac{(1 \pm \mathbf{x}^2)}{2\lambda}, \tag{13.55}$$

where

$$\lambda \equiv |x_0^2 - x_1^2 - x_2^2|^{1/2}. \tag{13.56}$$

Region A: $x_1^2 + x_2^2 - x_0^2 > 0$ $(\epsilon_0 = -1)$,

$$\begin{aligned}
p^0 &= \sinh b, & p^3 &= \sinh c \cos\gamma, \\
p^1 &= \cosh b \cos\beta, & p^4 &= \sinh c \sin\gamma, \\
p^2 &= \cosh b \sin\beta, & p^5 &= \cosh c,
\end{aligned} \tag{13.57a}$$

Region B: $x_1^2 + x_2^2 - x_0^2 < 0$ $(\epsilon_0 = +1)$,

$$\begin{aligned}
p^0 &= \cosh c, & p^3 &= \cosh b \cos\beta, \\
p^1 &= \sinh c \cos\gamma, & p^4 &= \cosh b \sin\beta, \\
p^2 &= \sinh c \sin\gamma, & p^5 &= \sinh b,
\end{aligned} \tag{13.57b}$$

$$0 \leqslant c < \infty, \quad -\infty < b < \infty,$$
$$0 \leqslant \gamma < 2\pi, \quad 0 \leqslant \beta < 2\pi.$$

Singular surface: $x_1^2 + x_2^2 = x_0^2.$ (13.58)

Invariant fields: $A = 0$ (13.59a)

$$\begin{aligned}
F &= \frac{M}{2} \epsilon_{ijk} p^i \, dp^j \wedge dp^k + \frac{N}{2} \epsilon_{abc} p^a \, dp^b \wedge dp^c \\
&= -M \cosh b \, db \wedge d\beta + N \sinh c \, dc \wedge d\gamma \\
&= \frac{\bar{M}}{2\lambda^3} [\epsilon_{lmn} x^l \, dx^m \wedge dx^n] \\
&\quad + \frac{\bar{N}}{\lambda^3} dx^3 \wedge [x_0 dx_0 - x_1 dx_1 - x_2 dx_2],
\end{aligned} \tag{13.59b}$$

where in region A:

$$(ijk) = (012), \quad (abc) = (345), \quad \bar{M} = M, \quad \bar{N} = N,$$

and in region B:

$$(ijk) = (534), \quad (abc) = (120), \quad \bar{M} = N, \quad \bar{N} = M,$$

and in both regions $(lmn) = (012).$

$$\begin{aligned}
G &= C(\eta_{ij} \, dp^i \, dp^j) + D(\bar\eta_{ab} \, dp^a \, dp^b) \\
&= C(db^2 - \cosh^2 b \, d\beta^2) + D(dc^2 + \sinh^2 c \, d\gamma^2) \\
&= \frac{C}{\lambda^2}\left(-\epsilon_0(dx_0^2 - dx_1^2 - dx_2^2) + \frac{(x_0 dx_0 - x_1 dx_1 - x_2 dx_2)^2}{\lambda^2}\right) \\
&\quad + \frac{D}{\lambda^2}\left(-\epsilon_0 dx_3^2 + \frac{(x_0 dx_0 - x_1 dx_1 - x_2 dx_2)^2}{\lambda^2}\right),
\end{aligned}$$

where (13.59c)

$$\eta_{ij} = \text{diag}(+1, -1, -1), \quad \bar\eta_{ab} = \text{diag}(+1, +1, -1)$$

in region A: $(ijk) = (012)$, $(abc) = (345)$, $\bar{C} = C$, $\bar{D} = D$,

in region B: $(ijk) = (534)$, $(abc) = (120)$, $\bar{C} = D$, $\bar{D} = C$,

$$\phi = L\lambda^d. \tag{13.59d}$$

If

$$C = -D, \quad G = \frac{C}{\lambda^2} g_M.$$

The values of the constants C, D, K, M, N need not be the same in regions A and B. If $O(2,1) \times O(2,1)$ invariance is required, then $M = N = 0$. The 2-form F, expressed in the $p^a(b, c, \gamma, \beta)$ coordinates is only locally defined on $H_{\pm}^2 \times H^2$ for the half-space defined by $\epsilon = +1$, but this covers the entire projective space $(H_{\pm}^2 \times H^2)/\mathbb{Z}_2$ and hence the regular orbits.

(ii) The representative leaving invariant a space of type $\{(0, y, z, 0, x \sinh\lambda, x \cosh\lambda)\}$

Orbits M_0: $H_{\pm}^2 \times H^2/\mathbb{Z}_2$ (as above; two regular orbits, one stratum):

$$q_0^2 - q_3^2 - q_4^2 = q_1^2 + q_2^2 - q_5^2 = \epsilon_0. \tag{13.60}$$

(Again, the orbits corresponding to $\epsilon_0 = -1$ and $\epsilon_0 = +1$ respectively will be referred to as region A and region B.)

Isotopy subgroup: $SO(2) \times SO(1, 1)$.

Coordinates:

$$x^\mu = \frac{a q^\mu}{q^4 + q^5}, \quad (a = e^\lambda), \tag{13.61}$$

$$q^\mu = \epsilon \frac{x^\mu}{\eta}, \quad q^{4/5} = \epsilon \frac{(a^2 \pm \mathbf{x}^2)}{2a\eta}, \tag{13.62}$$

where

$$\eta \equiv \left|x_1^2 + x_2^2 - \frac{a^2}{4}\left(1 - \frac{\mathbf{x}^2}{a^2}\right)^2\right|^{1/2}. \tag{13.63}$$

Region A:

$$x_1^2 + x_2^2 - \frac{a^2}{4}\left(1 - \frac{\mathbf{x}^2}{a^2}\right)^2 < 0, \quad \epsilon_0 = -1,$$

$$\begin{aligned}
q^0 &= \sinh b, & q^3 &= \cosh b \cos\beta, \\
q^1 &= \sinh c \cos\gamma, & q^4 &= \cosh b \sin\beta, \\
q^2 &= \sinh c \sin\gamma, & q^5 &= \cosh c.
\end{aligned} \tag{13.64a}$$

Region B:

$$x_1^2 + x_2^2 - \frac{a^2}{4}\left(1 - \frac{\mathbf{x}^2}{a^2}\right)^2 > 0, \quad \epsilon_0 = 1,$$

$$\begin{aligned}
q^0 &= \cosh c, & q^3 &= \sinh c \cos\gamma, \\
q^1 &= \cosh b \cos\beta, & q^4 &= \sinh c \sin\gamma, \\
q^2 &= \cosh b \sin\beta, & q^5 &= \sinh b,
\end{aligned} \tag{13.64b}$$

$$0 \leqslant c < \infty, \quad -\infty < b < \infty,$$
$$0 \leqslant \gamma < 2\pi, \quad 0 \leqslant \beta < 2\pi.$$

Singular surfaces:

$$(a \pm \sqrt{x_1^2 + x_2^2})^2 + x_3^2 = x_0^2, \tag{13.65}$$

Invariant fields: $A = 0$ (13.66a)

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$$F = \frac{M}{2}\epsilon_{ijk}q^i\,dq^j \wedge dq^k + \frac{N}{2}\epsilon_{abc}q^a\,dq^b \wedge dq^c$$

(13.66b)

$$= -M\cosh b\,db \wedge d\beta + N\sinh c\,dc \wedge d\gamma,$$

$$G = C(\eta_{ij}dq^i\,dq^j) + D(\bar{\eta}_{ab}dq^a\,dq^b)$$

$$= C[db^2 - \cosh^2 b\,d\beta^2] + D[dc^2 + \sinh^2 c\,d\gamma^2],$$ (13.66c)

$$\phi = L\eta^d,$$ (13.66d)

where

$$\eta_{ij} = \mathrm{diag}(+1,-1,-1), \quad \bar{\eta}_{ab} = \mathrm{diag}(+1,+1,-1),$$

in region A: $(ijk) = (034)$, $(abc) = (125)$,

in region B: $(ijk) = (512)$, $(abc) = (340)$.

If

$$C = -D, \quad G = \frac{C}{\mu^2}g_M.$$ (13.67)

The above expressions, within curvilinear coordinates, are identical to those for the previous case. They may be straightforwardly expressed in Cartesian coordinates, using Eqs. (13.63) and (13.64), but the expressions are cumbersome. All the comments made above for case (d) (i) apply equally here.

(iii) The representative leaving invariant a space of type $\{(x,y,0,z,z,x)\}$:

Orbits: $(H_+^2 \times H^2)/Z_2 \sim H_+^2 \times H^2$ (as above, two regular orbits, one stratum):

$$\tfrac{1}{2}(r_0 + r_5)^2 - \tfrac{1}{2}(r_3 + r_4)^2 - r_1^2 = r_2^2 - \tfrac{1}{2}(r_0 - r_5)^2 + \tfrac{1}{2}(r_3 - r_4)^2 = \epsilon_0.$$

(13.68)

(Again, values $\epsilon_0 = -1$ and $\epsilon_0 = +1$ identify regions A and B, respectively.)

Isotropy subgroup: $SO(2) \times SO(1,1)$.

Coordinates:

$$x^\mu = \frac{r^\mu}{r^4 + r^5},$$ (13.69)

$$r^\mu = \frac{\epsilon x^\mu}{\nu}, \quad r^{4/5} = \frac{\epsilon(1 \pm \mathbf{x}^2)}{2\nu},$$ (13.70)

where

$$\nu = \frac{1}{\sqrt{2}}\left|x_2^2 - x_1^2 - \mathbf{x}^2(x^0 + x^3) + (x^0 - x^3)\right|^{1/2}.$$

Region A: $x_2^2 - x_1^2 - \mathbf{x}^2(x^0 + x^3) + (x^0 - x^3) < 0$, $\epsilon_0 = -1$,

$$r^0 = \frac{1}{\sqrt{2}}(\sinh b + \cosh c), \quad r^3 = \frac{1}{\sqrt{2}}(\cosh b \cos\beta + \sinh c \cos\gamma),$$

$$r^1 = \cosh b \sin\beta, \quad r^4 = \frac{1}{\sqrt{2}}(\cosh b \cos\beta - \sinh c \cos\gamma),$$

$$r^2 = \sinh c \sin\gamma, \quad r^5 = \frac{1}{\sqrt{2}}(\sinh b - \sinh c).$$

(13.71a)

Region B: $x_2^2 - x_1^2 - \mathbf{x}^2(x^0 + x^3) + (x^0 - x^3) > 0$, $\epsilon_0 = +1$,

$$r^0 = \frac{1}{\sqrt{2}}(\sinh b + \cosh c), \quad r^3 = \frac{1}{\sqrt{2}}(\sinh c \cos\gamma + \cosh b \cos\beta),$$

$$r^1 = \sinh c \sin\gamma, \quad r^4 = \frac{1}{\sqrt{2}}(\sinh c \cos\gamma - \cosh b \cos\beta),$$

$$r^2 = \cosh b \sin\beta, \quad r^5 = \frac{1}{\sqrt{2}}(\sinh c - \sinh b),$$

(13.71b)

$$0 \le c < \infty, \quad -\infty < b < \infty, \quad 0 \le \gamma < 2\pi, \quad 0 \le \beta < 2\pi.$$

Singular surface:

$$x_2^2 - x_1^2 - \mathbf{x}^2(x^0 + x^3) + (x^0 - x^3) = 0.$$ (13.72)

Invariant fields (for region A: $\{ijk\} = \{0 + 5, 3 + 4, 1\}$, $\{abc\} = \{2, 3 - 4, 0 - 5\}$, for region B: $\{ijk\} = \{0 - 5, 2, 3 - 4\}$, $\{abc\} = \{3 + 4, 1, 0 + 5\}$):

$$A = 0,$$ (13.73a)

$$F = \frac{M}{2}\epsilon_{ijk}r^i\,dr^j \wedge dr^k + \frac{N}{2}\epsilon_{abc}r^a\,dr^b \wedge dr^c$$

(13.73b)

$$= -M\cosh b\,db \wedge d\beta + N\sinh c\,dc \wedge d\alpha,$$

$$G = C\eta_{ij}dr^i\,dr^j + D\bar{\eta}_{ab}dr^a\,dr^b$$

$$= C[db^2 - \cosh^2 b\,d\beta^2] + D[dc^2 + \sinh^2 c\,d\gamma^2],$$ (13.73c)

$$\phi = L\nu^d.$$ (13.73d)

If

$$C = -D, \quad G = \frac{C}{\nu^2}g_M.$$ (13.74)

Again, the comments above for cases (i) and (ii) apply equally here.

(e) The group O(4,1)

(i) The representative leaving invariant a space of type $\{(0,0,0,0, x\sinh\lambda, x\cosh\lambda)\}$:

Orbit M_0: H^4 [one-sheeted hyperboloid, signature $(4,1)$]

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 - t_0^2 = 1,$$ (13.75)

Isotropy subgroup: $O(3,1)$

Coordinates:

$$x^\mu = \frac{at^\mu}{1 + t^4} \quad (a = e^\lambda),$$ (13.76)

$$t^\mu = \frac{2ax^\mu}{a^2 - \mathbf{x}^2}, \quad t^{4/5} = \frac{a^2 + \mathbf{x}^2}{a^2 - \mathbf{x}^2},$$ (13.77)

$$t^0 = \sinh b, \qquad\qquad t^3 = \cosh b \sin\beta \cos\theta,$$

$$t^1 = \cosh b \sin\beta \sin\theta \cos\phi, \quad t^4 = \cosh b \cos\beta,$$ (13.78)

$$t^2 = \cosh b \sin\beta \sin\theta \sin\phi,$$

$$-\infty < b < \infty, \quad 0 \le \theta \le \pi,$$

$$0 \le \beta \le \pi, \qquad 0 \le \phi < 2\pi.$$

Singular surface:

$$|\vec{x}|^2 = x_0^2 - a^2.$$ (13.79)

Invariant fields:

$$A = 0,$$ (13.80a)

$$F = 0,$$ (13.80b)

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$$G = C(dt_0^2 - dt_1^2 - dt_2^2 - dt_3^2 - dt_4^2)$$

$$= C[db^2 - \cosh^2 b (d\beta^2 + \sin^2\beta d\theta^2 + \sin^2\beta \sin^2\theta \, d\phi^2)]$$

$$= \frac{4a^2 C}{(a^2 - \mathbf{x}^2)^2} \, \eta_{\mu\nu} dx^\mu \, dx^\nu,$$

$$\text{(13.80c)}$$

$$\phi = L(a^2 - \mathbf{x}^2)^d. \qquad \text{(13.80d)}$$

There are no nontrivial scalar fields, therefore C and L are constants. Note that G may be interpreted as the metric for the de Sitter space with constant positive curvature.

(ii) The representative leaving invariant a space of the type $\{(x, 0, 0, 0, 0, 0)\}$:

Orbit M_0: H^4 (one-sheeted hyperboloid)

$$\bar{l}_1^2 + \bar{l}_2^2 + \bar{l}_3^2 + \bar{l}_4^2 - \bar{l}_5^2 = 1. \qquad \text{(13.81)}$$

Isotropy subgroup: O(3, 1)

Coordinates:

$$x^i = \frac{\bar{l}^i}{\bar{l}^4 + \bar{l}^5}, \quad x^0 = \frac{1}{\bar{l}^4 + \bar{l}^5}, \qquad \text{(13.82)}$$

$$\bar{l}^i = \frac{x^i}{x^0}, \quad \bar{l}^{4/5} = \frac{1 \pm \mathbf{x}^2}{2x^0}. \qquad \text{(13.83)}$$

$$\bar{l}^1 = \cosh b \, \sin\beta \, \sin\theta \, \cos\phi, \quad \bar{l}^4 = \cosh b \, \cos\beta,$$

$$\bar{l}^2 = \cosh b \, \sin\beta \, \sin\theta \, \sin\phi, \quad \bar{l}^5 = \sinh b, \qquad \text{(13.84)}$$

$$\bar{l}^3 = \cosh b \, \sin\beta \, \cos\theta,$$

$$-\infty < b < \infty, \quad 0 \le \theta \le \pi,$$

$$0 \le \beta \le \pi, \qquad 0 \le \phi < 2\pi.$$

Singular surface:

$$x^0 = 0. \qquad \text{(13.85)}$$

Invariant fields:

$$A = 0, \qquad \text{(13.86a)}$$

$$F = 0, \qquad \text{(13.86b)}$$

$$G = C[d\bar{l}_5^2 - d\bar{l}_1^2 - d\bar{l}_2^2 - d\bar{l}_3^2 - d\bar{l}_4^2]$$

$$= C[db^2 - \cosh^2 b (d\beta^2 + \sin^2\beta \, d\theta^2 + \sin^2\beta \sin^2\theta \, d\phi^2)]$$

$$= \frac{C}{x_0^2} \eta_{\mu\nu} dx^\mu \, dx^\nu,$$

$$\text{(13.86c)}$$

$$\phi = L x_0^d. \qquad \text{(13.86d)}$$

The group in this case may be interpreted as the conformal group for a three-dimensional Euclidean space, embedded in a four-dimensional space by the addition of a scaling parameter x_0.

(f) The group O(3,2)

(i) The representative leaving invariant a space of type $\{(0, 0, 0, 0, x \cosh\gamma, x \sinh\gamma)\}$:

Orbit M_0: $\bar{H}^4$ [one-sheeted hyperboloid, signature (2, 3)]

$$s_5^2 + s_0^2 - s_1^2 - s_2^2 - s_3^2 = 1. \qquad \text{(13.87)}$$

Isotropy subgroup: O(3, 1)

Coordinates:

$$x^\mu = \frac{as^\mu}{1 + s^5} \quad (a = e^\lambda), \qquad \text{(13.88)}$$

$$s^\mu = \frac{2ax^\mu}{a^2 + \mathbf{x}^2}, \quad s^5 = \frac{a^2 - \mathbf{x}^2}{a^2 + \mathbf{x}^2}. \qquad \text{(13.89)}$$

$$s^0 = \cosh b \, \sin\beta, \qquad s^3 = \sinh b \, \cos\theta,$$

$$s^1 = \sinh b \, \sin\theta \, \cos\phi, \quad s^5 = \cosh b \, \cos\beta, \qquad \text{(13.90)}$$

$$s^2 = \sinh b \, \sin\theta \, \sin\phi,$$

$$0 \le b < \infty, \quad 0 \le \theta < \pi,$$

$$0 \le \beta < 2\pi, \quad 0 \le \phi < 2\pi.$$

Singular surface:

$$|\tilde{x}|^2 = x_0^2 + a^2. \qquad \text{(13.91)}$$

Invariant fields:

$$A = 0, \qquad \text{(13.92a)}$$

$$F = 0, \qquad \text{(13.92b)}$$

$$G = C(ds_5^2 + ds_0^2 - ds_1^2 - ds_2^2 - ds_3^2)$$

$$= -C(db^2 - \cosh^2 b d\beta^2 + \sinh^2 b d\theta^2 + \sinh^2 b \sin^2\theta \, d\phi^2)$$

$$= \frac{4a^2 C}{(a^2 + \mathbf{x}^2)} \, \eta_{\mu\nu} dx^\mu \, dx^\nu,$$

$$\text{(13.92c)}$$

$$\phi = L(a^2 + \mathbf{x}^2)^d. \qquad \text{(13.92d)}$$

Again, C and L are constants and G may be interpreted as the metric for the de Sitter space of constant negative curvature.

(ii) The representative leaving invariant the space $\{(0, 0, 0, x, 0, 0)\}$:

Orbit M_0: $\bar{H}^4$ [one-sheeted hyperboloid, signature (2, 3)]

$$\bar{s}_5^2 + \bar{s}_0^2 - \bar{s}_1^2 - \bar{s}_2^2 - \bar{s}_4^2 = 1. \qquad \text{(13.93)}$$

Isotropy group: O(3, 1)

Coordinates:

$$x^\mu = \frac{\bar{s}^\mu}{\bar{s}^4 + \bar{s}^5} \quad (\mu \ne 3), \quad x^3 = \frac{1}{\bar{s}^4 + \bar{s}^5}, \qquad \text{(13.94)}$$

$$\bar{s}^\mu = \frac{x^\mu}{x^3} \quad (\mu = 0, 1, 2), \quad \bar{s}^{4/5} = \frac{1 \pm \mathbf{x}^2}{2x^3}. \qquad \text{(13.95)}$$

$$\bar{s}^0 = \cosh b \, \sin\beta, \qquad \bar{s}^4 = \sinh b \, \cos\theta,$$

$$\bar{s}^1 = \sinh b \, \sin\theta \, \cos\phi, \quad \bar{s}^5 = \cosh b \, \cos\beta, \qquad \text{(13.96)}$$

$$\bar{s}^2 = \sinh b \, \sin\theta \, \sin\phi,$$

$$0 \le b < \infty, \quad 0 \le \theta < \pi,$$

$$0 \le \beta < 2\pi, \quad 0 \le \phi < 2\pi.$$

Singular surface:

$$x^3 = 0. \qquad \text{(13.97)}$$

Invariant fields:

$$A = 0, \qquad \text{(13.98a)}$$

$$F = 0, \qquad (13.98b)$$

$$G = C(d\vec{s}_5^2 + d\vec{s}_0^2 - d\vec{s}_1^2 - d\vec{s}_2^2 - d\vec{s}_4^2)$$

$$= -C(db^2 \cosh^2 b\, d\beta^2 + \sinh^2 b\, d\theta^2 + \sinh^2 b\, \sin^2\theta\, d\phi^2)$$

$$= \frac{C}{x_3^2}\, \eta_{\mu\nu}\, dx^\mu\, dx^\nu, \qquad (13.98c)$$

$$\phi = L x_3^d. \qquad (13.98d)$$

The group here may be interpreted as the conformal group for a three-dimensional Lorentzian space embedded in a four-dimensional space by the addition of a scaling parameter x_3.

(g) The group S(U(2,1) × U(1))

We consider the subgroup of SU(2,2) defined, in the diagonal representation with Hermitian form h (Secs. IV 10 (iii) and IV 11 (viii)], as leaving invariant the subspace of $\mathbb{C}^4$ consisting of vectors of the form $(0,0,0,\omega)$. From the decomposition of Eq. (11.18) and the representation on C^5 defined by Eqs. (11.20)—(11.22), we obtain the following results:

Orbit M_0: As observed in Sec. IV 11, the group acts transitively on $\overline{M}$; that is, there is one orbit covering the entire space $\overline{M} \sim (S^1 \times S^3)/Z_2$. We may thus coordinatize, as for example (a) [the group O(2)×O(4)] by Eqs. (13.5)—(13.9).

Isotropy subgroup: The group G_0 of Sec. IV 11 (iii), with elements of the form (11.12)—(11.14), or equivalently, (11.21) and (11.22) [where the isotropy point is taken as the origin $(0,0,0,0,1,1)$ under the embedding j of Eq. (7.6)].

Invariant fields: The linear isotropy conditions at the origin imply that there is no nonvanishing 1-form, 2-form, symmetric (0,2) tensor or scalar density field invariant under this group.

This is most directly seen by using the realization given by Eq. (8.31) in a neighborhood of the origin (for which the local transformation is sufficient) to evaluate the Jacobian of the isotropy subgroup transformations. On the other hand, nonzero invariant *vector* fields do exist, having the general form:

$$V = K\left(u^2\frac{\partial}{\partial u^1} - u^1\frac{\partial}{\partial u^2} - u^4\frac{\partial}{\partial u^3} + u^3\frac{\partial}{\partial u^4} + u^5\frac{\partial}{\partial u^0} - u^0\frac{\partial}{\partial u^5}\right)$$

$$= K\left[(x_0^2 - x^3 x^0 + \tfrac{1}{2}(1 - \mathbf{x}^2))\frac{\partial}{\partial x^0} + (x^0 x^1 - x^3 x^1 + x^2)\frac{\partial}{\partial x^1}\right.$$

$$\left. + (x^0 x^2 - x^3 x^2 - x^1)\frac{\partial}{\partial x^2} + (x^0 x^3 - x_3^2 - \tfrac{1}{2}(1 + \mathbf{x}^2))\frac{\partial}{\partial x^3}\right],$$

$$(13.99)$$

where K is constant. The fact that an invariant vector field can exist, while an invariant 1-form (covector) does not is due to the nonexistence of an invariant metric field [(0,2) tensor] which could relate the two. The vector field (13.99) is just the one induced by the invariant 1-parameter subgroup U(1).

(h) The group OPT(3,1)

For each of the two cases [leaving invariant spaces of the type $\{(x,0,0,x,\pm y,y)\}$], there is one regular orbit in $\overline{M}$ which is diffeomorphic to Minkowski space M plus a three-dimensional Euclidean space "at infinity." The isotropy subgroup is the direct product of a Euclidean group E(2) with a three-dimensional solvable group. There are no invariant tensor fields other than the trivial constant scalar.

(i) The group SIM(3,1)

The three representatives leave invariant the spaces $\{(0,0,0,0,-x,x)\}$, $\{(0,0,0,0,x,x)\}$, and $\{(x,0,0,-x,0,0)\}$. In each case, the regular orbit is diffeomorphic to M, the isotropy subgroup is O(3,1)×O(1,1) and there are no nontrivial tensor fields.

(j) Nonmaximal subgroups

In all the above cases there was only one regular stratum in $\overline{M}$, containing at most two orbits. Consequently, no nontrivial invariant scalar fields existed and the invariant fields obtained are distinguished only by the values taken for the constants K, L, M, N, C, and D. For nonmaximal subgroups, the strata may contain an infinity of orbits, making $G \backslash M_i$ (M_i the ith regular stratum) a manifold rather than a finite set of points, and hence there exist nontrivial scalar fields. We consider three illustrative examples below.

(α) The group O(4) $\subset$ O(4)×O(2) [contained in the maximal subgroup of example (a) above]:

The same coordinates may be used as for example (a), however the orbits are now diffeomorphic to a 3-sphere S^3. These are distinguished by the value of the single nontrivial scalar invariant, which may be taken as $u^0 = x^0/\tau = \cos\psi$ [see Eqs. (13.4)—(13.9) above for notation]. The isotropy group for this case is still O(3). Hence, the invariant fields are of the same form as those for the maximal group O(2)×O(4) [Eqs. (13.18) and (13.24) above], but with the constants K, L, C, and D replaced by arbitrary functions $K(u_0)$, $L(u_0)$, $C(u_0)$, and $D(u_0)$ of the scalar invariant.

(β) The homogeneous Lorentz group O(3,1) contained in the maximal subgroup SIM(3,1) which leaves invariant the space $\{0,0,0,0,-x,x\}$.

Here the orbital analysis is familiar and may be described entirely in Minkowski space M upon which SIM(3,1) acts as a global transformation group. The regular orbits are two-sheeted (timelike) or one-sheeted (spacelike) hyperboloids: the corresponding strata having O(3) and O(2,1), respectively, as isotropy subgroups. The single scalar invariant is $\mathbf{x}^2$ and the invariant fields are of the form:

$$A = K(\mathbf{x}^2)\eta_{\mu\nu} x^\mu\, dx^\nu, \qquad (13.100a)$$

$$F = 0, \qquad (13.100b)$$

$$g = C(\mathbf{x}^2)x_\mu x_\nu\, dx^\mu\, dx^\nu + D(\mathbf{x}^2)\eta_{\mu\nu}\, dx^\mu\, dx^\nu, \qquad (13.100c)$$

$$\phi = L(\mathbf{x}^2). \qquad (13.100d)$$

(γ) The group O(3,1)×O(1,1) contained in the maximal subgroup SIM(3,1):

This is just the homogeneous Lorentz group augmented by scaling transformations. The strata are the same as those for O(3, 1) given above, but each one contains just one orbit. There are therefore no non-trivial scalar invariants, and the form of the arbitrary functions in Eq. (13. 100) becomes uniquely determined, giving the following scalar fields:

$$A = K \frac{\eta_{\mu\nu} x^{\mu} dx^{\nu}}{|\mathbf{x}^2|^{1/2}} ,$$ (13. 101a)

$$F = 0,$$ (13. 101b)

$$g = C \frac{x_{\mu} x_{\nu}}{\mathbf{x}^4} dx^{\mu} dx^{\nu} + \frac{D}{\mathbf{x}^2} \eta_{\mu\nu} dx^{\mu} dx^{\nu},$$ (13. 101c)

$$\phi = L(\mathbf{x}^2)^{d/2} .$$ (13. 101d)

14. Invariance under P, T, and PT

As discussed in the Introduction, the choice of one representative of each conjugacy class under the Poincaré group for the maximal subgroups of $C(3, 1)$ is motivated by considerations of relativistic invariance, insofar as proper Lorentz transformations are concerned. For the discrete transformations P, T, and PT, however, the interpretation is not quite the same, since the physical equivalence of two different fields related only by such transformations implies further dynamical assumptions depending, for instance, on the invariance of the equations governing such fields. Therefore, we are led to examine how these transformations act upon the fields obtained above. By inspection of the six-dimensional forms of these fields, we can see that these transformations [vid. Eq. (7. 15)] map the fields A, G, and ϕ onto themselves in every case, therefore yielding no new invariant fields. For the case of the nonvanishing 2-forms F [i. e. for the groups SO(3)×SO(2, 1) and SO(2, 1)×SO(2, 1)], we see that these discrete transformations have the effect at most of changing the signs of the constants M and N, which are arbitrary. Therefore, again, no new types of fields are obtained for these cases, even though the invariance groups involved are not members of the same conjugacy class under the proper Poincaré group.

15. Discussion

With regard to physical interpretation of the fields obtained here, the following questions may be asked:

(a) Which, if any, of these fields may be identified as solutions of conformally invariant field equations, and in particular, can the 2-form fields be identified as satisfying Maxwell's equations?

(b) If the singularities of these fields can be identified as defining distributions of localized sources, how may these sources be characterized in terms of spatial extent and time dependence?

In fact, the reply to these questions is very easily obtained. First, we note that everywhere on the regular orbits, the nonvanishing 2-forms are locally exact, and hence closed. This may be seen explicitly in terms of the curvilinear coordinates introduced above:

(i) SO(3)×SO(2, 1): From Eq. (13. 45) [or (13. 52)], we have

$$F = dA,$$ (15. 1)

where

$$A \equiv M \cos\theta \, d\phi - N \sinh b \, d\alpha.$$

(ii) SO(2, 1)×SO(2, 1): From Eq. (13. 59), [or (13. 66), (13. 73)]

$$F = dA,$$ (15. 2)

where

$$A \equiv - M \sinh b \, d\beta + N \cosh c \, d\alpha.$$

Thus, the first set of Maxwell's equations,

$$dF = 0,$$ (15. 3)

apply everywhere on the regular strata, though not on the singular ones. To verify the second set of Maxwell's equations, we need the dual forms $*F$. This is most easily obtained by noting that the dual of a 2-form is unchanged under a conformal change in metric. Since the choice $C = - D$ for the symmetric (0, 2) tensors obtained above yields what can be identified as a conformally flat metric which has a very simple form in the curvilinear coordinates, the Hodge star duality can be applied with respect to this metric (indeed, it is quite simple to alternatively express the Minkowski metric g_M in the curvilinear system). In any case, the result is surprisingly simple; the dual of each F is just obtained by interchanging the two constants $M \leftrightarrow N$:

(i) SO(3)×SO(2, 1):

$$*F = N \sin\theta \, d\theta \wedge d\phi + M \cosh b \, d\alpha \wedge db,$$

(ii) SO(2, 1)×SO(2, 1):

$$*F = - N \cosh b \, db \wedge d\beta + M \sinh c \, dc \wedge d\alpha$$

and hence

$$d * F = 0.$$ (15. 4)

Therefore, the sources vanish everywhere, except on the singular strata. This is not surprising, since the singular strata exactly correspond to the singularities in all the fields. We now discuss these in particular:

(i) SO(3)×SO(2, 1): $\{0, x, y, z, 0, 0\}$

The singular stratum defined by Eq. (13. 44) corresponds to a point singularity at the origin. The field F may be interpreted as that of a static point electric charge, plus a point magnetic charge, located at the origin (i. e., Coulomb field plus magnetic monopole field), the two charges being proportional to M and N, respectively.

(ii) SO(3)×SO(2, 1): $\{0, x, y, 0, z \cosh\lambda, z \sinh\lambda\}$

The singular stratum is defined by Eq. (13. 51) and corresponds to a point source (e.g., electric plus magnetic) which is moving along the x_3 axis with a constant acceleration (in the relativistic sense that the 4-acceleration is Fermi transported along the trajectory remaining, of course, orthogonal to the 4-velocity).

(iii) SO(2, 1)×SO(2, 1): $\{x, y, z, 0, 0, 0\}$

The singular stratum is defined by Eq. (13. 58) and corresponds to a point source (e.g., electric plus magnetic) which is moving along the x_3 axis with a constant

(iv) $SO(2,1) \times SO(2,1)$: $\{0, y, z, 0, x \sinh\lambda, x \cosh\lambda\}$

The singular stratum is given by Eq. (13.65), which defines a toroid, symmetrical about the x_3 axis, with major radius a, and minor radius increasing with the velocity of light.

(v) $SO(2,1) \times SO(2,1)$: $\left\{ \frac{x}{\sqrt{2}}, y, 0, \frac{z}{\sqrt{2}}, \frac{z}{\sqrt{2}}, \frac{x}{\sqrt{2}} \right\}$

For this case, the singular stratum is defined by the cubic equation (13.72) with sections that are hyperbolas and cardioids respectively in the planes parallel to and normal to the $x_1 x_2$ plane. The other cases of singular surfaces will not be discussed further here. They all correspond to simple configurations such as lines, planes, spheres expanding with the velocity of light, etc.

However, since no invariant 2-form fields exist for these cases, it might be more natural to give a different interpretation to these singularities. Looking for instance at the invariant fields G for the groups $O(4,1)$ and $O(3,2)$, we obtain what may be interpreted as the metrics for de Sitter spaces of constant positive or negative curvature. However that is not consistent, of course, with the interpretation of $\{x^\mu\}$ as Cartesian coordinates. Instead, we must interpret these as singular coordinates for a curved space with metric G, the singularities at $\mathbf{x}^2 = \pm a^2$ being spuriously introduced through these coordinates. Within this interpretation, we obtain a list of metrics for spaces diffeomorphic to the regular orbits on C^5 whose isometry groups are the groups discussed above. In particular, the choices $C = -D$ for the constants defining these fields all yield conformally flat metrics. Finally, we may remark that the scalar density ϕ, with weight $d = -1$ obtained for the $O(3,2)$ de Sitter group is exactly that given in Ref. 16 as a solution of the conformal invariant field equation of the $\lambda \phi^4$ field theory. If this is taken as representing the classical ground state of the system, we have a spontaneous breakdown of the $C(3,1)$ symmetry.

VI. SUMMARY

The principle results of the present work are twofold. First we have obtained a characterization of the maximal subgroups of the conformal group of space—time through their conjugacy classes under the Poincaré group. In the process, a fairly general method for determination of such conjugacy classes has been utilized which has applicability both to the continued study of the conformal group and to other subgroup analyses. Secondly, we have determined the most general tensor fields, of the types frequently encountered in physics, that are invariant under these subgroups. Again, the differential geometric methods have been formulated in such a way as to be applicable to a wide range of similar studies. The particular results obtained may themselves be further utilized in a variety of problems, of which we mention a few here:

(i) *Spontaneous symmetry breaking*: As discussed in the Introduction, a knowledge of the invariance properties of solutions to conformally invariant field equations is basic to any study of the spontaneous breaking of conformal symmetry. At present, the only studies which

have involved an explicit examination of such solutions concerned $O(3,2)$ invariant scalar densities,[16] and $O(5)$ invariant Euclidean $SU(2)$ gauge fields.[18,22,53] The latter falls somewhat outside the scope of the results obtained here, since it involves the invariance of connections under "simultaneous" gauge and space—time transformations. However, it is quite simple to relate this to results regarding invariant tensor fields, as will be shown elsewhere.[49]

(ii) *Conserved quantities and integrals of motion*: If the fields studied here are regarded as influencing the motion of particles, either classically or quantum mechanically, through a suitable coupling, the symmetry group for the equations of motion will be determined by the symmetry of the fields. The invariants of this symmetry group will then be of importance in defining the integrals of motion (conserved quantum numbers) and its representations will be relevant to defining the quantum mechanical Hilbert space.[50]

(iii) *Separation of variables*: A knowledge of the subgroup structure of the invariance group for a symmetric (0, 2) tensor, interpreted as the group of isometries for the Riemannian space with the given tensor as metric, is central to the problem of separation of variables in the Laplace—Beltrami, Klein—Gordon and other differential equations defined in such a space.[51]

(iv) *Classification of G-structures*: Invariant tensor fields are directly related to the automorphisms of G-structures.[52] Therefore, if the symmetric tensor fields which have been obtained here are interpreted as metric tensors, their invariance group is precisely the automorphism group for the Riemannian (or pseudo-Riemannian) structures they define. The scalar densities similarly define volume structures, the 1-forms locally causal structures (given a Riemannian metric), the 2-forms symplectic or almost symplectic structures, and their corresponding invariance groups may likewise be regarded as automorphism groups of these structures.

(v) Finally, regarding generalizations of the methods used in the present work, two interesting possibilities that suggest themselves are: (i) the extension to nontensorial fields (e.g., spinor fields, other induced field representations of the conformal group,[3,9] or connections) and (ii) the study of fields which are *conformally* invariant (i.e., up to a multiplicative factor) under various space—time transformation groups.

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[53]Note added in proof: Since the writing of this manuscript, further examples have come to our attention. In particular, de Alfaro, Fubini, and Furlan, Phys. Lett. B 63, 163 (1976), have obtained solutions to both the scalar and Yang—Mills field equations which are invariant under O(4)×O(2), while M. Lüscher (Desy preprint) and B. M. Schechter, Phys. Rev. D 16, 3013 (1977), have studied all O(4) invariant solutions to the Yang—Mills equations.

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Super-convergent adiabatic invariants with resonant denominators by Lie transforms[a]

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Adiabatic invariants of motion for perturbed Hamitonian systems are very important in plasma physics. It has been shown how to deal with resonant denominators to first order in the perturbation by judicious choice of the zeroth order invariant. The method is extended to higher orders by solving the Liouville equation and by using the Lie transform technique. The resulting invariants are shown to be equivalent by using the operator algebra of the averaging method involving the Poisson bracket, integrating, and averaging operations. Partial Lie transforms are introduced to describe the internal structure of any resonance in any order. The super convergent expansions of Kolmogorov are easily illustrated in the Lie formalism and allow a quick analysis of high-order resonances to be made.

INTRODUCTION

Invariants of motion have played a key role in many plasma devices and continue to be of interest for the containment of plasma, the structure of magnetic surfaces in toroidal devices, and the propogation of large amplitude waves for plasma heating. Surprisingly, there is still something to say about these problems even after a century of researches since Poincaré. A major advance in the technology of carrying out the perturbation theories was the introduction by Deprit[1] of the Lie transform technique. This gives a recursive set of equations that can be solved to any desired order and is applied here to the calculation of resonant adiabatic invariants. The Lie transform technique turns out to be elegant and flexible and, although many of the results are not new, the properties of an adiabatic motion can be very easily analyzed. The principal results of the paper are now summarized.

Invariants of nearly periodic Hamiltonian systems have been calculated by various methods, including the Poisson bracket method of McNamara and Whiteman.[2,3] The theory of such invariants has been extended to resonant cases by Dunnett *et al.*[4] by a renormalization of the leading term, but they only calculated the first correction. A typical problem, due to Smith and Kaufman,[5] is outlined in Sec. 1 and is used throughout the paper for illustration.

The operator algebra of the averaging method[2,3] is extended in Sec. 2 to deal with resonant denominators. The results of the paper are expressed in terms of these operators that enable general solutions to be given, rather than just algorithms. The generalized resonant invariant is calculated to second order in Sec. 3 in terms of the operator algebra. Enough details are given there for the reader to learn the manipulation techniques. Little is known about the general convergence properties of these series, but the method of construction shows that the rest of the invariant is determined by a nonsingular, nonsecular equation.

The Lie transform method is presented in the canonical form due to Kamel, review by Nayfeh.[6] This canonical algorithm has been programmed in MACSYMA,[7] the M.I.T. algebraic manipulator, by Char[8] to carry out the algorithm to high orders. Time-dependent problems are dealt with in this paper by introducing the energy and time as canonically conjugate variables. A time-dependent invariant can then be calculated in this extended phase space. The averaged Hamiltonian and generating function for the Lie transformation are presented to third order in the operator algebra notation. The remarkable feature of the Lie transform is that any function of the new or old variables can be transformed into the other variables by the same generating function, which depends only on the final variables. An important difference from the treatment of McNamara and Whiteman[2] is that initial conditions do not appear in the transformations. Perhaps the most interesting result of this paper is that the resonant adiabatic invariant is just an appropriately chosen function of the averaged momenta, expressed in the original variables. The Lie transform approach gives a different looking invariant from that of Sec. 2, which can nevertheless be shown to be the same function by using the operator algebra.

The Lie form of the invariant is more convenient, as shown in Sec. 5, where the invariant for the sample problem is given explicitly to $O(\epsilon^2)$. As the invariant is calculated to higher and higher order the renormalization of the leading term must be altered to take account of new harmonic resonances arising at each new order. The rules for picking the leading term are given here and then the invariant is examined for the position and width of each resonance. This leads to a refinement of the often-used overlap condition for breakdown of the invariant into stochastic motion.

It turns out that the invariant indicates the breakup of resonant islands but does not give the correct internal structure for the resonances. In Sec. 6 a very simple classification of Lie transforms is given in terms of arbitrary generating functions. It is then shown in Sec. 7 why each order of the invariant describes trapped orbits fairly well in the current order of resonant harmonics and how to select a partial Lie transform that

will describe any selected resonance in any order. The sample problem reveals an essential difficulty with these trapped orbits: Because the slow drift frequency around the trapped orbits drops to zero at the separatrix between trapped and passing orbits, all possible resonances between harmonics of the primary frequency and harmonics of the drift frequency appear in $O(\epsilon^2)$. This is shown by making the action angle transformation to local trapped orbits. The leading predicted resonance agrees with exact orbit calculations, but the region near the separatrix is always stochastic due to the accumulation of resonances as it is approached. Following Smith,[9] an explicit estimate is given for the thickness of the stochastic layer.

The Lie transforms equivalent to the super-convergent set of canonical transformations first discussed by Kolmogorov (see Ref. 16) are displayed in Sec. 8. The corresponding super-convergent invariant only improves the accuracy of the description of the passing orbits. However, the technique, as shown in Sec. 9 for a time-dependent problem, is very useful for generating a qualitative understanding of the resonance structure in higher orders. The calculations are summarized in Sec. 10, and the general form of the Hamiltonian is noted for which these procedures can be carried out.

1. THE PROBLEMS OF RESONANT DENOMINATORS

The essentials of the Poisson bracket method were in the algebra of the poisson bracket, integrating, and averaging operators that enables the partial differential equations of the formulation to be simplified and solved. The required modifications are best illustrated by introducing a simple example, first studied by Smith and Kaufman.[5] The Hamiltonian can be reduced to

$$h = \tfrac{1}{2}p_z^2 + p_\Phi + \epsilon \sum_{n=-\infty}^{\infty} J_n(\sqrt{2p_\Phi}) \sin(z - n\phi) \equiv h_0 + \epsilon h_1, \qquad (1)$$

where J_n are Bessel functions, p_z, p_Φ are conjugate to z, ϕ. This describes the motion of a charged particle in a uniform magnetic field and a finite-amplitude electrostatic wave propagating obliquely to the magnetic field. The particles execute a fixed frequency ($\omega = 1$) cyclotron oscillation and see the electric fluctuations with a frequency proportional to p_z, the velocity of the particle along the field.

An invariant K is sought that satisfies the Poisson bracket equation,

$$\frac{dK}{dt} = [h, K] \equiv \sum_i \left(\frac{\partial h}{\partial p_i} \frac{\partial K}{\partial q_i} - \frac{\partial h}{\partial q_i} \frac{\partial K}{\partial p_i} \right) = 0, \qquad (2)$$

and is expanded as

$$K = K_0 + \epsilon K_1 + \frac{\epsilon^2}{2!} K_2 + \cdots + \frac{\epsilon^n}{n!} K_n + K_\infty. \qquad (3)$$

It will become evident that the expansion of the invariant can only be achieved to a finite order if one is to avoid pathological functions. This is quite satisfactory for many purposes, and the condition to be imposed on K_∞, the remainder of the asymptotically defined invariant, is that its time derivative is nonsingular and oscillatory. Inserting Eqs. (1) and (3) into Eq. (2) gives the sequence of equations

$$[h_0, K_0] = 0, \qquad (4)$$

$$[h_0, K_i] = [K_{i-1}, h_1], \qquad (5)$$

and

$$[h, K_\infty] = \epsilon^{n+1}[K_n, h_1]. \qquad (6)$$

It is immediately apparent that the demand that the last equation be at least well behaved and nonsingular imposes conditions on K_n and all previous orders. The lowest order equation becomes

$$p_z \frac{\partial K_0}{\partial z} + \frac{\partial K_0}{\partial \phi} = 0, \qquad (7)$$

and so $K_0 = K_0(p_z, p_\Phi, z - p_z\phi)$ and is independent of the distance along the zero order orbit, $Q = z + p_z\phi$. As usual, the invariant is to be constructed to be periodic in z and ϕ, period 2π and so cannot be a function of $z - p_z\phi$ either. [Notice that it would have been impossible to satisfy these periodic conditions if the Hamiltonian had been transformed to the simplest form, $h = p_1 + \epsilon H_1(P, Q)$ because the periodic terms in H_1 would have periods dependent on p_z that, when differentiated by the Poisson brackets, would have produced terms proportional to z^n, an unwanted secularity.] It is convenient for the present example to choose, $K_0 = K_0(p_z)$.

The first-order equation is then

$$[K_1, h_0] = -\frac{\partial K_0}{\partial p_z} \sum J_n \cos(z - n\phi) \equiv -K_0' \sum J_n \cos(z - n\phi). \qquad (8)$$

On integrating this equation one finds

$$K_1 = K_0' \sum \frac{J_n \sin(z - n\phi)}{(p_z - n)} + G_1(p_z, p_\Phi, z - p_z\phi), \qquad (9)$$

where G_1 is a constant of integration, independent of Q. Taylor and Laing[10] observed that the resonances or poles in K_1, which immediately destroy the convergence of the series at $O(\epsilon)$, can be removed by suitable choice of K_0,

$$\frac{\partial K_0}{\partial p_z} = p_z \prod_1^\infty \left[1 - \left(\frac{p_z}{n}\right)^2 \right] = \sin\pi p_z. \qquad (10)$$

The infinite product representation shows us what to do in a simpler example with a finite number of resonances. Taylor and Laing assumed G_1 can be zero and compared the resulting invariant to Smith and Kaufman's orbit calculations with some success. However, this solution does not behave well if one more integration is performed in going to the next order since

$$\int K_1 dQ = +K_0' \sum \frac{J_n \cos(z - n\phi)}{(p_z - n)^2} + G_1 Q, \qquad (11)$$

which has poles at $p_z = n$. Also, the result of Eq. (9) does not integrate nicely at the resonances,

$$k_n = \underset{p_z \to n}{L} (K_1) = -J_n \sin(z - n\phi) + G_1 \quad \text{and}$$

$$\int k_n dQ = [-J_n \sin(z - n\phi) + G_1]Q. \qquad (12)$$

In other words the processes $L_{p_z \to n}$ and $\int dQ$ are not interchangeable with this choice of K_0 but obviously would be if $K_0 = \sin^2 \pi p_z$. The same observations are true under differentiation with respect to p_z which raises the order of the poles in Eq. (9) or removes a zero

from K_0. It is clear that K_0 will have to be chosen carefully to get to $O(\epsilon^n)$ without introducing any poles in the solution. It will be shown that K_0 need not be chosen until $O(\epsilon^n)$ has been reached and that the equations for K_i can be solved in each order.

2. THE OPERATOR ALGEBRA

The basic operations are integrating and averaging functions periodic in several variables along the zero-order orbits, parametrized by Q. The operations are only uniformly valid if the integrand is zero at any resonance. Since there is no unique period in the general case, the averaging operator is defined to be

$$\bar{f} = \underset{Q \to \infty}{\angle \text{im}} \left(\frac{1}{Q} \int_0^Q f \, dQ' \right) . \tag{13}$$

A typical term in the S-K problem would be $f = K(p_z)$ $\cos(mz - n\phi)$ which gives

$$\bar{f} = \underset{Q \to \infty}{\angle \text{im}} \left(\frac{K}{Q} \left[\frac{\sin(mz - n\phi)}{mp_z - n} \right]_0^Q \right) \tag{14}$$

and is zero if $K(n/m) = 0$. In general, $\bar{f}$ contains no resonant functions and is zero at resonance. Two integrating operators were defined in Ref. (2) and need no modification provided $f \to 0$ at resonances. They are

$$\tilde{f} = \int_0^Q (f - \bar{f}) \, dQ' = \hat{f} - \hat{f}(0) \text{ and } \hat{f} = \int^Q (f - \bar{f}) \, dQ. \tag{15}$$

Observe that f needs only a single zero at each resonance since $\bar{f} \equiv 0$ at these points.

The initial condition terms do not affect any results since they would merely change the choice of lowest order constant of motion. They will be ignored henceforth, and only the following relationships are needed:

$$\overline{\tilde{f}} = 0, \quad \hat{\bar{f}} = 0, \quad \overline{\tilde{f}g} + \overline{\tilde{g}f} = 0, \quad \text{and} \quad (\hat{f}g)^{\wedge} + (f\hat{g})^{\wedge} = \hat{f}\hat{g} - \overline{\hat{f}\hat{g}}$$

$$+ \overline{\tilde{f}\hat{g}} + \hat{\tilde{f}}g. \tag{16}$$

These are valid if f or g have enough zeros at resonances to account for all the integrations, and are easily verified from the definitions and integration by parts. The term $\overline{\tilde{f}\hat{g}}$ represents only the nonresonant parts of $\tilde{f}\hat{g}$.

The Poisson bracket operator satisfies the well-known identities:

$$[f, g] + [g, f] = 0, \quad [f, [g, h]] + [h, [f, g]] + [g, [f, h]] = 0,$$

and

$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0 \tag{17}$$

However, the Poisson brackets differentiate any resonant denominators, and so the functions involved must have zeros of one order higher at the resonant points. In the S-K example one finds

$$\frac{\partial}{\partial p_z} (\bar{f}) = \overline{\frac{\partial f}{\partial p_z}} \quad \text{and} \quad \frac{\partial}{\partial p_z} (\hat{f}) = \left(\frac{\partial f}{\partial p_z} \right)^{\wedge} . \tag{18}$$

The following relations are useful when one function in a Poisson bracket is constant along the zero order orbit,

$$\overline{[K, f]} = [K, \bar{f}], \quad [K, f]^{\wedge} = [K, \hat{f}]. \tag{19}$$

Simple rules will be given for determining the number of zeros required in f to deal with the harmonics and resonances of various orders arising from the $[\ ]$, $\bar{}$, and $\hat{}$ operations.

3. THE GENERALIZED INVARIANT

The calculation of McNamara and Whiteman can now be repeated under the new rules to find a more general solution in which K_0 is an arbitrary function of the lowest order constant of motion and is independent of h_0. The Hamiltonian and invariant are assumed to be of the form

$$h = \sum_0^\infty \frac{\epsilon^n}{n!} h_n(p, q), \quad K = \sum_0^N \frac{\epsilon^n}{n!} K_n(p, q) + K_\infty. \tag{20}$$

These expansions are substituted into Eq. (2), which is solved order by order. In lowest order, K_0 satisfies

$$[h_0, K_0] = 0. \tag{21}$$

At $O(\epsilon)$ the equation gives

$$[h_0, K_1] = [K_0, h_1]. \tag{22}$$

The solution, K_1, will be nonsecular and nonsingular only if the average of the right-hand side over the zero-order orbits is zero,

$$[K_0, \bar{h}_1] = 0. \tag{23}$$

The solution is then

$$K_1 = [K_0, \hat{h}_1] + G_1. \tag{24}$$

The constant G_1 should be determined by ensuring that the next order equation is nonsecular. This is

$$[h_0, K_2] = [K_1, h_1] + [K_0, h_2], \tag{25}$$

and so, to avoid secularities,

$$\overline{[K_1, h_1]} + [K_0, \bar{h}_2] = 0 = \overline{[[K_0, \hat{h}_1], h_1]} + [G_1, \bar{h}_1] + [K_0, \bar{h}_2]. \tag{26}$$

The operator algebra can be used to reduce the first term as follows:

$$\overline{[[K_0, \hat{h}_1], h_1]} = -\overline{[[\hat{h}_1, h_1], K_0]} - \overline{[[h_1, K_0]\hat{h}_1]}$$

$$= \overline{[[\hat{h}_1, h_1], K_0]} + \overline{[[\hat{h}_1, K_0], h_1]},$$

$$\therefore \overline{[[K_0, \hat{h}_1], h_1]} = \frac{1}{2}[K_0, \overline{[\hat{h}_1, h_1]}]. \tag{27}$$

Since K_0 is a function of the zero-order constants of motion, it is orthogonal to functions averaged along the zero-order orbit. The equation for G_1 reduces to

$$[G_1, \bar{h}_1] = 0, \tag{28}$$

and $G_1 = 0$ is appropriate since K_0 already satisfies the same equation. The same is not true in the next order for which

$$K_2 = 2[K_1, h_1]^{\wedge} + [K_0, \hat{h}_2] + G_2$$

$$= 2[[K_0, \hat{h}_1], h_1]^{\wedge} + [K_0, \hat{h}_2] + G_2. \tag{29}$$

The secularity condition on G_2 is

$$\tfrac{1}{2}\overline{[K_2, h_1]} + \tfrac{1}{2}\ \overline{[K_1, h_2]} + \overline{[K_0, \bar{h}_3]} = 0$$

$$= [[\overline{[K_0, \hat{h}_1], h_1}]\hat{}\,h_1] + \tfrac{1}{2}\ \overline{[[K_0, \hat{h}_2], h_1]} + \tfrac{1}{2}\ \overline{[[K_0, \hat{h}_1], h_2]}$$

$$+ \overline{[G_2, \bar{h}_1]}. \tag{30}$$

The operator algebra now gets harder to use because it is not a clear recursive set of operations. However, the details are worth displaying as the same steps are used again in the Lie transform section.

The second and third terms cancel identically and the first term is rearranged as follows:

$$\overline{[[[K_0, \hat{h}_1], \hat{h}_1], h_1]}\hat{} = - \overline{[[[K_0, \hat{h}_1], h_1], \hat{h}_1]}$$

$$= \overline{[([[\hat{h}_1, h_1], K_0] + [[h_1, K_0], \hat{h}_1])], \hat{h}_1]}$$

$$= - \overline{[[K_0, \hat{h}_1], [\hat{h}_1, h_1]]} - \overline{[[[\hat{h}_1, [\hat{h}_1, h_1]], K_0]}$$

$$+ \overline{[[[h_1, K_0], \hat{h}_1], \hat{h}_1]}. \tag{31}$$

The second term vanishes identically. The term can also be rearranged in an alternative way with the $\hat{\,}\hat{\,}$ formula, Eq. (16):

$$[[K_0, \hat{h}_1], h_1]\hat{} = - [[K_0, h_1]\hat{}, \hat{h}_1] + [[K_0, \hat{h}_1], \hat{h}_1] - \overline{[[K_0, \hat{h}_1], \hat{h}_1]}$$

$$+ [[K_0, \hat{\hat{h}}_1], \bar{h}_1], \tag{32}$$

and the term involving $[K_0, \bar{h}_1]$ is already zero. The first term (i) now yields,

$$\overline{[(i), h_1]} = - \overline{[[[h_1, K_0], \hat{h}_1], \hat{h}_1]}, \tag{33}$$

which is the negative of the third term of Eq. (31). The second term becomes

$$\overline{[(ii), h_1]} = - \overline{[[\hat{h}_1, h_1], [K_0, \hat{h}_1]]} - \overline{[[\hat{h}_1, [K_0, \hat{h}_1]], \hat{h}_1]}$$

$$= - \overline{[[\hat{h}_1, h_1][K_0, \hat{h}_1]]} - \overline{[[[K_0, \hat{h}_1], h_1], h_1]}. \tag{34}$$

The fourth term yields

$$\overline{[(iv), h_1]} = - \overline{[[[K_0, \hat{h}_1], h_1], \hat{h}_1]}$$

$$= \overline{[[\bar{h}_1, \hat{h}_1], [K_0, \hat{h}_1]]} + \overline{[[\hat{h}_1, [K_0, \hat{h}_1]], \bar{h}_1]}. \tag{35}$$

Notice that the second term here is not necessarily zero because $\bar{h}_1$ still depends on coordinates not averaged out. Adding Eqs. (31) and (32) gives

$$3\overline{[[[K_0, \hat{h}_1], h_1]\hat{}, h_1]} = \overline{[[\hat{h}_1, [K_0, \hat{h}_1]], \bar{h}_1]}$$

$$+ \overline{[[\bar{h}_1, \hat{h}_1], [K_0, \hat{h}_1]]}. \tag{36}$$

The secularity condition finally becomes

$$\overline{[(G_2 - \tfrac{1}{2}\overline{[[K_0, \hat{h}_1], \hat{h}_1]}), \bar{h}_1]} = 0, \tag{37}$$

with the obvious solution

$$G_2 = \tfrac{1}{2}\overline{[[K_0, \hat{h}_1], \hat{h}_1]}. \tag{38}$$

If the development is terminated here, then

$$[h, K_\infty] = \frac{\epsilon^3}{2}\{\tfrac{1}{3}[K_0, h_3] + [K_1, h_2] + [K_2, h_1]\} + O(\epsilon^4). \tag{39}$$

The right-hand side of this equation has no secularities away from the resonance points regardless of the choice of K_0. At the resonances the right-hand side will have no poles, but K_0 should not be chosen to have additional zeros there because K_2 will then contain no information about the resonance regions. Since K_∞ is to be found by integration along the true nonlinear orbits, the presence of resonant terms will not cause any immediate singularity to develop in K_∞. These remarks will become clearer with an example. The best that can be said is that dK_∞/dt is nonsingular, nonsecular, and $O(\epsilon^3)$. The generalized invariant to $O(\epsilon^2)$ is finally

$$K^{[2]} = K_0 + \epsilon[K_0, \hat{h}_1] + \epsilon^2\{[K_0, \hat{h}_1]\hat{}, h_1] - \tfrac{1}{2}\overline{[[K_0, \hat{h}_1], \hat{h}_1]}$$

$$+ \tfrac{1}{2}[K_0, \hat{h}_2]\} + K_\infty, \tag{40}$$

where $K_0 = 0$ at any poles arising from the $\hat{\,}$ operations.

4. THE LIE TRANSFORM METHOD

The Lie transform method introduced by Deprit shows how to construct a generating function for a family of coordinate transformations and their inverses for any Hamiltonian system. The method and generalizations are reviewed by Nayfeh[6] and some recent developments and applications are given by Dewar.[11,12] The form of the algorithm in the notation of this paper is as follows: Given a Hamiltonian

$$h(p, q, t, \epsilon) = \sum_0^\infty \frac{\epsilon^n}{n!} h_n(p, q, t) \tag{41}$$

such that h_0 represents a soluble problem, one seeks to transform to a new set of coordinates (P, Q) in which the new Hamiltonian is

$$H(P, Q, t, \epsilon) = \sum_0^\infty \frac{\epsilon^n}{n!} H_n(P, Q, t) \tag{42}$$

with more desirable properties. Starting from the identity transformation in lowest order, so that

$$H_0 = h_0(P, Q, t), \tag{43}$$

one seeks a generating function

$$W(P, Q, t) = \sum_0^\infty \frac{\epsilon^n}{n!} W_{n+1}, \tag{44}$$

by solving the chain of equations,

$$\frac{\partial W_n}{\partial t} + [H_0, W_n] = H_n - h_n - \sum_{j=1}^{n-1} \left\{ \binom{n-1}{j-1} [W_j, h_{n-j}] \right.$$

$$\left. + \binom{n-1}{j} H_{j, n-j} \right\}, \tag{45}$$

where

$$H_{j, i} = [W_j, H_i] - \sum_{m=1}^{j-1} \binom{j-1}{m-1} [W_j, H_{j-m, i}]. \tag{46}$$

Equation (45) is solved for two functions; H_n is chosen at each order to remove undesirable terms from the right-hand side—in the case of adiabatic invariants, the average of the right-hand side. The equation is then solved for W_n by integration along the unperturbed orbits. It is easy to absorb any time dependence by introducing time and energy as conjugate variables so that, with $p_t \equiv H$, the new Hamiltonian in the extended phase space is

$$E = p_t + H(p, q, t) \equiv 0 \text{ and } E_0 \equiv p_t + H_0(p, q, t). \quad (47)$$

The solution of the equations can now be written in terms of the $[\ ]$, $\hat{\ }$, and $\overline{\ }$ operators to $O(\epsilon^3)$ as,

$$H = h_0(P, Q, t) + \epsilon \overline{h}_1 + \frac{\epsilon^2}{2}\{\overline{h}_2 + \overline{[\hat{h}_1, h_1]}\} + \frac{\epsilon^3}{3!}\{\overline{h}_3 + 3\overline{[\hat{h}_1, h_2]}$$

$$+ 2\overline{[\hat{h}_1, [\hat{h}_1 h_1]]} + \overline{[\hat{h}_1, [\hat{h}_1, \overline{h}_1]]}\}$$

and

$$\epsilon W = \epsilon \hat{h}_1 + \frac{\epsilon^2}{2}\{\hat{h}_2 + [\hat{h}_1, \hat{h}_1]^{\hat{}} + [\hat{\hat{h}}_1, \overline{h}_1]\} + \frac{\epsilon^3}{3!}\hat{H}_0^{(3)}, \quad (48)$$

where

$$H_0^{(3)} = h_3 + [(\hat{h}_2 + [\hat{h}_1, \hat{h}_1] + [\hat{h}_1, \overline{h}_1]), 2h_1 + \overline{h}_1]$$

$$+ [\hat{h}_1, (2\overline{h}_2 + 2[\hat{h}_1, h_1] + h_2 - [\hat{h}_1, \overline{h}_1])]. \quad (49)$$

The transformation operator and its inverse are given by:

$$F(p(P, Q, t), q(P, Q, t), t, \epsilon) = \exp\{+\epsilon([W(P, Q, t,),]$$

$$+ \partial/\partial\epsilon)\epsilon = 0\} \cdot F(P, Q, t)$$

$$= E_W \cdot F, \quad (50)$$

and

$$F(P(p, q, t), Q(p, q, t), t, \epsilon)$$

$$= \exp\{\epsilon(- [W(P, Q, t, \epsilon),] + \partial/\partial\epsilon)\epsilon = 0\} \cdot F(p, q, t) \quad (51)$$

$$= E_{-W} \cdot F.$$

When F is expandable in the parameter ϵ, this becomes

$$F = F_0(P, Q, t) + \epsilon F_1 + \frac{\epsilon^2}{2!}F_2 + \cdots$$

and

$$F(p, q, t) = F_0(p, q, t) + \epsilon\{F_1 + [F_0, W_1]\} + \frac{\epsilon^2}{2!}\{F_2$$

$$- [W_1, (2F_1 - [W_1, F_0])] + [F_0, W_2]\}$$

$$+ \frac{\epsilon^3}{3!}\{F_3 - [W_1, (3F_2 - 3[W_1, F_1] - 2[W_2, F_0]$$

$$+ [W_1, [W_1, F_0]])] - [W_2, (3F_1 - [W_1, F_0])]$$

$$- [W_3, F_0]\}. \quad (52)$$

These formulas give the coordinate transformations just by setting $F = P, Q$. The resonant invariant can also be calculated by taking a suitable function, K_0, of the zero-order constants of motion. Equations (48) and (52) give

$$K_0(P) = K_0(p) + \epsilon[K_0, W_1] + \frac{\epsilon^2}{2}\{[W_1, [W_1, K_0]] + [K_0, W_2]\}$$

$$+ \frac{\epsilon^3}{3!}\{3[W_2, [W_1, K_0]] - [W_1, [W_1, [W_1, K_0]]]$$

$$+ [K_0, W_3 + 2[W_1, W_2]]\} = K_0(p) + \epsilon[K_0, \hat{h}_1]$$

$$+ \frac{\epsilon^2}{2}[\hat{h}_1, [\hat{h}_1, K_0]] + \frac{\epsilon^2}{2}[K_0, (\hat{h}_2 + [\hat{h}_1, h_1]^{\hat{}} + [\hat{\hat{h}}_1, \overline{h}_1])]$$

$$+ \epsilon^3 K_3. \quad (53)$$

It is a straightforward matter to apply the operator algebra to show that the $O(\epsilon^2)$ term in Eq. (53) is identical with the result of the Poisson bracket method, Eq. (40). This expression is in fact more convenient, as will be shown.

The general algorithm of Eqs. (41)—(46) has been implemented on MACSYMA by Char[8] to generate W, H, and the transforms $(P, Q) \to (p, q)$ automatically for simple cases. When the problem allows a new Hamiltonian to be found that is independent of an angle variable, Q_1, then the conjugate momentum P_1 is an adiabatic invariant. In the resonant case it is only necessary to take a suitable function of the momenta whose conjugate angles have been averaged out, and Taylor expand the function to the desired order. This is then the resonant adiabatic invariant.

The formalism appears complicated but the result, Eq. (53) is quite straightforward to evaluate, at least to $O(\epsilon^2)$, even without MACSYMA.

5. A TIME-INDEPENDENT EXAMPLE

The invariant for the example of Smith and Kaufman is given to $O(\epsilon^2)$, by

$$K = K_0(p_z) + \epsilon K_0' \sum_{n=-\infty}^{\infty} \frac{J_n(\sqrt{2p_\phi})}{(p_z - n)} \sin(z - n_\phi)$$
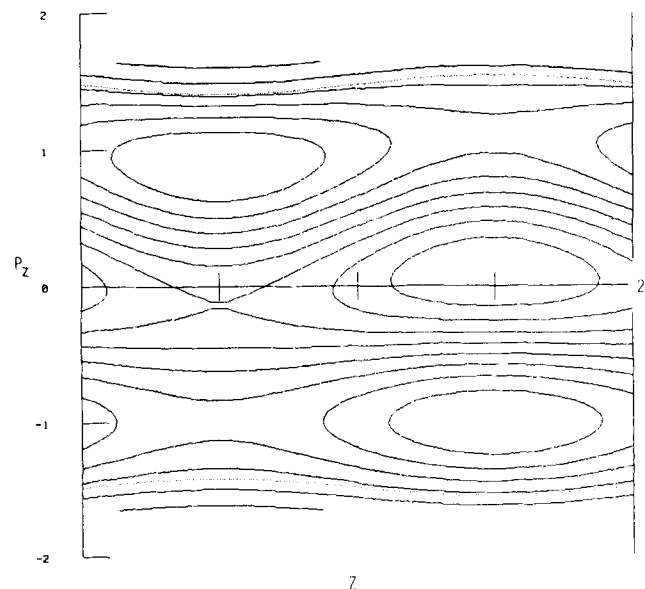

FIG. 1. Surface of section of the resonant invariant to $O(\epsilon)$ at $\epsilon = 0.1$. $H = 1.1$. The primary resonances are well represented in shape and position.

$$+ \epsilon^2 \left[\frac{K_0''}{2} \left(\sum_n \frac{J_n \sin(z - n_\phi)}{p_z - n} \right)^2 - K_0' \sum_n \sum_m \frac{J_m}{2(p_z - n)(p_z - m)} \right.$$

$$\times \left(\frac{J_n'}{p_z - n} + \frac{m J_n}{\rho} \right) \cos(m - n)\phi + \frac{K_0'}{2} \sum_n \sum_m$$

$$\times \frac{J_m}{(2p_z - (m + n))} \left(\frac{J_n}{(p_z - n)^2} + \frac{m(n - m)J_n'}{\rho(p_z - n)(p_z - m)} \right)$$

$$\left. \times \cos(2z - (m + n)\phi) \right], \tag{54}$$

where $K_0' = \partial K_0 / \partial p_z$, $J_n' = \partial J_n / \partial \rho$, $\rho = \sqrt{2p_\phi}$. The exact form of K now depends on K_0 which, in turn, depends on the order to which K is desired. The invariant is shown to $O(\epsilon)$ in Fig. 1 in the plane $\phi = \pi$ with p_ϕ determined from the Hamiltonian:

$$p_\phi = h - \frac{p_z^2}{2} - \epsilon \sin z \tag{55}$$

and $K_0' = \sin \pi p_z$, the choice used by Taylor and Laing.[10] The invariant is shown to $O(\epsilon^2)$ in Fig. 2 with K_0' $= \sin^2 \pi p_z \sin 2\pi p_z$. This choice of K_0' is arrived at by observing that K_2 has poles of order 3 at $p_z = n$ but only poles of order unity at $2p_z = n$, and these are multiplied by K_0'. The general rules for constructing K_0' can be readily deduced from the form of K, Eq. (53). In the Nth order the transform generator, W, introduces the integral $(\hat{\ })$ of h_1^N, $h_2 h_1^{N-1}$, and so on, producing harmonics and beats between all the resonances of h_1, h_2, etc. However, this term always appears multiplied only by first derivatives of K_0. Thus, each new order introduces new harmonics that require a zero of order unity in K_0'. In addition, the new order in ϵ has introduced one more integration $(\hat{\ })$ and one more Poisson bracket. Resonances already dealt with in lower orders now need two further zeros in K_0 for the new order.
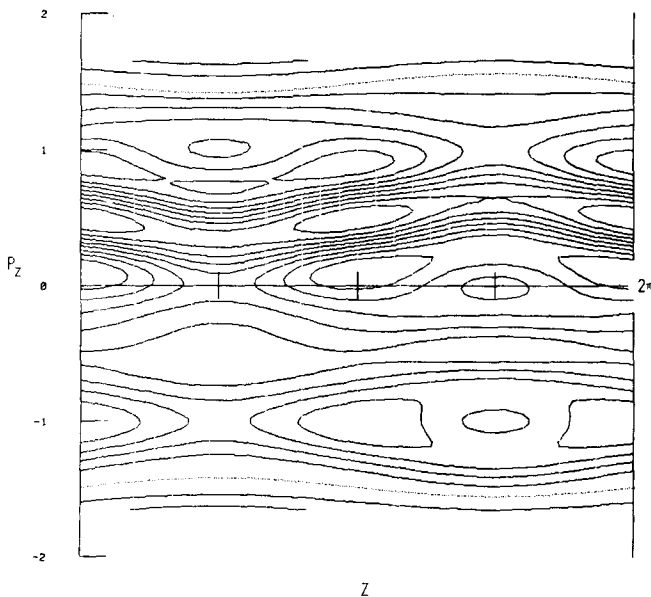

FIG. 2. Super convergent resonant invariant correct to $O(\epsilon^2)$ at $\epsilon = 0.1$. The second harmonics are well represented but the breakup of the primary resonances is quite wrong. These figures should be compared with the orbit computations by Smith and Kaufman.

The Smith and Kaufman case to $O(\epsilon^3)$ would require K_0' $= \sin^3(2\pi p_z) \sin(3\pi p_z)$, giving four zeros at $p_z = n$, three at $2p_z = n$, and one at $3p_z = n$. A curious result of proceeding to higher orders is that K vanishes to all orders except the Nth at each resonance. The Nth term describes the resonances and their harmonics to the Nth order.

The invariant may be used to analyze the position and nature of the fixed points of the motions. Thus, in Fig. 2, one would solve

$$\left. \frac{\partial K}{\partial z} \right|_{\phi = \pi} = 0, \quad \left. \frac{\partial K}{\partial p_z} \right|_{\phi = \pi} = 0 \tag{56}$$

for the fixed points shown. In this example, Eq. (55) near the resonance, $p_z = N$ gives

$$\frac{\partial K}{\partial z} = \epsilon \left(\frac{K_0'}{p_z - N} \right) (-1)^N J_N \cos z + \frac{\epsilon^2}{2} \left(\frac{K_0'}{p_z - N} \right) \frac{J_N^2 \sin 2z}{(p_z - N)}. \tag{57}$$

This locates the fixed points on the lines $z = \pi/2$, $3\pi/2$, and the second derivatives of K identify the elliptic and hyperbolic points. Along the line $p_z = N$, the invariant becomes

$$K \rightarrow \epsilon(-1)^N \frac{K_0'}{(p_z - N)} J_N \sin z - \frac{\epsilon^2 J_N^2}{2} \left[\left(\frac{K_0'}{(p_z - N)^2} \right) \right.$$

$$\left. - \frac{1}{(p_z - N)} \left(\frac{K_0'}{p_z - N} \right) \cos 2z \right]. \tag{58}$$

The change in K between an elliptic point and its separatrix in order ϵ is

$$\Delta K_1 = \epsilon \left| \frac{K_0'}{(p_z - N)} J_N \right|. \tag{59}$$

Expanding K in $p_z = N + \delta_N$ about the elliptic points gives

$$K = K_E + \frac{\delta_N^2}{2} K_0'' + \epsilon \frac{\delta_N^2}{2} K_1'' + \cdots, \tag{60}$$

and so the width of the resonance is $\delta_N^2 K_0'' / 2 = \Delta K_1$, and

$$\delta_N^2 = \frac{2\epsilon}{K_0''} \left| \frac{K_0'}{p_z - N} J_N \right|. \tag{61}$$

If K is only calculated to $O(\epsilon)$, then $K_0' \sim k(p_z - N)$, and so

$$\delta_N = \sqrt{2\epsilon J_N}, \tag{62}$$

independent of the choice of K_0. If K is calculated to $O(\epsilon^2)$, then K_1 will vanish at $p_z = N$, and so ΔK will be given by K_2,

$$\Delta K_2 = \epsilon^2 \frac{J_N^2}{4} \frac{1}{(p_z - N)} \left(\frac{K_0'}{(p_z - N)} \right)'. \tag{63}$$

However, K_0'' also vanishes in Eq. (60) and so the width of the resonance is determined by K_1''. Of course, this finally yields the original expression, Eq. (62), for the width.

The construction of K to order M requires a single additional zero in K_0' for each new harmonic in the $O(\epsilon^M)$ term. The width of these resonances is therefore given by

$$\frac{\delta^2}{2} K_0'' = \epsilon^M \Delta K_n,$$ (64)

i.e., $\delta = O(\epsilon^{M/2})$. In particular, the resonance widths at $2p_z = N$ are

$$\delta_{N/2} = \epsilon \left[\sum_n \left(\frac{J_n J_{N-n}}{(p_z - n)^2} + \frac{(N-n)(2n-N)J_n' J_{N-n}}{\rho(p_z - n)(p_z - N + n)} \right) \right]^{1/2}$$ (65)

An improved criterion for breakdown of the invariant due to overlapping of neighboring resonances is

$$\delta_N + \delta_{N-1} + 2\delta_{N/2} = 1.$$ (66)

The story is not yet complete. The invariant of Eq. (52) can be compared with the intersection of exact particle orbits with the plane $\phi = \pi$ in Fig. 3. The invariant does not describe properly the breakup of the trapped orbits around the primary resonances at $p_z = n$ but does give a good description of the size and position of the next harmonics at $2p_z = n$. Comparison of the invariant, as shown in Figs. 1 and 2 shows a drastic change in topology of the primary resonances. The conclusions are that the resonant invariant to order ϵ^n describes the Nth harmonic resonances quite well, that the untrapped orbits will be adequately described, but that the resonances of orders ϵ to ϵ^{n-1} will be spuriously broken up. In defence of the invariant the value of ϵ used in Figs. 1 and 2 leads to stochastic behavior for about 75% of the true orbits. At small enough values of ϵ, when the orbits are adiabatic, the agreement is much better.

The procedure for choosing seems a little *ad hoc* and it is not immediately clear why K should describe trapped orbits at all. The breakup of the resonance regions needs further development of the Lie transform theory.
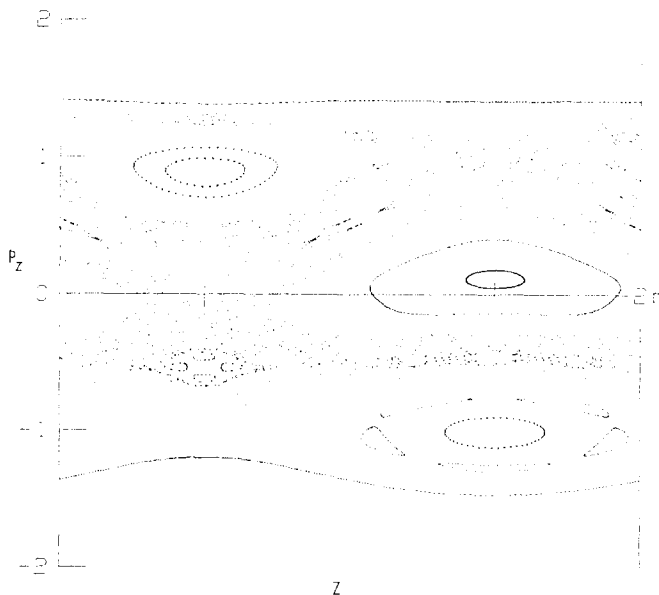
FIG. 3. Surface of section of numerically computed orbits shows (i) primary islands with five internal islands due to drift resonances and (ii) stochastic regions between islands. Similar plots were given in Refs. 5 and 9. Figs. 1, 2, and 3 agree much better at smaller ϵ, before the occurrence of the breakup shown here.

6. PARTIAL TRANSFORMS TO RESONANT REGIONS

A key point in the development of the Lie transform[13,14] is that any family of canonical transformations, dependent on a parameter ϵ, for which the identity transformation is given at $\epsilon = 0$, can always be written in terms of a single scalar function, $W(P, Q, \epsilon)$ as

$$\frac{\partial p}{\partial \epsilon} = [p, W], \quad \frac{\partial q}{\partial \epsilon} = [q, W].$$ (65a)

These equations can only be solved as a power series in ϵ or when W is especially simple. However, any function $W(P, Q)$ generates a canonical transformation that can be written formally as

$$p = \exp(-\epsilon[W, \ ]) \cdot p(P, Q, \epsilon), \quad q = \exp(-\epsilon[W, \ ])q.$$ (66a)

This suggests that W be chosen as only part of the generating function in Eq. (48) so as to average out only chosen terms or harmonics of the perturbation. It turns out to be very simple to describe any resonance or harmonic resonance in detail.

7. ANALYSIS OF RESONANCES

These ideas are best understood in the context of a specific example like that of Smith and Kaufman. The integrations leading to the invariant Eq. (52) were along the zero-order orbits, $\dot{z} = p_z$, $\dot{\phi} = 1$. As the periods in z and ϕ are in general incommensurate, the averaging operator, Eq. (13), eliminates z and ϕ completely. The averaged Hamiltonian for this problem, as given by Eq. (46), is just $H(p_z, p_\phi)$, and represents a surface average of h over the invariant surfaces K, and says nothing about the drift frequencies of trapped orbits. A resonance at $p_z = N$ should therefore be analyzed by first making a coordinate transformation to $p_z = N + p_z^*$ so that the zero-order orbits to be integrated along are given by $\dot{z} = N$, $\dot{\phi} = 1$. The term $k_1 = \epsilon J_N \sin(z - N\phi)$ is then constant along this orbit and is the leading term in the local invariant. This result was shown by McNamara and Whiteman[2] for simply periodic systems or could be shown here by making the transformation to the resonant orbits and choosing $K_0 = h_0$ when

$$K = h_0 + \epsilon[h_0, \hat{h}_1] + O(\epsilon^2) \simeq h_0 + \epsilon(h_1 - \overline{h}_1) = h - \epsilon\overline{h}_1.$$ (67)

The adiabatic invariant differs from the Hamiltonian, which is already a constant of motion, only by the term $\epsilon\overline{h}_1$. This is exactly the term picked up by the resonant invariant Eq. (54) as $p_z \to N$ and it is now apparent why the trapped orbits are adequately described by the generalized invariant.

The resonant breakup of the trapped orbits around the primary resonance at $p_z = N$ is described by first transforming out the other resonances. Choosing the transformation function

$$W_1 = \hat{h}_1^* = + \sum_{n \neq N} \frac{J_n \cos(Z - n\Phi)}{(p_z - n)}$$ (68)

as the definition of a complete canonical transformation, gives the new Hamiltonian

$$H = h_0(P, Q) + \epsilon\{h_1(P, Q) + [h_0, \hat{h}_1^*]\} + \frac{\epsilon^2}{2!}[\hat{h}_1^*, h_1 + \bar{h}_1]$$

$$+ O(\epsilon^3)$$

$$\doteq \frac{p_z^2}{2} + P_\phi + J_N \sin(Z - N\Phi) + \frac{\epsilon^2}{2!} H_2, \qquad (69)$$

where H_2 contains all the interactions of the Nth term with the rest of h_1. The trapped orbits are given by

$$\dot{Z} = P_z \quad \dot{\Phi} = 1 + \epsilon(J_N'/\rho)\sin(Z - N\Phi),$$

$$\dot{P}_z = -\epsilon J_N \cos(Z - N\Phi), \quad \dot{P}_\phi = \epsilon N J_N \cos(Z - N\Phi), \qquad (70)$$

with fixed points at $\cos(Z - N\Phi) = 0$, $P_z = N(1 + \epsilon(J_N'/\rho)\sin(Z - N\Phi))$. It is a trivial matter to find a mixed variable-generating function for the transformation to these points where, in the new coordinates, $q_2 = \bar{q}$,

$$-F(P, q) = \frac{1}{2}(P_\phi - NP_z)(q_2 - \bar{q}) + \frac{1}{2}(P_\phi + NP_z)$$

$$\times (q_1 - q_2)\frac{-N^2}{2}q_1 + N_2 q_2, \qquad (71)$$

giving the linear coordinate transformations

$$z = \frac{N}{2}(q_1 - 2q_2 - \bar{q}), \quad NP_z = N^2 + p_2, \qquad (72)$$

$$\Phi = \frac{1}{2}(q_1 + \bar{q}), \quad P_\phi = 2p_1 - p_2.$$

The Hamiltonian then becomes

$$H = \frac{N^2}{2} + 2p_1 + \frac{p_2^2}{2N^2} - \epsilon J_N(\sqrt{2(2p_1 - p_2)})\cos Nq_2$$

$$+ \frac{\epsilon^2}{4}\sum_n{}'\sum_m{}^* \left[\left(\frac{J_n J_m}{(N-n)^2} + \frac{m J_n' J_m}{\rho(N-n)}\right)(\cos\psi_{nm} + \cos\chi_{nm})\right.$$

$$\left. - \frac{n J_n' J_m'}{\rho(N-n)}(\cos\psi_{nm} - \cos\chi_{nm})\right] + O(\epsilon^3)$$

$$= H^{(1)} + \frac{\epsilon^2}{2!}H_2 + O(\epsilon^3), \qquad (73)$$

where

$$\psi_{nm} = \left(N - \frac{(m+n)}{2}\right)q_1 - 2Nq_2 - \left(N + \frac{m+n}{2}\right)\bar{q},$$

$$\chi_{nm} = \frac{(m-n)}{2}(q_1 + \bar{q}),$$

and resonant denominators in H_2 are given by $p_z = N$ to this order.

An important point to notice is that $H_2(q_2)$ depends only on $2Nq_2$. The resonant Hamiltonian $H^{(1)}$, correct to $O(\epsilon)$, describes a simple pendulum motion in (p_2, q_2) and will generally arise in a discussion of resonances. The following standard analysis closely follows that of Smith.[9] The resonant Hamiltonian $H^{(1)}$ is transformed to action-angle variables (I_2, θ_2) defined by

$$I_2 = \frac{\pi}{2}\oint p_2 dq_2 = \frac{\pi}{2}\oint [2(P_2 + \epsilon J_N \cos Nq_2)]^{1/2} d(Nq_2), \qquad (74)$$

where $P_2 = P_2(I_2, P_1)$.

The dependence of J_N or p_2 is dropped since $p_2 \sim O(\sqrt{\epsilon})$ at the resonance. The integral in Eq. (74) can be

evaluated in terms of complete elliptic integrals E, K and so[15]

$$I_2 = \frac{8}{\pi}\sqrt{\epsilon J_N}\, [E(k) - (1 - k^2)K(k)] \text{ and } \theta_2 = \frac{\pi}{2K}\frac{1}{\sqrt{\epsilon J_N}}$$

$$\times \text{ sn}^{-1}(\xi, k), \qquad (75)$$

where $\kappa^2 = (p_2 + \epsilon J_N)/\epsilon J_N$ and $\sin\xi = 2\kappa \sin(Nq_2/2)$.

The resonant Hamiltonian becomes

$$H^{(1)} = \frac{N^2}{2} + 2P_1 + P_2(I_2, p_1) \qquad (76)$$

but it is now necessary to express H_2 as a Fourier series in θ_2. Fortunately, to lowest order, H_2 is independent of P_2 and only the $\cos\psi_{nm}$ need be expanded,

$$\cos\psi_{nm} = \cos(\Phi_{nm} - 2Nq_2) = \sum_{k=-\infty}^{\infty} u_k \cos(\Phi_{nm} - k\theta_2),$$

and

$$u_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\Phi_{nm} - 2Nq_2)\cos(\Phi_{nm} - k\theta_2)d\theta_2. \qquad (77)$$

The perturbation now contains all possible resonances between the fast cyclotron phase Q_1 and the slow drift phase θ_2 and each term with a factor $\cos([N - (m+n)/2]Q_1 - k\theta_2 - [N + (m+n)/2\bar{q}])$ leads to a resonant denominator

$$d_{kmn} = \left(2 + \frac{\partial p_2}{\partial p_1}\right)\left[N - \left(\frac{m+n}{2}\right)\right] - k\omega_D \text{ and}$$

$$\omega_D = \frac{\partial P_2}{\partial I_2}. \qquad (78)$$

The smallest value of k for resonance is at $k = (\partial P_2/\partial I_2)^{-1}$ that, close to the elliptic fixed point, becomes

$$k = N^{-1}(\epsilon J_N)^{-1/2} = \omega_D^{-1}. \qquad (79)$$

At the parameters used by Smith and Kaufman this value is 4.24, and so the first drift resonance is at the $k = 5$ harmonic. The technique of removing resonances in the adiabatic invariant series can no longer be applied directly since all possible rational ratio harmonics are present in H_2. The only choice of $K_0(I_2)$, which is zero at all these places, is zero! However, as $L = 2(N - (m+n)/2)$ increases, k increases as $L/\epsilon^{1/2}$ and the amplitudes of the terms drop rapidly. The only significant contributions are from $L = \pm 1$ and the perturbation H_2 becomes

$$H_2 \doteq \sum_k \alpha_k \cos(Q_1 - k\theta_2 - (2N - \frac{1}{2})\bar{q}). \qquad (80)$$

An invariant, L, local to the resonant island, can be constructed in the usual way,

$$L = L_0(I_2) + \frac{\epsilon^2}{2!}[L_0, \hat{H}_2], \qquad (81)$$

with $L_0' = \sin(\pi/\omega_D)$. The height of the kth drift resonance is

$$\Delta L = \left| \epsilon^2 \frac{L_0' k\alpha_k}{d_k} \right|, \qquad (82)$$

and so the width is [cf. Eq. (59)]

$$\Delta I_2 = \epsilon \left| \frac{2L_0' k\alpha_k}{L_0'' d_k} \right|^{1/2} = \epsilon \left| 2k\alpha_k \bigg/ \frac{\partial\omega_D}{\partial I_2} \right|^{1/2}. \qquad (83)$$

Note that the shear in the drift surfaces, $\partial \omega_D / \partial I_2$, appears in ωI_2. Since there is an infinite sequence of resonances near the separatrix, it is important to find where they begin to overlap and hence the thickness of the layer, near the separatrix, which may be expected to become stochastic. The width and separation of the harmonics is most easily expressed in terms of the frequencies $\omega = 1/k$ so that the frequency separation between neighboring resonances is

$$\omega_{k+1} - \omega_k = 1/k(k+1). \tag{84}$$

The width of a resonance is $\Delta \omega = \omega I_2 (\partial \omega_D / \partial I_2)$ and

$$\frac{\partial \omega_D}{\partial I_2} = \frac{\partial \omega_D}{\partial \kappa} \frac{\partial \kappa}{\partial P_2} \frac{\partial P_2}{\partial I_2} = \frac{\partial \omega_D}{\partial \kappa} \cdot \frac{1}{2\omega_0^2} \cdot \omega_D. \tag{85}$$

As $K \to 1$ the drift frequency becomes[15]

$$\frac{\pi}{2} \frac{\omega_0}{\omega_D} = \ln\left(\frac{4}{\sqrt{1 - \kappa^2}}\right), \tag{86}$$

and so

$$\frac{\partial \omega_D}{\partial I_2} = \frac{1}{16\pi} \left(\frac{\omega_D}{\omega_0}\right)^3 \exp(\pi \omega_0 / \omega_D). \tag{87}$$

The width becomes equal to the frequency separation when

$$\left(\frac{\Delta \omega}{\omega_k - \omega_{k+1}}\right)^2 = \frac{\epsilon^2 \alpha_k}{8\pi \omega_0^3} (k+1)^2 \exp(\pi k \omega_0) = 1. \tag{88}$$

This is a very rapidly increasing function of k, and so only the lowest order drift resonances should be seen in orbit calculations. However, the frequency ω_D drops so rapidly at the edge of the resonant region that the width of the stochastic layer is rather small compared with the size of the region. At any given ϵ, Eq. (88) determines the edge of the stochastic region. A similar region exists just outside the separatrix, and the total width of this layer should be added to the overlap criterion, Eq. (66), for breakdown of the whole phase space between the primary resonances. An interesting question arises as to the stability of drift surfaces between the primary resonances. The Kolmogorov— Arnold—Moser (KAM) theorem[16] shows that the higher harmonics of the primary resonances do not destroy the drift surfaces at small enough ϵ.

8. THE SUPER-CONVERGENT INVARIANT

The basic technique used by Kolmogorov (see Ref. 16) to show the topological stability of invariant surfaces far from a resonance is easily developed in the Lie transform formalism. Consider the Hamiltonian

$$H = h_0(p) + \epsilon h_1(p, q) + \frac{\epsilon^2}{2!} h_2 + \frac{\epsilon^3}{3!} h_3 \tag{89}$$

in which h_1, h_2 are periodic in each of the angle variables q, with frequencies, $\omega = \partial h_0 / \partial p$, all of the same order. The Lie transform generator, $W_1 = \hat{h}_1$, will average the Hamiltonian to $O(\epsilon)$ and generate an infinite series of correction terms as follows from Eq. (66a),

$$H = h_0(P) + \epsilon \bar{h}_1(P) + \frac{\epsilon^2}{2!}\{h_2(P, Q) + [\hat{h}_1, h_1 + \bar{h}_1]\}$$

$$+ \frac{\epsilon^3}{3!}\{[\hat{h}_1[\hat{h}_1, (2h_1 + \bar{h}_1)]] + 3[\hat{h}_1, h_2] + h_3 (P, Q)\} + \cdots$$

$$= H^{(1)} + \frac{\epsilon^2}{2!} H_2 + \frac{\epsilon^3}{3!} H_3 + \frac{\epsilon^4}{4!} H_4 + \cdots. \tag{90}$$

The average of h_1 is now independent of all the angle variables, Q, and the motion given by $H^{(1)} = h_0 + \epsilon \bar{h}_1$ is again completely soluble, with new frequencies $\Omega = \omega + \epsilon \partial \bar{h}_1 / \partial P$. A second Lie transform can now be performed using $H^{(1)}$ to describe the unperturbed orbits. The operators for these orbits will be written as $\sim$, $\langle \ \rangle$ to distinguish them, and the transform generator is

$$\frac{\epsilon^2}{2!} W_2 = \frac{\epsilon^2}{2} \tilde{H}_2 + \frac{\epsilon^3}{3!} \tilde{H}_3. \tag{91}$$

This transform now depends on ϵ explicitly, and so the full operator E_W must be evaluated in generating the expansion. However, the linear term in E_W merely gives $\langle H_2 \rangle$ and $\langle H_3 \rangle$, while the quadratic and higher terms can only generate corrections of $O(\epsilon^4)$ and above. The averaged Hamiltonian becomes

$$H = H^{(1)} + \frac{\epsilon^2}{2!} \langle H_2 \rangle + \frac{\epsilon^3}{3!} \langle H_3 \rangle + \frac{\epsilon^4}{4!} H_4 + \cdots$$

$$= H^{(3)} + \frac{\epsilon^4}{4!} H_4 + \cdots. \tag{92}$$

Thus, the transform Eq. (91) has dealt with two terms at once, and Hamiltonian Eq. (92) describes the orbits far from any resonance to $O(\epsilon^4)$. The next transform would use $H^{(3)}$ for the unperturbed orbits and a generator

$$W = \sum_{n=4}^{7} \frac{\epsilon^n}{n!} \tilde{H}_n \tag{93}$$

which deals with four terms at once. This super-convergent process is equivalent to Newton's tangent method for finding zeros of a function. The description of nonresonant orbits is improving in accuracy faster than new resonant denominators are being generated. Arnold[17] and Moser[18] showed how to estimate from below the smallness of the resonant denominators that represent a set of frequencies of motion of measure zero. The Kolmogorov—Arnold—Moser theorem states that for almost all frequencies (the irrationals) of the zeroth-order motion of Hamiltonians like Eq. (89), the invariant surfaces are only slightly distorted by a small enough perturbation. This means that the breakdown of an invariant can only arise from the interaction of the resonant orbits, since the intervening phase space is topologically stable. It is also worth remarking that despite this, the elliptic fixed points of the primary resonances are the most stable parts of the phase space, in general, and the last to disappear as the perturbation amplitude is increased. It has been speculated that the planets all lie near such elliptic fixed points of a basic resonance of the solar system.

At this point it is convenient to note the form of the partial transform needed to analyze a resonance of harmonics of the primary frequencies, e.g. $2p_z = 2k + 1$ in the basic problem. For a resonance in H_2 of Eq. (90) it would be simply $W_2 = H_2^*$, where the desired resonance is omitted from the transform generator and thereby not averaged away.

At each step of the super convergent expansion the latest approximation to the averaged Hamiltonian is used for the unperturbed orbits. The ϵ dependence of these orbits is formally ignored, equivalent to solving

a related problem with parameter δ, which is set back to ε after the transformation. In calculating a super convergent adiabatic invariant it is apparent that the inverse transformations will eventually generate the series given by Eq. (53) but with orbits for the ^ and ¯ operations given by the Hamiltonian averaged to the order of the calculation, in this case $H^{(3)}$ of Eq. (92). The choice of K_0 must now be modified to take into account the shift in the position of the resonances. In the basic example a suitable choice is

$$K'_0 = \omega'(p_z)\sin^2\pi\omega\sin2\pi\omega, \tag{94}$$

where

$$\omega(p_z) = \frac{\partial}{\partial p_z}\left(h_0 + \frac{\epsilon^2}{2!}\overline{[\tilde{h}_1, h_1]}\right),$$

and so

$$K_0 = \frac{1}{2\pi}\sin^4\pi\omega,$$

which contains a further correction of $O(\epsilon^2)$ to the description of the invariant.

9. A TIME-DEPENDENT EXAMPLE WITH HIGH-ORDER RESONANCES

Rather simple Hamiltonians often give rather complicated orbits, and it is useful to try and understand the resonance structure without calculating the terms in detail. One such example, given by Smith,[19] is the perturbed pendulum

$$H = \frac{p^2}{2} - \cos q - \epsilon\cos(q - \Omega t). \tag{95}$$

A surface of section at $\epsilon = 0.2$, $\Omega = 0.5$ around the trapped orbits near $p = q = 0$ showed a resonant island chain at $3\omega_b = 5\Omega$, where ω_b is the oscillation frequency of the pendulum. This implies that the term $O(\epsilon^5)$ is large enough to be visible. At the value of ϵ used in the calculations, no islands were seen for $\omega_b \gtrsim 0.5$, as this is too close to the separatrix between trapped and rotating orbits. Fourier expansion of the perturbation in Eq. (95) in terms of the angle ϕ conjugate to the pendulum action J gives an infinite sum

$$\epsilon h_1 = -\epsilon\sum u_n(J)\cos(n\phi - \Omega t). \tag{96}$$

By introducing the momentum P_t conjugate to time, the Hamiltonian Eq. (95) can be cast in the form used in this paper,

$$H = H_0(J) + P_t + \epsilon h_1(J, \phi, t). \tag{97}$$

Following the super convergent expansion method of Sec. 8 yields the following form for H on the first transformation,

$$H = H_0 + P_t + \frac{\epsilon^2}{2!}H_2 + \frac{\epsilon^3}{3!}H_3 + \cdots, \tag{98}$$

where H_3 will have terms of the form $\cos(n\phi - 3\Omega t)$. The next transform generator, Eq. (91), will have resonant denominators in H_3 of the form $(m\omega_b(J) - 3\Omega)$, which will enter the coefficients of cross terms between H_2, H_3 of $O(\epsilon^5)$. In the units of Eq. (95), $\omega_b(0) = 1$, and the resonance most likely to be seen is with $\omega_b = n\,\Omega/m$ closest to 1, the primary resonances in Fig. 4, with a large enough coefficient! The first important resonance
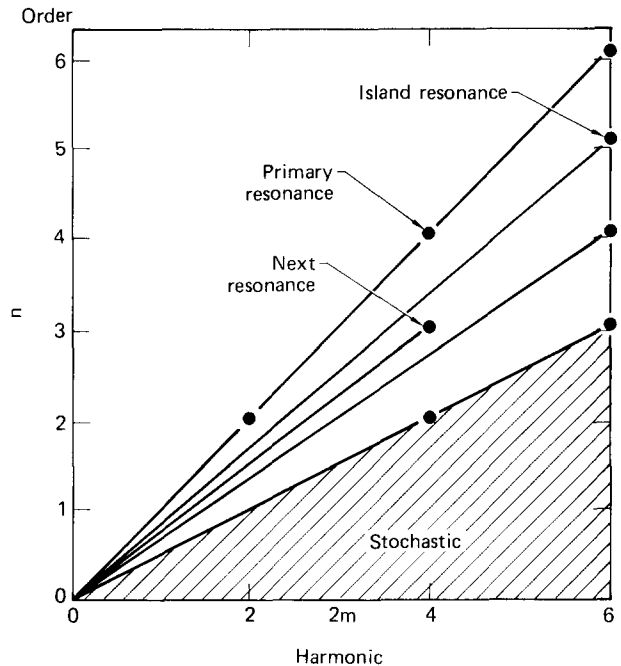

FIG. 4. Sketch relates resonances between harmonics (m) of the pendulum frequency with harmonics of the perturbation in Eq. (95) in various orders (n) of the Lie transform.

in $O(\epsilon^3)$ is at $\omega_b = \frac{3}{4}$, the next resonance in Fig. 4, but is closer to the separatrix than the resonance in $O(\epsilon^5)$ at $\omega_b = \frac{5}{6}$, the island resonance in Fig. 4. This resonance has a coefficient with denominator $2\omega_b - 3\Omega = \frac{1}{6}$, which is numerically smaller than ϵ. This resonance is observed and the $O(\epsilon^3)$ one is not.

A key difference between time-dependent and time-independent problems, illustrated by this example, is that the momentum p_t appears alone in the Hamiltonian. The single resonant perturbation in Eq. (95), which could have been picked out of a series of perturbations, can only generate harmonics with frequency $m\Omega$ in the $O(\epsilon^m)$ term of the Lie series expansions. Karney and Bers[20,21] have treated the problem of a magnetized charged particle in a finite amplitude electric wave traveling exactly perpendicular to the magnetic field. At very short wavelength they find a complicated structure of fourth, fifth, and higher harmonics of the perturbation. An analytic description of this behavior would require a correspondingly high order calculation.

10. SUMMARY

This paper has given explicit forms for adiabatic invariants with resonant denominators for time-dependent and independent examples as well as local invariants near the elliptic fixed points of a particular resonance. The super-convergent algorithm of Kolmogorov is easily expressed as a sequence of Lie transforms and shows that the breakdown of the invariant is due to the resonance structure. The procedure for estimating when the invariant disintegrates is to locate the resonances, calculate their widths, and then find the thickness of the stochastic layer about the separatrix regions of each resonance. This gives sufficient information to evaluate the fraction of phase space that has become

stochastic for any level of nonlinearity. In many practical cases it turns out that the first resonances of interest arise in quite high orders of the perturbation theory. Thus, although particular results are readily understood it is difficult to predict the behavior analytically without a great deal of labor. The Lie transform methods given here have been coded for MACSYMA but are not yet efficient enough to reach high order in the general case.

It is easy to see that the theory given here can be generalized to Hamiltonian systems in which other coordinates vary slowly during the adiabatic motions. Of course, this makes it more complicated to keep track of the various orders induced by the Poisson bracket operations, as described by Whiteman and McNamara[3] to $O(\epsilon^2)$. The general form of Hamiltonian for which these perturbation expansions will work is

$$H = H_0(p_1, p_2, p_3, \epsilon q_3) + \epsilon H_1(p_1, q_1, p_2, q_2, p_3, \epsilon q_3, \epsilon). \quad (99)$$

The lowest order term, H_0, is allowed to vary slowly in q_3, but is independent of q_1, q_2. The perturbation is allowed to be periodic in q_1, q_2, period 2π, but must still vary slowly in q_3. These statements can be generalized to more dimensions and, of course, p_2, q_2 could be energy and time.

The subject of nonadiabatic jumps has not been addressed in this paper, but it should be evident from the KAM theorem that these jumps will arise in the resonant regions only. The island invariant Eq. (81) should be subject to small jumps, nonexpandable in either ϵ or $\sqrt{\epsilon}$, due to the interaction of the cyclotron motion with the drift motion. The eventual stabiltiy of the resonant regions of phase space is determined by a super-adiabatiaty condition on the jumps themselves.

There are many further questions to be answered on the behavior of these invariants. In particular, the self-consistent behavior of a magnetized plasma carrying a finite amplitude wave may be strongly affected by the existence of the invariants discussed here.

ACKNOWLEDGMENTS

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The bound states for the symmetric shifted Coulomb potential

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We investigate the bound states for the symmetric one-dimensional shifted Coulomb potential, $V(x) = -2s(|x| + d)^{-1}$. Explicit approximate expressions for the infinite number of bound-state energies are obtained. For small s, the ground-state energy is $O(s^2 \ln^2 sd)$, whereas the energies of the excited states are $O(s^2)$. We prove that the square roots of the binding energies form approximately a harmonic progression both for the even solutions and for the odd solutions. This is also true for the sequence of *all* solutions when sd is not small. However, when $sd \ll 1$, this sequence shows an interesting odd–even staggering phenomenon.

1. INTRODUCTION

In this paper we shall deduce explicit approximate expressions for the binding energies for the symmetric one-dimensional shifted Coulomb potential,

$$V(x) = -2s(|x| + d)^{-1}, \quad s > 0, \quad -\infty < x < \infty, \quad (1.1)$$

where d is some positive parameter. For $x > 0$ the odd boundstate wavefunctions for this symmetric potential are, if we write r for x, proportional to the $l = 0$ radial bound-state wavefunctions for the three-dimensional shifted Coulomb potential,

$$V(r) = -2s(r+d)^{-1}. \quad (1.2)$$

We take units such that $\hbar = 2m = 1$. Moreover, for the treatment of the Coulomb potential it is convenient to use Sommerfeld's parameter $\gamma = -s/k$, where k is the square root of the energy E.

In this investigation an important role will be played by the Jost solution.[1,2] The pure Coulomb Jost solution for $l = 0$ is given by

$$f_C(k,r) = e^{ikr + \pi\gamma/2} U(i\gamma, 0, -2ikr), \quad (1.3)$$

where U is an irregular solution of the confluent hypergeometric differential equation.[3] The asymptotic behavior of f_C is given by

$$\lim_{r \to \infty} f_C(k,r) \exp[-ikr + i\gamma \ln(2kr)] = 1. \quad (1.4)$$

It easily follows that the function

$$f(k,r) \equiv e^{-ikd} f_C(k,r+d)$$
$$= e^{ikr + \pi\gamma/2} U[i\gamma, 0, -2ik(r+d)] \quad (1.5)$$

is a solution of the Schrödinger equation with the shifted potential (1.2). Since its asymptotic behavior is the same as that of $f_C(k,r)$, we may call $f(k,r)$ the Jost solution for the shifted Coulomb potential (1.2).

We consider negative energies, so we put $k = i\kappa$ with $\kappa > 0$, the corresponding energies being $E = -\kappa^2$. Then the bound-state wavefunctions we are looking for follow from $f(i\kappa,r)$ by imposing the appropriate boundary conditions at $r = 0$. The odd wavefunctions satisfy

$$\lim_{r \to 0} f(i\kappa,r) = 0, \quad (1.6)$$

whereas for the even wavefunctions we have

$$\lim_{r \to 0} \frac{d}{dr} f(i\kappa,r) = 0. \quad (1.7)$$

From Eq. (1.6) we shall deduce a discrete set $\kappa = \kappa_n^{(-)}$, $n = 1, 2, \ldots$, for which the equation holds. The $\kappa_n^{(+)}$ will be called the *odd* solutions. For $d = 0$ we get the well-known pure Coulomb solutions $\kappa_n = s/n$. The values $\kappa = \kappa_n^{(+)}$, $n = 0, 1, 2, \ldots$, for which Eq. (1.7) holds will be called the *even* solutions. The even ground-state solution, $\kappa_0^{(+)}$, plays a special role.

The way in which we are going to derive approximate expressions for $\kappa_n^{(-)}$ and $\kappa_n^{(+)}$ depends critically on the magnitude of sd. Therefore, in Sec. 2 we restrict ourselves to the case $sd \ll 1$. Mehta and Patil[4] have given an expression for $\kappa_n^{(-)}$ in the case of small sd. Our expression [Eq. (2.8)] is in agreement with theirs [Ref. 4, Eq. (27)]. We can understand why the *product* sd plays an important role by transforming the Schrödinger equation according to $u(r) = v(y)$, $y = r(2s/d)^{1/2}$, which yields

$$v'' + [1 + y(2sd)^{-1/2}]^{-1} v = v\kappa^2 d/(2s). \quad (1.8)$$

This equation has again the form of a Schrödinger equation, with a potential which depends only on sd. Note also that the first-order perturbation with respect to the pure Coulomb potential $(d = 0)$ depends on sd,

$$V(r) = \frac{-2s}{r+d} = \frac{-2s}{r} + \frac{2sd}{r^2} - \frac{2sd^2}{r^2(r+d)}. \quad (1.9)$$

In Sec. 3 we consider the case when sd is not small, of the order 1 or bigger. Here we discuss some simple explicit expressions for the lowest odd and even states. We prove that

$$\lim_{sd \to \infty} \kappa (2s/d)^{-1/2} = 1 \quad (1.10)$$

for all binding energies. Note that Eq. (1.8) implies $\kappa^2 \leqslant 2s/d$.

We point out that for intermediate values of $sd < 1$, approximate values for the binding energies can be obtained by means of interpolation. In Sec. 4 some interesting special points will be considered. In particular we discuss an interesting phenomenon which occurs in the complete sequence $\{\kappa_n^{(-)}, \kappa_n^{(+)}\}$. It turns out that this sequence shows a very curious odd-even staggering for small values of sd. Sec. 5 contains a short summary.

2. THE CASE $sd \ll 1$

In this section we shall derive approximate expressions for the odd solutions $\kappa_n^{(-)}$ and for the even solutions $\kappa_n^{(+)}$, assuming throughout that $sd \ll 1$.

The odd solutions follow from Eqs. (1.5) and (1.6), i.e., we have to find those values of $\kappa > 0$ for which the equation

$$U(-s/\kappa, 0, 2\kappa d) = 0 \tag{2.1}$$

holds. First we shall prove that this equation has no solution $\kappa > s$. To this end we use the integral representation
$\Gamma(a+1)U(a,0,z)$

$$= \int_0^\infty [a+1+z(1+t)] t^a (1+t)^{-a-2} e^{-zt} dt, \; \mathrm{Re}\, z > 0,$$

$\mathrm{Re}\, a > -1.$ $\tag{2.2}$

Since for $a = -s/\kappa > -1$ the integrand is positive for all t, it follows that Eq. (2.1) has no solution in this case [cf. also Eq. (1.8)]. So we may restrict ourselves to $\kappa \leqslant s$, i.e., $\kappa d \leqslant sd$, so $\kappa d \ll 1$ since $sd \ll 1$.

In order to solve Eq. (2.1), we use

$$U(a,c,z) = z^{1-c} U(a+1-c, 2-c, z), \tag{2.3}$$

and Eq. (13) of Ref. 5 (Vol. 1, p. 261), and obtain
$\Gamma(a)U(a,0,z) = a^{-1} + z \ln z \; {}_1F_1(a+1;2;z)$

$$+ \sum_{n=0}^\infty [\psi(a+n+1) - \psi(n+1) - \psi(n+2)]$$

$$\times z^{n+1}(a+1)_n / [n!(n+1)!]. \tag{2.4a}$$

Here ψ is the logarithmic derivative of the gamma function, $\psi(z) = \Gamma'(z)/\Gamma(z)$. For small z we have

$$\Gamma(a)U(a,0,z) \simeq a^{-1} + z \ln z + z\psi(a+1), \; z \to 0. \tag{2.4b}$$

We use the fact that the product of a and z, which is here $-2sd$, is close to zero. By combining Eqs. (2.1), (2.4a), and (2.4b) we see that we have to solve the equation

$$(2sd)^{-1} \simeq \ln(2\kappa d) + \psi(1 - s/\kappa). \tag{2.5}$$

This equation can only have a solution when $1 - s/\kappa$ is near one of the poles of ψ. By using

$$\psi(z) \simeq -\frac{1}{z+n} + \psi(n+1), \; \text{if}\, z \simeq -n, n = 0,1,2,\cdots \tag{2.6}$$

and

$$\psi(n+1) \simeq \ln n, \; n \to \infty,$$

we obtain from Eq. (2.5),

$$(2sd)^{-1} \simeq \ln(2n\kappa d) - (n - s/\kappa)^{-1}.$$

Therefore, the approximate odd solutions $\kappa_n^{(-)}$ follow from

$$s/\kappa_n^{(-)} \simeq n + 2sd \; [1 + 2sd \; \ln(2sd)], \tag{2.7}$$

and are explicitly given by

$$\kappa_n^{(-)} \simeq sn^{-1} - 2s^2 dn^{-2}[1 + 2sd \; \ln(2sd)]. \tag{2.8}$$

This expression is in agreement with Eq. (2.7) of Mehta and Patil.[4] We note that the logarithmic term in Eqs. (2.7) and (2.8) may be neglected since $\lim_{z\to 0} z \ln z = 0$, so we have

$$s/\kappa_n^{(-)} \simeq n + 2sd, \; n = 1,2,\cdots \tag{2.9}$$

and

$$\kappa_n^{(-)} \simeq s/n - 2s^2 d /n^2, \; n = 1,2,\cdots. \tag{2.10}$$

It is interesting to note that the bound-state energies which we have deduced here, can also be obtained by applying first-order perturbation theory. On the one hand, we have from Eq. (2.10)

$$E_n = -\kappa_n^2 \simeq -s^2/n^2 + 4s^3 d /n^3. \tag{2.11}$$

On the other hand, we can consider $2sd /r^2$ as a perturbation of the pure Coulomb potential, according to Eq. (1.9). The first-order perturbation of the energy is then

$$\Delta E_n = 2sd \langle r^{-2} \rangle_n = 2sd \cdot 2s^2/n^3,$$

and the total energy, $E_n = -s^2/n^2 + \Delta E_n$, is in agreement with Eq. (2.11).

Now we turn to the *even* solutions $\kappa_n^{(+)}$, which have to be deduced from Eq. (1.7). Let us first rewrite this equation. By using Eq. (1.5) and well-known properties of the function U we obtain

$$\frac{d}{dr} f(k,r) = -ike^{ikr + \pi\gamma/2} \{ U[i\gamma, 0, -2ik(r+d)]$$

$$-2U[i\gamma, 1, -2ik(r+d)] \}. \tag{2.12}$$

Therefore, we have to find the solutions $\kappa = \kappa_n^{(+)}$ of the equation

$$U(-s/\kappa, 0, 2\kappa d) = 2U(-s/\kappa, 1, 2\kappa d). \tag{2.13}$$

First we shall show that this equation has no solution when κd is not small. To this end, we suppose that $\kappa d \gg sd$ (recall that $sd \ll 1$), so $\kappa \gg s$. We now use the integral representation
$\Gamma(a+1)U(a,1,z)$

$$= \int_0^\infty [a + z(1+t)] t^a (1+t)^{-a-1} e^{-zt} dt,$$

$\mathrm{Re}\, z > 0, \; \mathrm{Re}\, a > -1,$ $\tag{2.14}$

together with Eq. (2.2). It follows that we must have $1 - 2a\ln z \to 0$ when $a \to 0$ for the solution of Eq. (2.13), i.e., $z \equiv 2\kappa d$ must be small since $a \equiv -s/\kappa \simeq 0$. Therefore, we may restrict ourselves to the case that κd is small [cf. also Eq. (1.8)].

In order to solve Eq. (2.13), we use Eq. (2.4) and the expansion (Ref. 5, Vol. 1, p. 261)
$\Gamma(a)U(a,1,z) = -\ln z \; {}_1F_1(a;1;z)$

$$- \sum_{n=0}^\infty [\psi(a+n) - 2\psi(1+n)](a)_n z^n/(n!)^2, \tag{2.15a}$$

which becomes for small z,

$$\Gamma(a)U(a,1,z) \simeq -\ln z - \psi(a). \tag{2.15b}$$

By applying Eqs. (2.4) and (2.15) to Eq. (2.13), and using

$$\psi(a) = \psi(1+a) - a^{-1}, \tag{2.16}$$

and the fact that κd is small, we find that our equation to be solved turns out to be

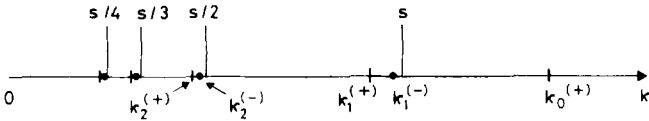$$\kappa/(2s) + \ln(2\kappa d) + \psi(1 - s/\kappa) \simeq 0. \tag{2.17}$$

FIG. 1. The odd solutions $\kappa_n^{(-)}$ (dots) and the even solutions $\kappa_n^{(+)}$ (small vertical lines) for the shifted Coulomb potential in the case $sd \ll 1$, as compared with the pure Coulomb solutions $\kappa_n = s/n$.

It is useful to distinguish two cases, either s/κ is small (e.g., $s/\kappa < \frac{1}{2}$), or not.

(i) In the first case, when s/κ is small, we may neglect $\psi(1 - s/\kappa)$ and Eq. (2.17) may be rewritten as

$$\kappa d \simeq -2sd \ln(2\kappa d). \tag{2.18a}$$

Put for convenience $2\kappa d \equiv z$ and $4sd \equiv \lambda$. Now let $z_0 = z_0(\lambda)$ be the solution of the equation

$$z = -\lambda \ln z. \tag{2.18b}$$

For $0 < \lambda < e^{-1}$, this solution z_0 is easily calculated numerically by means of iteration. Graphically we can prove that

$$f(\lambda) \equiv \frac{\lambda \,|\ln \lambda| - z_0(\lambda)}{\lambda \,\ln|\ln \lambda|} \tag{2.18c}$$

is a decreasing function of λ, even for $0 < \lambda < 1$. We have $f(0) = 1$, $f(e^{-1}) = \frac{1}{2}$, and $f(1) = 0$. It follows that the approximate solution of Eq. (2.18a) is

$$\kappa_0^{(+)} \simeq -2s \,\ln(4sd). \tag{2.19}$$

This is the ground-state solution, which is for this reason denoted by $\kappa_0^{(+)}$. We note that indeed $s/\kappa_0^{(+)}$ is small because $sd \ll 1$, so that our above condition is satisfied, and therefore this $\kappa_0^{(+)}$ is indeed the correct solution.

If we now take d fixed, and let s go to zero, we have

$$\lim_{s \to 0} \kappa_0^{(+)}/(s \ln sd) = -2. \tag{2.20}$$

On the other hand we can fix s and let d go to zero. In this case we have

$$\lim_{d \to 0} \kappa_0^{(+)}/\ln sd = -2s,$$

and therefore,

$$\lim_{d \to 0} \kappa_0^{(+)} = +\infty. \tag{2.21}$$

So the binding energy of the ground state for the one-dimensional shifted Coulomb potential becomes infinite for $d \to 0$. This is in agreement with the well-known fact that the Hamiltonian with a pure Coulomb potential is not bounded below in the one-dimensional case.

(ii) In the second case, when s/κ is not small, we may neglect the term $\kappa/(2s)$ in Eq. (2.17). Since $-\ln(2\kappa d)$ is large, it follows that $1 - s/\kappa$ must be near one of the poles of the ψ function. We use Eq. (2.6) as before and furthermore we have

$$\ln(2\kappa d) + \psi(n) \simeq \ln(2n\kappa d) \simeq \ln(2sd),$$

since $n\kappa$ approximately equals s. In this way we obtain from Eq. (2.17),

$$(n - s/\kappa)^{-1} \simeq \ln(2sd),$$

and therefore

$$s/\kappa_n^{(+)} \simeq n - [\ln(2sd)]^{-1}, \; n = 1,2..., \tag{2.22}$$

or

$$\kappa_n^{(+)} \simeq \frac{s}{n} + \frac{s}{n^2\ln(2sd)}, \; n = 1,2,\ldots . \tag{2.23}$$

It is instructive to compare the expressions for the even solutions with those for the odd solutions, see Fig. 1. From Eqs. (2.10) and (2.23) one easily verifies that

$$\kappa_0^{(+)} \gg \kappa_1^{(-)} > \kappa_1^{(+)} > \kappa_2^{(-)} > \kappa_2^{(+)} > \cdots. \tag{2.24}$$

The solutions $\kappa_n^{(\pm)}$ are somewhat smaller than the corresponding pure Coulomb solutions $\kappa_n = s/n$,

$$\kappa_n^{(+)} < \kappa_n^{(-)} < s/n, \; n = 1,2,\cdots. \tag{2.25}$$

This is clear since our potential is everywhere less attractive than the pure Coulomb potential. The phenomenon that both $\kappa_n^{(+)}$ and $\kappa_n^{(-)}$ are close to s/n for all $n = 1,2,\cdots$ may be called "odd-even staggering," see Fig. 1.

Finally we note that our Eqs. (2.10), (2.20), and (2.23) show that the conjecture of Blankenbecler *et al.* (Ref. 6, p. 70), at least for this potential which satisfies $V(r) = O(r^{-1})$, $r \to \infty$, is correct. Indeed, for the ground-state energy we have

$$E_0 = O(s^2\ln^2 sd), \; s \to 0, \tag{2.26}$$

and for the excited states,

$$E_n = O(s^2), \; s \to 0. \tag{2.27}$$

3. THE CASE WHEN *sd* IS NOT SMALL

When sd is not small, the series expansions of the functions U [Eqs. (2.4) and (2.15)] are not very useful. In this case a different approach is more appropriate. We fix s/κ at some integer value,

$$s/\kappa = n, \; n = 1,2,\cdots, \tag{3.1}$$

and we look for those values of d for which the bound states occur. Once these have been found, we shall deduce an approximate expression for $\kappa_n^{(\pm)}$ by means of interpolation. We shall restrict ourselves to the case that $sd \simeq \frac{1}{2}$ or bigger.

We use the equality

$$U(-n,\nu + 1,x) = (-)^n n! L_n^{(\nu)}(x)$$

$$= (-)^n (\nu + 1)_n \,_1F_1(-n;\nu + 1;x), \tag{3.2}$$

where $L_n^{(\nu)}$ is the Laguerre polynomial ($L_n^{(0)} \equiv L_n$). For the odd solutions the required values of d are then obtained from [cf. Eq. (2.1)]

$$L_n^{(-1)}(2sd/n) = 0, \tag{3.3}$$

and the even solutions from [cf. Eq. (2.13)]

$$L_n^{(-1)}(2sd/n) = 2L_n(2sd/n). \tag{3.4}$$

By using the equality

$$nL_n^{(-1)}(x) = -xL_{n-1}^{(1)}(x),$$

the equation for the odd solutions is rewritten as

$$L_{n-1}^{(1)}(2sd/n)=0,\qquad(3.5)$$

and the equation for the even solutions as

$$L_n(2sd/n)=-sdn^{-2}L_{n-1}^{(1)}(2sd/n),\qquad(3.6)$$

or

$$L_n(2sd/n)+L_{n-1}(2sd/n)=0.\qquad(3.7)$$

Let us first restrict ourselves to large n ($n\gg sd$). In this case the zeros of the Laguerre polynomials are closely related to the zeros of the Bessel function J_ν, see Szegö (Ref. 7, pp. 127–129): Let $x_{nm}(\nu)$, $m=1,2,\cdots n$, be the zeros of $L_n^{(\nu)}(x)$ in increasing order ($\nu>-1$), and let $j_{\nu,m}$, $m=1,2\cdots$, be the positive zeros of $J_\nu(z)$ in increasing order. Then we have

$$x_{nm}(\nu)>(\tfrac{1}{2}j_{\nu,m})^2/(n+(\nu+1)/2),\ m=1,2,...,n,\quad(3.8)$$

and

$$\lim_{n\to\infty}nx_{nm}(\nu)=(\tfrac{1}{2}j_{\nu,m})^2,\ m=1,2,\cdots.\qquad(3.9)$$

We deduce from Eqs. (3.5) and (3.9) that the approximate odd solutions are, for large n, given by

$$2\kappa dn=2sd\simeq(\tfrac{1}{2}j_{1,m})^2,\ m=1,2,\cdots.\qquad(3.10)$$

In order to find the even solutions, we consider Eq. (3.6) and use the asymptotic behavior of the Laguerre polynomials, see Szegö (Ref. 7, p. 198). We conclude that, for sufficiently large n, Eq. (3.6) may be replaced by

$$L_n(2sd/n)\simeq0.\qquad(3.11)$$

By combining this with Eq. (3.9), we find that the approximate even solutions are given by

$$2\kappa dn=2sd\simeq(\tfrac{1}{2}j_{0,m})^2,\ m=1,2,\cdots.\qquad(3.12)$$

Finally we obtain the approximate solutions for general values of sd ($>\tfrac{1}{2}$) by means of interpolation. From Eq. (3.10) we deduce the odd solutions,

$$\frac{s}{\kappa_n^{(-)}}\simeq n+m+\frac{(8sd)^{1/2}-j_{1,m}}{j_{1,m+1}-j_{1,m}},\qquad(3.13)$$

where the integer m is defined by the inequality

$$j_{1,m}\leqslant(8sd)^{1/2}<j_{1,m+1}.\qquad(3.14)$$

In case $(8sd)^{1/2}$ is smaller (but not much) than $j_{1,1}=3.83\cdots$, we have

$$\frac{s}{\kappa_n^{(-)}}\simeq n+(8sd)^{1/2}/j_{1,1}.\qquad(3.15)$$

Similarly, for the even solutions we deduce from Eq. (3.12),

$$\frac{s}{\kappa_n^{(+)}}\simeq n+m+\frac{(8sd)^{1/2}-j_{0,m}}{j_{0,m+1}-j_{0,m}},\qquad(3.16)$$

where m is defined by

$$j_{0,m}\leqslant(8sd)^{1/2}<j_{,m+1}.\qquad(3.17)$$

When $(8sd)^{1/2}$ is smaller (but not much) than $j_{0,1}=2.40\cdots$, we have

$$s/\kappa_n^{(+)}\simeq n+(8sd)^{1/2}/j_{0,1}.\qquad(3.18)$$

It is interesting to compare Eqs. (3.15) and (3.18) with the corresponding expressions, valid for small sd, i.e., Eqs. (2.9) and (2.22):

$$s/\kappa_n^{(-)}\simeq n+2sd,\ n=1,2,...,\quad sd\ll1$$

and

$$s/\kappa_n^{(+)}\simeq n+(-\ln2sd)^{-1},\ n=1,2,...,\quad sd\ll1.$$

When m is large, we may use the asymptotic expressions for the zeros of the Bessel functions J_1 and J_0,

$$j_{1,m}\simeq\pi(m+\tfrac{1}{4}),\qquad(3.19)$$

$$j_{0,m}\simeq\pi(m-\tfrac{1}{4}).\qquad(3.20)$$

From published tables [e.g., Abramowitz (Ref. 8, p. 409)] one can check that even for $m=1$ these approximate values are already within a few percent of the exact values. By inserting (3.19) into (3.13) and (3.20) into (3.16) we obtain

$$s/\kappa_n^{(-)}\simeq n-\tfrac{1}{4}+(8sd)^{1/2}/\pi\qquad(3.21)$$

and

$$s/\kappa_n^{(+)}\simeq n+\tfrac{1}{4}+(8sd)^{1/2}/\pi.\qquad(3.22)$$

Finally we shall consider some special cases. For $n=1$ (so $\kappa=s$), Eq. (3.5) obviously has no solution, and Eq. (3.6) or (3.7) has the unique solution $sd=1$. We can easily interpret this solution by looking back at Fig. 1, Sec. 2, where sd is assumed to be small. When we let d increase, our potential $V(r)=-2s(r+d)^{-1}$ becomes less attractive. Therefore, the binding energies will decrease, i.e., the κ_n's will decrease. In Fig. 1 we see that all κ_n's are smaller than s with the exception of $\kappa_0^{(+)}$. So it follows that there is no odd solution satisfying $\kappa=s$, and also that the even solution $\kappa=s$ must be the ground-state solution $\kappa_0^{(+)}$.

So we have proved that the even ground-state solution is exactly given by

$$\kappa_0^{(+)}=s,\ sd=1.\qquad(3.23)$$

For the corresponding bound-state wavefunction we find

$$u_0^{(+)}(r)=\text{const }(r+d)\exp(-r/d),$$

$$\kappa_0^{(+)}=s=\frac{1}{d}.\qquad(3.24)$$

For comparison we give the approximate expression of Eq. (3.22) in the case $sd=1$, $\kappa_{0\,\text{appr}}^{(+)}\simeq s/1.15$. By taking $n=2$ in Eq. (3.5) we find in a similar fashion,

$$\kappa_1^{(-)}=s/2,\ sd=2.\qquad(3.25)$$

The corresponding odd wavefunction is given by

$$u_1^{(-)}(r)=\text{const}r(r+d)e^{-r/d},$$

$$\kappa_1^{(-)}=s/2=1/d.\qquad(3.26)$$

In this case we obtain from Eq. (3.21), $\kappa_{1\,\text{appr}}^{(-)}\simeq s/2.02$.

We can make some additional comparisons by taking $n=2$ in Eq. (3.7) and $n=3$ in Eq. (3.5). For $sd=3+5^{1/2}\simeq5.24$ we have $\kappa_0^{(+)}=s/2$, to be compared with the

approximate value $\kappa_{0\,\text{appr}}^{(+)} \simeq s/2.31$. When $sd = 3 - 5^{1/2} \simeq 0.76$ we have $\kappa_1^{(+)} = s/2$ and $\kappa_{1\,\text{appr}}^{(+)} \simeq s/2.04$. Finally, when $sd = 3(3 + 3^{1/2})/2 \simeq 7.10$ we get $\kappa_1^{(-)} = s/3$, $\kappa_{1\,\text{appr}}^{(-)} \simeq s/3.15$, and for $sd = 3(3 - 3^{1/2})/2 \simeq 1.90$ we find $\kappa_2^{(-)} = s/3$, $\kappa_{2\,\text{appr}}^{(-)} \simeq s/2.99$. We conclude from these comparisons that the approximate expressions (3.21) and (3.22) are already quite accurate as soon as $\frac{1}{2} < sd < n$.

Now we are going to study the limit of $\kappa_n^{(-)}$ for $sd \to \infty$. In view of Eq. (3.5) we need an expression for the largest zeros of $L_{n-1}^{(1)}(2sd/n)$. Szegö (Ref. 7, p. 132) gives

$$y_{nm} = 4n + O(n^{1/3}), \quad n \to \infty, \quad m \text{ fixed}, \tag{3.27}$$

where y_{nm} ($m = 1,2,\cdots$) are the zeros in *decreasing* order of $L_n^{(\nu)}(y)$. We deduce from Eq. (3.27) the remarkable fact that

$$\lim_{sd \to \infty} (2s/d)^{-1/2} \kappa_n^{(-)} = 1.$$

Exactly the same limit holds for $\kappa_n^{(+)}$, because on the one hand we have $\kappa_n^{(-)} < \kappa_{n-1}^{(+)}$, and on the other hand,

$$\kappa_0^{(+)} \leqslant (2s/d)^{1/2}, \tag{3.28}$$

which follows from Eq. (1.8). Therefore,

$$\lim_{sd \to \infty} (2s/d)^{-1/2} \kappa_n^{(\pm)} = 1, \tag{3.29}$$

for any fixed n. From this equation we obtain a rough estimate of the accuracy of our approximate expressions (3.21) and (3.22) in the case when $sd \gg n$. Clearly we have

$$\lim_{sd \to \infty} (2s/d)^{-1/2} \kappa_{n\,\text{appr}}^{(\pm)} = \pi/4, \tag{3.30}$$

for any fixed n. So even in this case the approximate expressions are not too bad.

4. DISCUSSION

In this section we shall discuss some interesting points concerning our results of Secs. 2 and 3.

(i) In the first place we note that the odd and even solutions are interlacing, i.e.,

$$\kappa_0^{(+)} > \kappa_1^{(-)} > \kappa_1^{(+)} > \cdots > \kappa_n^{(-)}$$
$$> \kappa_n^{(+)} > \kappa_{n+1}^{(-)} > \kappa_{n+1}^{(+)} > \cdots. \tag{4.1}$$

This is a general property, which can be proved for any (sufficiently regular) symmetric potential. Both the sequences $\{\kappa_n^{(+)}\}$ and $\{\kappa_n^{(-)}\}$ are for large n approximately a harmonic progression, i.e., a sequence of terms whose reciprocals form an arithmetic progression. The common difference of this related arithmetic progression is s^{-1}, which follows from Eqs. (2.9), (2.22), (3.21), and (3.22). One might expect that also the sequence

$$\kappa_n^{(-)}, \quad \kappa_n^{(+)}, \quad \kappa_{n+1}^{(-)}, \quad \kappa_{n+1}^{(+)}, \cdots \tag{4.2}$$

will approximate a harmonic progression for large n. However, this is in general not the case. Only when sd is not small, we have from Eqs. (3.21) and (3.22),

$$s/\kappa_{n+1}^{(-)} - s/\kappa_n^{(+)} \simeq \tfrac{1}{2} \tag{4.3}$$

and

$$s/\kappa_n^{(+)} - s/\kappa_n^{(-)} \simeq \tfrac{1}{2}. \tag{4.4}$$

So we see that the common difference of the arithmetic progression related to (4.2) is $\frac{1}{2}s^{-1}$ in this case. This can be shown more explicitly by putting

$$\kappa_n^{(-)} \simeq s/(n+a),$$

$$\kappa_n^{(+)} \simeq s/(n+\tfrac{1}{2}+a),$$

where a is a constant, irrelevant for the present discussion. However, when sd is very small, both $\kappa_n^{(-)}$ and $\kappa_n^{(+)}$ are very close to s/n for all $n = 1,2,\cdots$, according to Eqs. (2.9) and (2.22). So in this case the sequence (4.2) is far from being harmonic.

In Ref. 9 we have found a similar phenomenon for the bound states for the potential $V(x) = c(|x| + d)^{-2}$. In that case, when $c < -\frac{1}{4}$, both $\{\kappa_n^{(+)}\}$ and $\{\kappa_n^{(-)}\}$ are approximately *geometric* progressions, but *only* when the potential strength c is sufficiently negative the sequence (4.2) is approximately geometric, too. Therefore, we conjecture that the so-called "odd-even staggering" is a generally occurring phenomenon.

(ii) We compare the odd bound-state solutions given by Eq. (3.21), with those for the potential

$$V(r) = -2s/r + (2sd)/r^2. \tag{4.5}$$

It is well known that, in the case $2sd = l(l+1)$, $l = 0,1,2,\cdots$, the odd bound-state solutions are given by

$$\kappa = s/m, \quad m = l+1, l+2, \cdots.$$

Therefore, we have approximately for large sd,

$$s/\kappa_n \simeq n - \tfrac{1}{2} + (2sd)^{1/2}. \tag{4.6}$$

This expression is very similar to the one of Eq. (3.21),

$$s/\kappa_n^{(-)} \simeq n - \tfrac{1}{4} + \frac{2}{\pi}(2sd)^{1/2}. \tag{4.7}$$

The main difference is the appearance of the factor $2/\pi$ which is smaller than 1. We can understand this by noting that

$$-\frac{2s}{r+d} = -\frac{2s}{r} + \frac{2sd}{r^2} - \frac{2sd^2}{r^2(r+d)}. \tag{4.8}$$

Since the last term on the right-hand side is negative, the potential $-2s/(r+d)$ is more attractive than the potential (4.5), so the κ's corresponding to (4.8) will be bigger than the κ's for the potential (4.5). Indeed we have

$$n - \tfrac{1}{4} + (2/\pi)(2sd)^{1/2} < n - \tfrac{1}{2} + (2sd)^{1/2}.$$

(iii) The interlacing property (4.1) of the κ_n's does hold for the approximate expressions of Eqs. (3.13) and (3.16) (i.e., if we take the right-hand sides). We can prove this by using a theorem concerning the zeros of the Bessel functions J_ν. First we give the well-known interlacing property of these zeros (Ref. 5, Vol. 2, p. 58),

$$0 < j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < l_{\nu+1,2} < j_{\nu,3} \cdots, \quad \nu > -1. \tag{4.9}$$

Now let sd be such that for some m,

$$j_{1,m} < j_{0,m+1} \leqslant (8sd)^{1/2} < j_{1,m+1}.$$

Then the right-hand side of Eq. (3.13) is smaller than $n+m+1$, while the right-hand side of Eq. (3.16) is at least $n+m+1$. So in this case Eq. (4.1) is satisfied. The only remaining possibility for sd is, that for some m the inequality

$$j_{0,m} < j_{1,m} \leqslant (8sd)^{1/2} < j_{0,m+1}$$

holds. In this case we have to prove that

$$\frac{(8sd)^{1/2} - j_{1,m}}{j_{1,m+1} - j_{1,m}} < \frac{(8sd)^{1/2} - j_{0,m}}{j_{0,m+1} - j_{0,m}}. \qquad (4.10)$$

One can easily show that Eq. (4.10) holds by using the fact that the sequence $j_{1,m+1} - j_{1,m}$ is decreasing to π ($m = 1,2,\cdots$), whereas the sequence $j_{0,m+1} - j_{0,m}$ is increasing to π. This, in turn, follows from an interesting property of the positive zeros $z_{\nu,n}$ (in ascending order) of any real solution Z_ν of Bessel's differential equation. In the Appendix we prove that for $\nu^2 < \frac{1}{4}$ the sequence $\{z_{\nu,n}\}$ is convex, i.e., $\{z_{\nu,n+1} - z_{\nu,n}\}$ is increasing, and that for $\nu^2 > \frac{1}{4}$ the sequence $\{z_{\nu,n}\}$ is concave, i.e., $\{z_{\nu,n+1} - z_{\nu,n}\}$ is decreasing. From the asymptotic behavior of Z_ν it is easily seen that $\{z_{\nu,n+1} - z_{\nu,n}\}$ converges to π for all real ν. In the case $\nu = \pm\frac{1}{2}$, $\{z_{\nu,n+1} - z_{\nu,n}\}$ is neither increasing nor decreasing. We have for all n,

$$z_{\nu,n+1} - z_{\nu,n} = \pi.$$

When ν is purely imaginary, the sequence $\{z_{\nu,n}\}$ is convex. So this holds in particular for the zeros of the modified Bessel function $K_{i\mu}(z)$ (which is real for real μ and z), which have been studied recently.[9]

5. SUMMARY

We have obtained explicit approximate expressions for the binding energies for the symmetric shifted Coulomb potential $V(x) = -2s(|x|+d)^{-1}$. For this potential Mehta and Patil[4] have given the binding energies corresponding to the odd wavefunctions only and for small sd only. In. Sec. 2 we restricted ourselves to $sd \ll 1$, and in Sec. 3 we considered the case when sd is not small. We proved that for the ground state, which is even, we have $E_0 = O(s^2 \ln^2 sd)$ for $s \rightarrow 0$, and for all other bound states $E_n = O(s^2)$. Furthermore we have proved that $\lim_{sd\to\infty} E_n d/s = -\frac{1}{2}$ for any fixed n. Explicit approximate expressions for the binding energies have been given in Eqs. (2.10), (2.19), (2.23), (3.13), (3.16), (3.21), and (3.22), valid for various cases. In Eqs. (3.23)–(3.26) ff. we have given some simple *exact* expressions.

In Sec. 4 we have discussed some interesting special points. One of these concerns the interlacing of the odd and even solutions. Both $\{\kappa_n^{(+)}\}$ and $\{\kappa_n^{(-)}\}$ are approximately harmonic progressions, but the combined sequence of even and odd solutions is "staggering," except when sd is not small.

APPENDIX

Let Z_ν be any real solution of Bessel's differential equation, and let $z_{\nu,n}$, $n = 1,2,...$, be its positive zeros in ascending order. The function u,

$$u(x) \equiv x^{1/2} Z_\nu(kx), \quad x > 0,$$

is for $k \neq 0$ a solution of (e.g., Szegö,[7] p 17)

$$\frac{d^2 u}{dx^2} + \left(k^2 + \frac{\frac{1}{4} - \nu^2}{x^2} \right) u = 0. \qquad (A1)$$

In this appendix we prove, by applying a theorem of Szegö[7] which is given below, that the sequence $\{z_{\nu,n}\}$ is convex when $\nu^2 < \frac{1}{4}$, i.e., $\{z_{\nu,n+1} - z_{\nu,n}\}$ is increasing, and that $\{z_{\nu,n}\}$ is concave when $\nu^2 > \frac{1}{4}$, i.e., $\{z_{\nu,n+1} - z_{\nu,n}\}$ is decreasing; cf. Ref. 7, p. 381.

Theorem [Szegö (Ref. 7, pp. 19 and 20)]: Let $\varphi(x)$ be continuous and decreasing in $x_0 < x < X_0$, and let $y(x)$ (not identically zero) be a solution of

$$y'' + \varphi(x)y = 0. \qquad (A2)$$

Then $x' < x'' < x'''$ being three consecutive zeros of $y(x)$, we have

$$x'' - x' < x''' - x'',$$

that is, the sequence of the zeros of $y(x)$ is convex.

In order to prove $x'' - x' \equiv h < x''' - x''$, we compare Eq. (A2) with

$$Y'' + \varphi(x-h)Y = 0,$$

which has the solution $Y(x) = y(x-h)$. According to a well-known comparison theorem of Sturm's type, $Y(x)$ has a zero in the open interval (x'',x'''). This proves Eq. (A3).

When $\varphi(x)$ is continuous and *increasing*, the sequence of the zeros of $y(x)$ is concave. This follows by observing that in this case $\varphi(-x)$ is a decreasing function of x.

The required result is now obtained by taking

$$\varphi(x) = k^2 + (\tfrac{1}{4} - \nu^2)x^{-2},$$

cf. Eqs. (A1) and (A2).

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Bound states for r^{-2}-like potentials in one and three dimensions

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We investigate the bound states for a symmetric one-dimensional potential $V(x) = c(|x| + d)^{-2}$ for $|x| > R \geq 0$, $V(x)$ arbitrary for $|x| < R$, and the bound states for its three-dimensional analog. We prove that the number of bound states is finite when $c \geq -1/4$ and infinite when $c < -1/4$. For the borderline case $c = -1/4$ this is a new result. For $R = 0$ we obtain explicit approximate expressions for many binding energies. In this case, when $-1/4 \leq c < 0$, there is exactly one bound state in the one-dimensional case and no bound state in the three-dimensional case. The one-dimensional ground-state energy is continuous in c at $c = -1/4$, whereas the excited-state energies are not. In the case $c < -1/4$ the binding energies form approximately a geometric progression, both for the odd solutions and for the even solutions. The sequence of *all* binding energies is approximately a geometric progression when $R = 0$ and the potential strength is sufficiently large. However, in general the complete sequence of binding energies shows an odd–even staggering.

1. INTRODUCTION

Recently there has been much interest in bound states for various potentials in nonrelativistic quantum mechanics. Many inequalities for the number of two-body bound states as a function of the potential have been established, for which we refer to the review papers by Simon[1] and by Martin,[2] and to the recent paper by Glaser *et al.*[3] More specifically one can consider a potential λV, $\lambda > 0$, and study the number of bound states and the energies as a function of the coupling constant λ, in particular for $\lambda \to \infty$ (e.g., Chadan,[4] Simon[5]) and for $\lambda \to 0$. In one and in two dimensions there is, under certain rather general conditions on the potential V, just one bound state if λ is sufficiently small.[6] For this case, explicit expressions for the ground-state energy up to orders of λ have been deduced by Simon,[6] and, for the one-dimensional case only, by Blankenbecler *et al.*[7] and Klaus.[8]

An interesting problem is, how to determine the two classes of Hamiltonians $H = -\Delta + V$ for which the number of bound states in finite and infinite, respectively. For local, nonsingular, central potentials $V(r)$ (in three dimensions) which go to zero for $r \to \infty$, it is well known that the bound-state energies can accumulate only at $E = 0$. It is also known that only the *tail* of the potential is important for the bound-state energies near $E = 0$. Simon[3] proved, by extending certain results in the classical work of Courant and Hilbert,[10] that a potential with a tail smaller than cr^{-2} with $c < -\frac{1}{4}$ has infinitely many bound states, and that a potential with a tail larger than cr^{-2} with $c > -\frac{1}{4}$ has finitely many bound states.

In this paper we deduce explicit approximate expressions for the bound-state energies for the (spherically) symmetric potential $V(r) = c(r + d)^{-2}$ in one and three dimensions in the case when $c + \frac{1}{4}$ is small, and for the one-dimensional ground-state energy for $-\frac{1}{4} \leq c < 0$. More generally, for a potential whose *tail* equals $c(r + d)^{-2}$, we prove that the number of bound states is finite when $c \geq -\frac{1}{4}$ and infinite when $c < -\frac{1}{4}$. In this way we have extended a result obtained by Simon,[9] because in this case d can be negative, and we include the borderline case $c = -\frac{1}{4}$. In the case $c < -\frac{1}{4}$ we have

derived approximate explicit expressions for the bound-state energies close to $E = 0$.

In Sec. 2 some preliminaries are given. In Sec. 3 we study the three-dimensional case and in Sec. 4 the one-dimensional case of the pure $c(r + d)^{-2}$ potential. In Sec. 5 we consider some generalizations, in particular the potentials with tail $c(r + d)^{-2}$. Finally in Sec. 6 the main results are summarized and discussed. Here we consider the *interlacing* property of the odd solutions $\kappa_n^{(-)}$ and the even solutions $\kappa_n^{(+)}$, where the κ_n's are the square roots of the binding energies. When $c < -\frac{1}{4}$, the infinite sequences $\{\kappa_n^{(-)}\}$ and $\{\kappa_n^{(+)}\}$ are approximately geometric progressions. This is compared with the case of the symmetric shifted Coulomb potential, for which we have obtained *harmonic* progressions instead.[11] The sequence $\{\kappa_n^{(-)}, \kappa_n^{(+)}\}$ of *all* solutions shows staggering in general.

2. PRELIMINARIES

We consider the spherically symmetric three-dimensional potential

$$V(r) = c(r + d)^{-2}, \qquad (2.1)$$

with c real and $d > 0$. A solution $u(r)$ of the $l = 0$ radial Schrödinger equation which energy $E = k^2$ has the form

$$(r + d)^{1/2} Z_\nu(kr + kd), \qquad (2.2)$$

where Z_ν is any solution of Bessel's differential equation[12] and the order ν follows from

$$\nu^2 = c + \tfrac{1}{4}. \qquad (2.3)$$

We shall also consider the symmetric one-dimensional potential

$$V(x) = c(|x| + d)^{-2}. \qquad (2.4)$$

Denoting $x > 0$ by r for convenience, we can write any solution of the one-dimensional Schrödinger equation also in the form (2.2). However, the one- and three-dimensional wavefunctions have to satisfy different boundary conditions.

Since we are interested in bound states, we put $E = -\kappa^2$, $k = i\kappa$, with $\kappa > 0$. Then we have for both the one- and three-dimensional bound-state wavefunction,

$$u(r) = \text{const}(r + d)^{1/2} K_\nu(\kappa r + \kappa d), \tag{2.5}$$

where K_ν is a modified Bessel function. Some of its properties, useful for our purposes, are

$$K_\nu(z) = K_{-\nu}(z), \tag{2.6}$$

$$K_\nu^*(z) = K_{\nu*}(z^*), \tag{2.7}$$

and

$$K_\nu(z) \sim \left(\frac{2z}{\pi}\right)^{-1/2} e^{-z}, \quad z \to \infty. \tag{2.8}$$

We note that $K_\nu(z)$ has a branch cut along the negative real z axis. For Re $z > 0$ the following integral representation will be useful, [12,13]

$$K_\nu(z) = \int_0^\infty \exp(-z \cosh t) \cosh \nu t \, dt, \tag{2.9}$$

which holds for Re $z > 0$ for any ν and, if $\nu = 0$, also for Re $z = 0$. It follows from this expression that, for real positive z, $K_\nu(z)$ is real for real ν as well as for purely imaginary ν.

3. THE THREE-DIMENSIONAL CASE

In this case the s-wave bound-states are obtained by imposing the well-known boundary condition at the origin, $u(0) = 0$, on the wavefunction (2.5). This yields the bound-state condition,

$$K_\nu(\kappa d) = 0. \tag{3.1}$$

If $c \geqslant -\frac{1}{4}$ we have $\nu \geqslant 0$. It follows from Eq. (2.9) that K_ν has no (real nonnegative)zero in this case, cf also. [12]

If, on the other hand, $c < -\frac{1}{4}$, we see from Eq. (2.3) that ν is purely imaginary. It is convenient to use in this case the equality

$$K_\nu(z) = \frac{\pi}{2 \sin \pi \nu} [I_{-\nu}(z) - I_\nu(z)], \tag{3.2}$$

where

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1 + \nu)} {}_0F_1(1 + \nu; z^2/4). \tag{3.3}$$

We substitute Eqs. (3.2) and (3.3) into Eq. (3.1), put $\kappa d = y$ and $\nu = i\mu$ with $\mu > 0$. We note that $K_{i\mu}(z)$ is a real function. The bound-state condition (3.1) is equivalent to

$$(y/2)^{2i\mu} = \frac{\Gamma(1 + i\mu)}{\Gamma(1 - i\mu)} \frac{{}_0F_1(1 - i\mu; y^2/4)}{{}_0F_1(1 + i\mu; y^2/4)}. \tag{3.4}$$

Since $\Gamma^*(z) = \Gamma(z^*)$ we can write

$$\Gamma(1 + i\mu)/\Gamma(1 - i\mu) = \exp(2i\omega), \tag{3.5}$$

where $\omega = \arg \Gamma(1 + i\mu)$ is a real function of μ. For similar reasons we have

$$\frac{{}_0F_1(1 - i\mu; y^2/4)}{{}_0F_1(1 + i\mu; y^2/4)} = \exp(2i\varphi), \tag{3.6}$$

where φ is a real function of μ and y^2. We take $\varphi = 0$ when $\mu = 0$ or when $y = 0$. Insertion of Eqs. (3.5) and (3.6) into Eq. (3.4) yields

$$\exp(2i\mu \ln(y/2)) = \exp(2i\omega + 2i\varphi),$$

or

$$\mu \ln(y/2) = -\pi m + \omega + \varphi. \tag{3.7}$$

We have to investigate for which (integral) values of m Eq. (3.7) has a solution. For this purpose we shall now study the functions ω and φ.

The (generalized) hypergeometric functions ${}_0F_1$, and therefore also φ, can be expanded in a power series in y^2. We obtain

$$\varphi(\mu, y^2) = \mu(1 + \mu^2)^{-1}(y/2)^2 + O(y^4), \quad y \to 0. \tag{3.8}$$

In order to find a more explicit expression for ω, we differentiate both sides of Eq. (3.5) with respect to μ and obtain

$$\frac{d\omega}{d\mu} = \frac{1}{2}(\psi(1 + i\mu) + \psi(1 - i\mu)) = \text{Re } \psi(1 + i\mu) = \text{Re } \psi(i\mu). \tag{3.9}$$

Here $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function. The following two known expressions will be useful, [13]

$$\text{Re } \psi(i\mu) = -C + \sum_{n=1}^\infty n^{-1}(1 + n^2/\mu^2)^{-1}, \quad |\mu| < \infty$$

$$= -C + \int_0^\infty \frac{1 - \cos \mu t}{e^t - 1} dt, \quad |\mu| < \infty. \tag{3.10}$$

Here $C = 0.5772\ldots$ is Euler's constant. By integrating $d\omega/d\mu$ and noting that $\omega = 0$ when $\mu = 0$ we get

$$\omega(\mu) = -C\mu + \int_0^\infty \frac{\mu t - \sin \mu t}{e^t - 1} \frac{dt}{t}. \tag{3.11}$$

For $|\mu| \leqslant 1$ we have from Eq. (3.12) the expansion

$$\omega(\mu) = -C\mu + \sum_{n=1}^\infty (-)^{n+1} \mu^{2n+1} \zeta(2n+1)/(2n+1),$$

$$|\mu| \leqslant 1, \tag{3.12}$$

where $\zeta(z)$ is Riemann's zeta function. From Eq. (3.10) we have furthermore,

$$\omega(\mu) = -C\mu + \sum_{n=1}^\infty [\mu/n - \arctan(\mu/n)], \quad |\mu| < \infty \tag{3.13}$$

We deduce from the above expressions that $\omega(\mu)$ is negative for small μ and becomes positive when μ increases. From Eqs. (3.9) and (3.10) we have $d^2\omega/d\mu^2 > 0$, so $d\omega/d\mu$ is strictly increasing.

For $\mu \to \infty$ the following asymptotic series is known, [13]

$$\text{Re } \psi(i\mu) \sim \ln \mu - \sum_{n=1}^\infty (-\mu^2)^{-n} B_{2n}/(2n), \tag{3.14}$$

where B_{2n} are Bernoulli's numbers. Since it is permissible to integrate an asymptotic series termwise, we easily get an asymptotic series for ω,

$$\omega(\mu) \sim \mu \ln \mu - \mu + c_0 + \sum_{n=1}^\infty \frac{(-)^n \mu^{1-2n} B_{2n}}{2n(2n-1)}, \quad \mu \to \infty. \tag{3.15}$$

Here c_0 is the constant of integration, for which we have obtained $c_0 = \pi/4$.

Now we are in a position to derive the approximate solutions to Eq. (3.7). Let us first restrict ourselves to small μ. Then we have

$$\ln(y/2) = (-\pi m + \omega + \varphi)/\mu$$

$$\approx -\pi m/\mu - C + O(y^2). \tag{3.16}$$

For $m = 1, 2, \cdots$, y must be small, so that $O(y^2)$ may

be neglected, and the approximate solutions follow at once from Eq. (3.16). For the case $m = 0$ we use the integral representation of Eq. (2.9),

$$K_{i\mu}(y) = \int_0^\infty \exp(-y\cosh t)\cos\mu t\, dt. \qquad (3.17)$$

It can be seen from this expression that for small μ any solution y of the equation $K_{i\mu}(y) = 0$, if it exists, must be very small. It follows then from Eq. (3.16) that there is no solution at all for $m = 0$. We conclude that for $m = 0$, and a fortiori for $m < 0$, Eq. (3.7) has no solution. (In this connection it may be recalled that for $\mu = 0$ there is no bound state at all.) Therefore, the ground state corresponds to the case $m = 1$.

So we have proved that, for small μ, all solutions to Eq. (3.1) are given approximately by

$$\kappa_m \sim (2/d)\exp(-\pi m/\mu - C), \quad \mu \text{ small}, \quad m = 1, 2, \cdots. \qquad (3.18)$$

Now we let μ increase to larger values. The solutions κ_m are continuous with respect to μ for $\mu > 0$, so we conclude that, for any $\mu > 0$, all bound states are uniquely determined by the solutions of Eq. (3.7) for $m = 1, 2, \cdots$. For each m Eq. (3.7) has a unique solution $y_m = \kappa_m\, d/2$ and we have

$$\kappa_1 > \kappa_2 > \kappa_3 > \cdots. \qquad (3.19)$$

When m/μ is sufficiently large, we see from Eqs. (3.7) and (3.8) that y will be sufficiently small so that φ may be neglected. Therefore, we have the explicit expression

$$\kappa_m \simeq (2/d)\exp(-\pi m/\mu + \omega/\mu), \quad \mu > 0, \quad m/\mu \text{ large}. \qquad (3.20)$$

Note the difference with Eq. (3.18) where ω/μ has been replaced by $-C$. It follows from Eq. (3.20) that

$$\lim_{m\to\infty} \kappa_m/\kappa_{m+1} = \exp(\pi/\mu), \quad \text{for all } \mu > 0, \qquad (3.21)$$

so the κ_m's form approximately a geometric progression.

We note that κ_m must be an increasing function of μ. The binding energy will increase with increasing potential strength $|c|$. We can prove this for small μ with the help of Eq. (3.20). We have

$$\frac{d\kappa_m}{d\mu} \simeq \left(\frac{\pi m}{\mu^2} + \frac{d}{d\mu}\frac{\omega}{\mu}\right)\kappa_m,$$

which is indeed positive for small μ.

Finally we deduce from Eqs. (3.7) and (3.8) the more detailed equality,

$$\lim_{m\to\infty}[\mu\ln(y_m/2) + \pi m - \omega]\exp(2\pi m/\mu) = \mu(1 + \mu^2)^{-1} \\ \times \exp(2\omega/\mu). \qquad (3.22)$$

Therefore,

$$\lim_{m\to\infty}[\mu\ln(y_m/y_{m+1}) - \pi]\exp(2\pi m/\mu) \\ = \mu(1 + \mu^2)^{-1}\exp(2\omega/\mu)[1 - \exp(-2\pi/\mu)]. \qquad (3.23)$$

The right-hand side is for small μ approximately $\mu \times \exp(-2C)$, whereas for large μ it approaches the number $2\pi\exp(-2)$.

4. THE ONE-DIMENSIONAL CASE

In Sec. 2 we have seen that the one-dimensional bound-state wavefunction is given by Eq. (2.5) with $r = x > 0$. Since the potential is symmetric, the bound-state wavefunctions must be either odd or even. The odd wavefunctions satisfy $u(0) = 0$ and are therefore for $x > 0$ proportional to the $l = 0$ radial solutions for the three-dimensional case which has been studied in Sec. 3. The corresponding "odd solutions" κ are denoted by $\kappa_m^{(-)}$. These are equal to the solutions κ_m of Eq. (3.1) discussed in the preceding section.

The even wavefunctions are obtained from Eq. (2.5) by imposing the boundary condition $u'(0) = 0$. In this way we obtain the following bound-state condition,

$$-2\kappa d K_\nu'(\kappa d) = K_\nu(\kappa d). \qquad (4.1)$$

The so-called even solutions κ of this equation will be denoted by $\kappa_m^{(+)}$. We shall discuss the cases $c < -\frac{1}{4}$ and $c \geq -\frac{1}{4}$ separately.

(i) For $c < -\frac{1}{4}$ the situation is very similar to the three-dimensional case. With the help of Eq. (2.6) and

$$K_\nu' = -K_{1-\nu} - \frac{\nu}{\kappa d}K_\nu,$$

we find that Eq. (4.1) is equivalent to

$$(\tfrac{1}{2} - \nu)K_\nu(\kappa d) = \kappa d K_{1-\nu}(\kappa d). \qquad (4.2)$$

We put $\kappa d = y$ as before, use Eqs. (3.2) and (3.3) and

$$I_{1-\nu}(y) = I_{-1-\nu}(y) + (2\nu/y)I_{-\nu}.$$

In this way we obtain the following bound-state condition,

$$(y/2)^{2\nu}$$
$$= \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\frac{(1+2\nu)\,_0F_1(1-\nu;y^2/4) - 4\nu\,_0F_1(-\nu;y^2/4)}{(1-2\nu)\,_0F_1(1+\nu;y^2/4) + 4\nu\,_0F_1(\nu;y^2/4)}. \qquad (4.3)$$

By again putting $\nu = i\mu$ with $\mu > 0$ we can rewrite Eq. (4.3),

$$(y/2)^{2i\mu} = \exp(2i\,\omega)\exp(2i\chi). \qquad (4.4)$$

The function ω has been discussed extensively in the preceding section and χ is determined by

$$\exp(2i\chi)$$
$$= \frac{(1+2i\mu)\,_0F_1(1-i\mu;y^2/4) - 4i\mu\,_0F_1(-i\mu;y^2/4)}{(1-2i\mu)\,_0F_1(1+i\mu;y^2/4) + 4i\mu\,_0F_1(i\mu;y^2/4)}, \qquad (4.5)$$

and by taking $\chi = 0$ when $\mu = 0$. Because we obviously have

$$[\exp(2i\chi)]^* = \exp(-2i\chi),$$

χ is a real function of μ and y^2. It can be expanded in a power series in y^2,

$$\chi(\mu, y^2) = -\arctan(2\mu)$$
$$+ (y/2)^2\frac{\mu(13 + 4\mu^2)}{(1 + \mu^2)(1 + 4\mu^2)} + O(y^4), \quad y \to 0. \qquad (4.6)$$

We obtain from Eq. (4.4)

$$\mu\ln(y/2) = -\pi m + \omega + \chi. \qquad (4.7)$$

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This bound-state condition is equivalent to Eq. (4.1). It looks very much like Eq. (3.7) for the three-dimensional case, but there is one important difference. In contrast to Eq. (3.7), the present Eq. (4.7) has a unique solution not only for $m = 1, 2, \cdots$, but also for $m = 0$. This $m = 0$ solution is the ground-state, since Eq. (4.7) has no solution for negative values of m.

We prove this as follows. First we take μ small.

In the case $m < 0$, we use the integral representation (3.17) for $K_{i\mu}(y)$, and derive from Eq. (4.1) the following bound-state condition,

$$\int_0^\infty (2y\cosh t - 1)\exp(-y\cosh t)\cos\mu t \, dt = 0. \qquad (4.8)$$

A careful investigation of this equation leads to the conclusion that any solution y must be *small* if μ is small. [Note that $\exp(-y\cosh t)$ is an extremely rapidly decreasing function of t.] But a solution y of Eq. (4.7), which is equivalent to Eq. (4.8), can only be *large* if m is negative. This implies that there is no solution at all for negative values of m.

For $m = 0$ we have from Eqs. (4.6) and (4.7),

$$\ln(y/2) \sim -C - 2 + 13(y/2)^2 + \cdots. \qquad (4.9)$$

This equation does indeed have a unique solution. The point is that y is sufficiently small in order that terms of $O(y^4)$ may be neglected. Since the solution is continuous with respect to μ, we can consider the limit for $\mu \to 0$. A single iteration yields for the ground-state solution y_0,

$$y_0 \sim 2\exp(-C - 2 + 13\exp[-2(C+2)]) = 0.164\cdots. \qquad (4.10)$$

After some more iterations of the approximate Eq. (4.9) we get the more precise value $y_0 \sim 0.166$.

When we let μ increase to larger values, the ground-state solution will increase continuously. So it is for all $\mu > 0$ defined as the unique solution of Eq. (4.7) with $m = 0$. Of course, the approximate expression (4.10) is no longer valid for large μ.

The discussion of the case $m = 1, 2, \cdots$ is similar to the one of Sec. 3. A solution y of Eq. (4.7) which is small can be approximated by [cf Eq. (4.6)]

$$\mu\ln\left(\frac{y}{2}\right) \sim -\pi m + \omega - \arctan(2\mu). \qquad (4.11)$$

This expression is valid for $m = 1, 2, \cdots$ if m/μ is large. A similar approximation for the case when μ is small has been given by Blankenbecler *et al.*[7] in a different form. Their result contains an error as we shall now show. We deduce from Eq. (4.4) [cf Eq. (4.11)]

$$\tan\left[\mu\ln\left(\frac{2}{y}\right)\right] = \frac{2\mu - \tan\omega}{1 + 2\mu\tan\omega} + O(y), \qquad (4.12)$$

where

$$\tan\omega = \tan\arg\Gamma(1 + i\mu) = \operatorname{Im}\Gamma(1 + i\mu)/\operatorname{Re}\Gamma(1 + i\mu).$$

So for small μ we obtain from these expressions,

$$(C + 2)\mu = \tan[\mu\ln(2/y)] + O(y), \quad y \to 0. \qquad (4.13)$$

The right-hand side is the same as in Eq. (2.8) of

Blankenbecler *et al.*,[7] but on the left-hand side they only have the term 2μ.

We note that Eqs. (4.12) and (4.13) do hold even with $O(y)$ replaced by $O(y^2)$. However, in order to determine the behavior of the solutions $y = y_m$ for fixed μ, it is preferable to use $\lim_{m\to\infty}$ rather than $O(y)$, cf Eqs. (3.22) and (3.23). We deduce from Eqs. (4.6) and (4.7) the more detailed equation,

$$\lim_{m\to\infty}[\mu\ln(y_m/2) + \pi m - \omega + \arctan(2\mu)]\exp(2\pi m/\mu)$$
$$= \mu(1 + \mu^2)^{-1}(1 + 4\mu^2)^{-1}(13 + 4\mu^2)\exp\{[2\omega$$
$$- 2\arctan(2\mu)]/\mu\}. \qquad (4.14)$$

Therefore,

$$\lim_{m\to\infty}[\mu\ln(y_m/y_{m+1}) - \pi]\exp(2\pi m/\mu)$$
$$= \mu(1 + \mu^2)^{-1}(1 + 4\mu^2)^{-1}(13 + 4\mu^2)\exp\{[2\omega \qquad (4.15)$$
$$- 2\arctan(2\mu)]/\mu\}[1 - \exp(-2\pi/\mu)].$$

The right-hand side is for small μ approximately $13\mu \times \exp(-4 - 2C)$, whereas for large μ it approaches the number $2\pi c^{-2}$.

So we have obtained the following results which are similar to the ones of Sec. 3, i.e., similar to the results for the odd solutions $\kappa_m^{(-)}$. *All* solutions of Eq. (4.1) are given by the unique solutions of Eq. (4.7), with $m = 0, 1, 2, \cdots$, respectively. These solutions $\kappa_m^{(+)}$ are decreasing,

$$\kappa_0^{(+)} > \kappa_1^{(+)} > \kappa_2^{(+)} > \cdots.$$

For small μ we have the approximate expressions,

$$\kappa_m^{(+)} \sim \left(\frac{2}{d}\right)\exp(-\pi m/\mu - C - 2), \quad \mu \text{ small},$$

$$m = 0, 1, 2, \cdots, \qquad (4.16)$$

whereas for arbitrary $\mu > 0$ and large m we have

$$\kappa_m^{(+)} \sim \left(\frac{2}{d}\right)\exp\left[\frac{(-\pi m + \omega + \chi)}{\mu}\right]$$

$$\sim \left(\frac{2}{d}\right)\exp\left\{\frac{[-\pi m + \omega - \arctan(2\mu)]}{\mu}\right\}, \qquad (4.17)$$

$$\mu > 0, \quad m/\mu \text{ large}.$$

Furthermore, just as for the odd solutions,

$$\lim_{m\to\infty}\kappa_m^{(+)}/\kappa_{m+1}^{(+)} = \exp(\pi/\mu), \quad \text{all } \mu > 0, \qquad (4.18)$$

so the $\kappa_m^{(+)}$'s form approximately a geometric progression, too. In Eqs. (4.14) and (4.15) we have given more detailed expressions. We note that only the ground state remains when we let μ go to zero,

$$\kappa_0^{(+)} \to \sim (2/d)\exp(-C - 2), \quad \mu \to 0, \qquad (4.19)$$

$$\kappa_m^{(+)} \to 0, \quad m > 0, \quad \mu \to 0. \qquad (4.20)$$

(ii) We shall now consider the case $-\frac{1}{4} \le c < 0$, i.e., $0 \le \nu < \frac{1}{2}$. By considering Eq. (4.1) for $\kappa \to 0$ and for $\kappa \to \infty$ it easily follows that there is *at least* one bound state for all $d > 0$ (cf[7]).

We shall prove that there exists *exactly one* bound state for all $d > 0$. This proof has not been given by Blankenbecler *et al.* Note that both sides of Eq. (4.1)

are positive and monotonic strictly decreasing functions of κ, because

$$(yK'_\nu)' = yK''_\nu + K'_\nu = (y + \nu^2/y)K_\nu > 0, \quad y > 0, \qquad (4.21)$$

where we have used that K_ν satisfies the modified Bessel differential equation. Therefore, we have to find a different, equivalent bound-state condition in order to prove that there is *only* one solution.

For this purpose our Eq. (4.2) is suitable, which we rewrite as

$$f(y) = \tfrac{1}{2} - \nu, \qquad (4.22)$$

with $y \equiv \kappa d$ and

$$f(y) \equiv yK_{1-\nu}(y)/K_\nu(y). \qquad (4.23)$$

We are going to prove that $f'(y) > 0$ for $y > 0$. With the help of Eq. (2.6), the equality

$$K'_\nu = - K_{1-\nu} - (\nu/y)K_\nu, \qquad (4.24)$$

and the same relation with ν replaced by $1 - \nu$ we derive

$$f'(y) = y(K_{1-\nu}K_{1+\nu} - K_\nu^2)/K_\nu^2. \qquad (4.25)$$

By substituting the integral representation[13]

$$K_{\nu-\lambda}(y)K_{\nu+\lambda}(y)$$
$$= 2\int_0^\infty K_{2\nu}(2y\cosh t)\cosh(2\lambda t)dt, \quad \mathrm{Re}y > 0, \qquad (4.26)$$

for the products of the K functions we then prove

$$f'(y) > 0, \quad y > 0. \qquad (4.27)$$

Furthermore we have $f(0) = 0$ and $f(y) = O(y)$ for $y \to \infty$, so f is strictly increasing from 0 to ∞. In this way we have proved that Eq. (4.22) has exactly one solution if $0 \leqslant \nu < \tfrac{1}{2}$.

Now we shall derive an upper bound to the solution y of Eq. (4.22). From Eq. (2.9) we obtain ($y > 0$),

$$\frac{d}{d\nu}K_\nu(y) > 0 \quad \text{if } \nu > 0, \qquad (4.28)$$
$$= 0 \quad \text{if } \nu = 0.$$

Therefore, $K_\nu(y)$ is a strictly increasing function of ν for $\nu \geqslant 0$. This implies that

$$K_{1-\nu}(y) > K_\nu(y), \quad 0 \leqslant \nu < \tfrac{1}{2}, \; y > 0, \qquad (4.29)$$

so that the solution y of Eq. (4.22) satisfies

$$0 < y < \tfrac{1}{2} - \nu. \qquad (4.30)$$

For $\nu \to \tfrac{1}{2}$, the potential strength c goes to zero and therefore also $y \to 0$, i.e., the binding energy goes to zero. For this case we have proved that

$$\lim_{\nu \uparrow 1/2} y^{-1}(\tanh[\nu\ln(2/y)] - 2\nu) = 1, \qquad (4.31)$$

where we should remember that $y = y(\nu)$ is a function of ν. This result is somewhat more explicit than the one of Blankenbecler *et al.* [Ref. 7, Eq. (2.6)]. Furthermore we have from Eq. (4.3) the approximation,

$$\left(\frac{y}{2}\right)^{2\nu}$$
$$= \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\frac{1-2\nu}{1+2\nu} + \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\frac{2\nu(13-4\nu^2)}{(1-\nu^2)(1+2\nu)^2}\left(\frac{y}{2}\right)^2$$
$$+ O(y^4), \qquad (4.32)$$

for $\nu \to \tfrac{1}{2}$. Because of the dependence of y on ν, the symbol O in Eq. (4.32) should be handled with care. It is *not* possible to *fix* ν and give an approximate formula up to orders of y^2, as we did before in the case $c < -\tfrac{1}{4}$ (ν imaginary), cf Eq. (4.12) ff. By using Eq. (4.32) we have obtained for the solution $y = y(\nu)$,

$$\lim_{\nu \uparrow 1/2} (1 - 2\nu)^{-2}[2y(\nu) - (1 - 2\nu) - (1 - 2\nu)^2\ln(1 - 2\nu)]$$
$$= C = 0.5772\cdots. \qquad (4.33)$$

This result confirms and extends the corresponding result obtained by Blankenbecler *et al.* [Ref. 7, Eq. below (2.6) (corrected for the misprint) and Eq. (2.7)].

The maximum value of κ is attained of course for $\nu = 0$, i.e., for $c = -\tfrac{1}{4}$ since the attraction is then maximal. It is interesting to calculate the value of κ for this critical potential strength. The bound-state condition Eq. (4.2) for $\nu = 0$ is

$$K_0(\kappa d) = 2\kappa d K_1(\kappa d). \qquad (4.34)$$

This equation can be solved with the help of tables for Bessel functions, which yields the rough estimate

$$\kappa_0^{(+)}d \lesssim 0.17. \qquad (4.35)$$

This is approximately the same result as we have obtained in Eq. (4.10) for the case $\mu \to 0$. Below, in Eq. (4.38) we shall find the more accurate value $0.165721514(2)$.

We have discussed above the even solutions $\kappa_m^{(+)}$ for the two regions $c < -\tfrac{1}{4}$ and $c > -\tfrac{1}{4}$. A very interesting phenomenon may be pointed out which occurs at the critical value $c = -\tfrac{1}{4}$. If we let c increase from below to $-\tfrac{1}{4}$, an infinite number of bound states disappear abruptly, and only the ground state remains. We have seen that $\kappa_m^{(+)} \to 0$, $m = 1, 2, \cdots$, for $c \uparrow -\tfrac{1}{4}$. However, for $c \downarrow -\tfrac{1}{4}$ these $\kappa_m^{(+)}$'s are nonexistent, so they are not continuous with respect to c at $c = -\tfrac{1}{4}$.

We shall now prove that the ground-state solution $\kappa_0^{(+)}$ *is* continuous with respect to c at $c = -\tfrac{1}{4}$. In the case $c \uparrow -\tfrac{1}{4}$, we have from Eq. (4.7) with $m = 0$,

$$\lim_{\mu \to 0}\ln(y/2) = \lim_{\mu \to 0}(\omega + \chi)/\mu,$$

where $y = \kappa_0^{(+)}d$. It is easy to see that

$$\lim_{\mu \to 0}\chi/\mu = \lim_{\mu \to 0}[\exp(2i\chi) - 1]/(2i\mu).$$

We now substitute for $\exp(2i\chi)$ the expression of Eq. (4.5) and take the limit for $\mu \to 0$. It is easily seen that the way in which μ goes to zero is irrelevant, so we may also let the real variable $\nu = i\mu$ go to zero. This means that for $c \downarrow -\tfrac{1}{4}$ we find exactly the same expression as for $c \uparrow -\tfrac{1}{4}$. After some manipulations, where we use

$$(\nu)_n = \nu(\nu + 1)_{n-1},$$

and

$$\nu \, _0F_1(\nu;z) = \nu + z \, _1F_2(1;2,1+\nu;z), \qquad (4.36)$$

we find the following exact equation for the ground-state solution y, for both cases $c \uparrow -\tfrac{1}{4}$ and $c \downarrow -\tfrac{1}{4}$,

$$\ln\left(\frac{y}{2}\right) = -C$$

$$+\frac{-4+2\,{}_0F_1(1;y^2/4) - {}_0F_1{}'(1;y^2/4) - y^2\,{}_1F_2{}'(1;2,1;y^2/4)}{{}_0F_1(1;y^2/4) + y^2\,{}_0F_1(2;y^2/4)}.$$

$$(4.37)$$

Here we have used the abbreviations, for this occasion only,

$${}_0F_1{}'(1;y^2/4) \equiv \left(\frac{d}{d\nu}\right){}_0F_1(1+\nu;y^2/4)\big|_{\nu=0},$$

and

$${}_1F_2{}'(1;2,1;y^2/4) \equiv \left(\frac{d}{d\nu}\right){}_1F_2(1;2,1+\nu;y^2/4)\big|_{\nu=0}.$$

Note that Eq. (4.37) has the same unique solution y as Eq. (4.34), i.e., $K_0(y) = 2yK_1(y)$. However, Eq. (4.37) is more suitable for obtaining an accurate value for the solution. We expand the hypergeometric functions on the right-hand side of Eq. (4.37). By means of an iteration procedure we then find for the even ground-state solution,

$$\kappa_0^{(+)}d = y_0 = 0.165721514(2), \quad \nu = \mu = 0, \quad c = -\tfrac{1}{4}. \quad (4.38)$$

Let us recall some related results from the above discussion,

$$\kappa_0^{(+)}d < \tfrac{1}{2} - \nu, \qquad 0 \le \nu < \tfrac{1}{2}, \qquad (4.39)$$

$$\kappa_0^{(+)}d \le 0.165721514(2), \quad 0 \le \nu < \tfrac{1}{2}, \qquad (4.40)$$

$$\kappa_0^{(+)}d \simeq \tfrac{1}{2} - \nu, \qquad \nu \simeq \tfrac{1}{2}. \qquad (4.41)$$

We note that Eq. (4.33) gives a more detailed expression for the behavior of $\kappa_0^{(+)}d$ near $\nu = \tfrac{1}{2}$.

5. GENERALIZATIONS

In this section we shall discuss some generalizations of the results of Secs. 3 and 4. In the three-dimensional case we take

$$V(r) = c(r+d)^{-2}, \quad r > R \quad (R > 0, \ R+d > 0), \qquad (5.1)$$

and for $0 < r < R$ an arbitrary potential, satisfying

$$\int_0^R |V(r)|r\,dr < \infty. \qquad (5.2)$$

In the one-dimensional case we take

$$V(x) = c(|x|+d)^{-2}, \quad |x| > R \quad (R > 0, \ R+d > 0), \qquad (5.3)$$

and for $|x| < R$ an arbitrary symmetric potential satisfying

$$\int_0^R |V(x)|\,dx < \infty. \qquad (5.4)$$

This condition on the potential is more restrictive than Eq. (5.2). Note that the parameter d may be negative in this section, as long as $R+d > 0$.

We shall prove that also in this case the number of bound states is finite when $c \ge -\tfrac{1}{4}$, and infinite when $c < -\tfrac{1}{4}$. In the latter case we shall give explicit approximate expressions for $\kappa_m^{(+)}$ and $\kappa_m^{(-)}$, for large m.

Let us first consider the three-dimensional case. For $l > 0$ we can only obtain solutions if d is chosen to be zero. The derivation will be similar to the one of Sec. 3. We shall restrict ourselves throughout to $l = 0$. For $r > R$ the s-wave bound-state wavefunction is given by

Eq. (2.5),

$$u(r) = \text{const}(r+d)^{1/2}K_\nu(\kappa r + \kappa d), \quad r > R. \qquad (5.5)$$

For $0 < r < R$ it satisfies the Volterra integral equation[14,15]

$$u(r) = \kappa^{-1}\sinh\kappa r + \kappa^{-1}\int_0^r \sinh(\kappa r - \kappa r')V(r')u(r')dr', \qquad (5.6)$$

apart from an unimportant normalization constant. It follows that the logarithmic derivative u'/u is a real-analytic function of the energy $k^2 \equiv -\kappa^2$. (Actually, u'/u is real-meromorphic, so it can accidentally have a pole just at $\kappa = 0$. However the following discussion can easily be adjusted in this exceptional case.) We have to match the wave functions for $r < R$ and $r > R$ by requiring u'/u to be continuous at $r = R$. Now we introduce the new variable

$$z \equiv \kappa(R+d),$$

which is more convenient than the variable $y \equiv \kappa d$ which we used before. We then have

$$\lim_{r \uparrow R} u'/u = \kappa K_\nu'(z)/K_\nu(z) + \tfrac{1}{2}(R+d)^{-1}. \qquad (5.7)$$

It is therefore convenient to introduce the real-analytic function $A(k^2) \equiv A(-\kappa^2)$ by putting

$$\lim_{r \uparrow R} u'/u = (\tfrac{1}{2}+A)/(R+d). \qquad (5.8)$$

In this way we obtain from Eqs. (5.7) and (5.8) the following bound-state condition,

$$AK_\nu(z) = zK_\nu'(z). \qquad (5.9)$$

As before, ν is real when $c \ge -\tfrac{1}{4}$ and ν is imaginary when $c < -\tfrac{1}{4}$. We convert Eq. (5.9) into

$$(A+\nu)K_\nu(z) = -zK_{1-\nu}(z). \qquad (5.10)$$

The solutions κ_m of this equation are just the odd solutions $\kappa_m^{(-)}$ for the one-dimensional case. The even solutions $\kappa_m^{(+)}$ follow from a different equation which, however, has exactly the same form as Eq. (5.10). The only difference is in the function A, which is defined by Eq. (5.8) where now $u'(0) = 0$ instead of $u(0) = 0$. This is proved as follows.

The even bound-state wavefunction $u(x)$ is for $x > R$ given by [cf Eq. (5.5)]

$$u(x) = \text{const}(x+d)^{1/2}K_\nu(\kappa x + \kappa d), \quad x > R. \qquad (5.11)$$

For $0 < x < R$ it satisfies the Volterra integral equation (apart from an unimportant normalization constant),

$$u(x) = \cosh\kappa x + \kappa^{-1}\int_0^x \sinh(\kappa x - \kappa x')V(x')u(x')dx'. \quad (5.12)$$

It follows that the logarithmic derivative u'/u is again a real-analytic function of the energy at $\kappa = 0$ (in general), just as before in the "odd case." Since the explicit form of this function is irrelevant for our discussion, we may denote it again by A. Obviously the even bound-state condition is then also given by Eq. (5.10).

We note that we must get back the bound-state conditions of Secs. 3 and 4 if we let R go to zero. Indeed, by taking $A = \infty$ in Eq. (5.10) we obtain Eq. (3.1) for the

odd solutions, and by taking $A = -\frac{1}{2}$ we obtain Eq. (4.2) for the even solutions.

In order to find the solutions of Eq. (5.10) we proceed in the same way as before, and obtain the equivalent equation [cf. Eq. (4.3)],

$$\left(\frac{z}{2}\right)^{2\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \frac{(A-\nu)\,{}_0F_1(1-\nu;z^2/4) + 2\nu\,{}_0F_1(-\nu;z^2/4)}{(A+\nu){}_0F_1(1+\nu;z^2/4) - 2\nu\,{}_0F_1(\nu;z^2/4)}.$$

(5.13)

It can be shown that this equation can have only a finite number of solutions when ν is a real number, i.e., when $c \geq -\frac{1}{4}$. In the case $c < -\frac{1}{4}$, ν is purely imaginary so we put for convenience $\nu = i\mu$ as before. We rewrite Eq. (5.13) as follows,

$$(z/2)^{2i\mu} = \exp(2i\omega)\exp(2i\xi).$$

(5.14)

The function ω depends only on μ [cf. Eq. (3.13)],

$$\omega(\mu) = \arg\Gamma(1+i\mu).$$

The function ξ is defined by

$$\exp(2i\xi)$$

$$= \frac{(A-i\mu)\,{}_0F_1(1-i\mu;z^2/4) + 2i\mu\,{}_0F_1(-i\mu;z^2/4)}{(A+i\mu)\,{}_0F_1(1+i\mu;z^2/4) - 2i\mu\,{}_0F_1(i\mu;z^2/4)},$$

(5.15)

and by taking $\xi = 0$ when $\mu = 0$. Just as the functions φ and χ of Eqs. (3.6) and (4.5), ξ is real. Since A is a real-analytic function of κ^2, and therefore of $z^2 \equiv \kappa^2(R+d)^2$, we can expand ξ as follows,

$$\xi = \arctan[\mu/A(0)] + O(z^2), \quad z \to 0.$$

(5.16)

From Eq. (5.14) we have

$$\mu \ln\left(\frac{z}{2}\right) = -\pi m + \omega + \xi.$$

(5.17)

When m, or m/μ, is sufficiently large, there exist solutions z which are very small. Then we may neglect the term $O(z^2)$ on the right-hand side of Eq. (5.16). So we find the following infinite series of bound-state solutions $[z \equiv \kappa(R+d)]$,

$$\kappa_m^{(\pm)} \simeq (2/(R+d))\exp(\{-\pi m + \omega + \arctan[\mu/A^{(\pm)}(0)]\}/\mu),$$

$$m/\mu\,\text{large},$$

(5.18)

where we now use the superscripts in $A^{(+)}$ and $A^{(-)}$ to denote the even and the odd case, respectively. By taking $R = 0$ we get back the expressions of Secs. 3 and 4. The odd solutions $\kappa_m^{(-)}$ of Eq. (3.20) are recovered by taking $A^{(-)}(0) = \infty$, and the even solutions $\kappa_m^{(+)}$ of Eq. (4.17) by taking $A^{(+)}(0) = -\frac{1}{2}$.

It follows from Eq. (5.18) that the tail of the potential almost completely determines the bound states near the origin. The only effect of the potential for $0 < r < R$ is contained in the constant $A^{(\pm)}(0)$. In particular we have, just as in Secs. 3 and 4, for all $\mu > 0$,

$$\lim_{m \to \infty} \kappa_m^{(-)}/\kappa_{m+1}^{(-)} = \exp(\pi/\mu),$$

(5.19)

$$\lim_{m \to \infty} \kappa_m^{(+)}/\kappa_{m+1}^{(+)} = \exp(\pi/\mu).$$

(5.20)

We note that $\kappa_m^{(\pm)}$ of Eq. (5.18) need not be the mth or

the $(m+1)$th κ in the sequence of solutions κ in decreasing order. That is, the counting will in general be different from the counting in Secs. 3 and 4. In this connection we note that for $m = 1, 2, \cdots$ we have $\kappa_m \to 0$ when $\mu \to 0$. In addition there can be extra bound states which have to be denoted by $\kappa_0, \kappa_{-1}, \kappa_{-2}, \cdots$. The number of these is finite and depends on the inner part of the potential.

6. SUMMARY AND DISCUSSION

In Sec. 3 we have deduced the s-wave bound-state energies $E_m = -\kappa_m^2$ for the spherically symmetric three-dimensional potential,

$$V(r) = c(r+d)^{-2}.$$

(6.1)

When $c \geq -\frac{1}{4}$ there is no bound state at all, but in the case $c < -\frac{1}{4}$ there are infinitely many bound states. We have obtained [cf. Eqs. (3.18) and (3.20)],

$$\kappa_m \simeq (2/d)\exp(-\pi m/\mu + \omega/\mu), \quad m = 1, 2, \cdots,$$

$$m/\mu\,\text{large},$$

(6.2)

where $\mu = (-c - \frac{1}{4})^{1/2} > 0$ and $\omega = \arg\Gamma(1 + i\mu)$. A more detailed expression for κ_m has been given in Eq. (3.22).

The $l = 0$ wave functions are, if we write r for $x > 0$, proportional to the *odd* wavefunctions for the one-dimensional symmetric potential,

$$V(x) = c(|x| + d)^{-2}.$$

(6.3)

The *even* bound-state wavefunctions and the corresponding energies $-(\kappa_m^{(+)})^2$ for this potential have been investigated in Sec. 4. For $-\frac{1}{4} \leq c < 0$ we have exactly one bound state, $m = 0$. This result has also been obtained by Blankenbecler et al.,[7] although they did not give the proof. For $c < -\frac{1}{4}$ we have bound states for $m = 0, 1, 2, \cdots$. We have proved that $\kappa_0^{(+)}$ is a continuous function of c in particular at $c = -\frac{1}{4}$. For this case ($\mu = 0$, $\nu \equiv (c + \frac{1}{4})^{1/2} = 0$) we have in Eq. (4.38) obtained $y_0 = \kappa_0^{(+)}d = 0.165721514(2)$. This is the solution of Eq. (4.34), and also of Eq. (4.37) which has a very different form. Some relations for $\kappa_0^{(+)}$ in the case $-\frac{1}{4} \leq c < 0$ have been given in Eqs. (4.39)—(4.41). In particular for the behavior of $\kappa_0^{(+)}$ for $\nu \to \frac{1}{2}$, i.e. $c \to 0$, we have deduced a precise expression, Eq. (4.33).

The even bound-state solutions with $m = 1, 2, \cdots$, which occur for $c < -\frac{1}{4}$, are approximately given by [cf. Eq. (4.17)]

$$\kappa_m^{(+)} \simeq (2/d)\exp([-\pi m + \omega - \arctan(2\mu)]/\mu),$$

$$m = 1, 2, \cdots, \quad m/\mu\,\text{large}.$$

(6.4)

In Eq. (4.14) we have obtained a more detailed expression.

In Sec. 5 we considered the more general case of an arbitrary symmetric potential which equals (6.1) or (6.3) only for $r > R$ or $|x| > R$, respectively. We proved that also in this case the number of bound states is finite when $c \geq -\frac{1}{4}$, and infinite when $c < -\frac{1}{4}$. In the latter case we have obtained the approximate expression [cf. Eq. (5.18)],

$$\kappa_m^{(\pm)} \simeq (2/(R+d)) \exp(\{-\pi m + \omega + \arctan[\mu/A^{(\pm)}(0)]\}/\mu),$$

m/μ large, $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (6.5)

for the odd and the even solutions.

So we have proved that any (spherically) symmetric potential with a tail $c(r+d)^{-2}$ yields, for any $d > -R$, infinitely many bound states when $c < -\frac{1}{4}$ and finitely many when $c \geq -\frac{1}{4}$. In this way we have extended proposition 3 of a paper by Simon,[9] which states that the number of bound states is finite for a potential which satisfies $V(r) \geq cr^{-2}$ for some $c > -\frac{1}{4}$. Note that the tail $c(r+d)^{-2}$ still yields only finitely many bound states if $c \geq -\frac{1}{4}$, despite the fact that when d is negative, it is more attractive than the tail cr^{-2}. Our result for the case $c < -\frac{1}{4}$ is in fact equivalent with proposition 2 of Ref 9, which says that, for a potential V with $V(r) < cr^{-2}$ $(r > R$, $c < -\frac{1}{4})$, there are infinitely many bound states. The equivalence can be seen by noting that in this case c belongs to the *open* interval $(-\infty, -\frac{1}{4})$.

We have denoted the ground-state solution in the one-dimensional case by $\kappa_0^{(+)}$, and in the three-dimensional case by κ_1. Our motive for this is that $\kappa_0^{(+)}$ is special since it exists for all $c < 0$, i.e., for arbitrarily small potential strength, while κ_1 (for the three-dimensional case) does not. This is a particular case of a general theorem proved by Simon,[6] which has recently been generalized by Klaus.[8] Simon proved that, under the condition

$$\int_{-\infty}^{\infty} (1+x^2)|V(x)|dx < \infty, \quad V \text{ not a.e. zero}, \qquad (6.6)$$

$-d^2/dx^2 + \lambda V$ has a bound state for all small positive λ if and only if

$$\int_{-\infty}^{\infty} V(x)dx \leq 0.$$

Klaus[8] proved this under the condition [weaker than (6.6)]

$$\int_{-\infty}^{\infty} (1+|x|)|V(x)|dx < \infty, \quad V \text{ not a.e zero},$$

which had been conjectured by Simon. A similar theorem for the two-dimensional case has also been proved by Simon.[6] An interesting consequence is that, in one and two dimensions, $-\Delta + \lambda V$ has at least one bound state for *all* $\lambda \neq 0$, if $\int_{-\infty}^{\infty} V(x)dx = 0$. If λ is small there is only one bound state.[6]

An interesting phenomenon comes to light when we consider the interrelationship between the odd solutions $\kappa_m^{(-)}$ and the even solutions $\kappa_m^{(+)}$. It can be proved that for an arbitrary (nonsingular) symmetric potential these κ's interlace, i.e.,

$$\kappa_0^{(+)} > \kappa_1^{(-)} > \kappa_1^{(+)} > \ldots \kappa_m^{(-)} > \kappa_m^{(+)} > \kappa_{m+1}^{(-)} > \kappa_{m+1}^{(+)} > \cdots \quad (6.7)$$

Both the sequence $\{\kappa_m^{(-)}\}$ and the sequence $\{\kappa_m^{(+)}\}$ are, for large m, approximately geometric progressions, with common ratio $\exp(-\pi/\mu)$. One might expect that also the sequence

$$\kappa_m^{(-)}, \ \kappa_m^{(+)}, \ \kappa_{m+1}^{(-)}, \ \kappa_{m+1}^{(+)}, \cdots \qquad (6.8)$$

will approximate a geometric progression, with common ratio $\exp[-\pi/(2\mu)]$, when m is large. However, in general this is not the case. Let us for convenience restrict ourselves to the simple case $R = 0$. Then we have from Eqs. (6.2) and (6.4), for large m,

$$\kappa_m^{(-)}/\kappa_m^{(+)} \simeq \exp[\mu^{-1}\arctan(2\mu)] > 1, \qquad (6.9)$$

and

$$\kappa_m^{(+)}/\kappa_{m+1}^{(-)} \simeq \exp([\pi - \arctan(2\mu)]/\mu) > 1. \qquad (6.10)$$

Indeed, for large μ both (6.9) and (6.10) approach $\exp(\pi/(2\mu))$, but for small μ the ratio (6.9) becomes c^2, and the ratio (6.10) becomes $\exp(\pi/\mu - 2)$. So we see that the sequence (6.8) is approximately a geometric progression only for the case of large μ, that is, for large potential strength.

In this connection it is interesting to note that a similar phenomenon occurs for the case of the symmetric shifted Coulomb potential,

$$V(x) = -2s(|x|+d)^{-1}.$$

Recently we have proved[11] that for this potential both the sequences of the odd solutions $\kappa_n^{(-)}$ and of the even solutions $\kappa_n^{(+)}$ form, for large n approximately a *harmonic* progression, i.e., a sequence of terms whose reciprocals form an arithmetic progression. The common difference of this related arithmetic progression turns out to be s^{-1}. For the case of small sd we have deduced,

$$s/\kappa_n^{(-)} \simeq n + 2sd, \quad n = 1, 2, \cdots,$$

$$s/\kappa_n^{(+)} \simeq n + (-\ln(2sd))^{-1}, \quad n = 1, 2, \cdots.$$

Clearly the sequence

$$\kappa_n^{(-)}, \ \kappa_n^{(+)}, \ \kappa_{n+1}^{(-)}, \ \kappa_{n+1}^{(+)}, \cdots \qquad (6.11)$$

is not a harmonic progression in this case. However, for the case when sd is not small, i.e., for sufficiently large potential strength, we have proved that (6.11) is for large n indeed a harmonic progression, approximately. The common difference of its related arithmetic progression is of course just $\frac{1}{2}s^{-1}$.

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Solitons and rational solutions of nonlinear evolution equations

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Rational solutions of certain nonlinear evolution equations are obtained by performing an appropriate limiting procedure on the soliton solutions obtained by direct methods. In this note specific attention is directed at the Korteweg–de Vries equation. However, the methods used are quite general and apply to most nonlinear evolution equations with the isospectral property, including certain multidimensional equations. In the latter case, nonsingular, algebraically decaying, soliton solutions can be constructed.

1. INTRODUCTION

In recent years much attention has been focused on a certain class of nonlinear evolution equations. The most famous example is, of course, the Korteweg–deVries (KdV) equation, with its many remarkable properties. The first indication that the KdV equation was truly unusual was the discovery (numerically) by Zabusky and Kruskal,[1] that the KdV equation admitted soliton solutions. This discovery motivated further studies out of which grew a method of solving the KdV equation.[2] The method, referred to as the inverse scattering transform, (I.S.T.), analytically recovers the special soliton solutions, as well as the dispersive tail, and requires knowledge of inverse scattering. Importantly, the ideas in Ref. 2 can be extended to many other physically interesting nonlinear evolution equations, including the cubic nonlinear Schrödinger, modified KdV, sine–Gordon, etc. equations (see, for example, Ref. 3).

A complementary approach to I.S.T. exists and enables us to generate the soliton solutions. It was first proposed by Hirota[4] for the KdV equation. This technique has the advantage of being applicable directly upon the equation. One-dimensional soliton solutions for all of the above equations, as well as those of the Boussinesq and multidimensional Kadomtsev–Petviashvili (K–P) equations can be obtained by this procedure. [See, for example, Refs. 5 and 6; it should be mentioned that for both of these latter equations: (a) they possess the important isospectral property, (b) the inverse scattering has not yet been completely done. (c) alternatively, one-dimensional soliton solutions can be found from the integral equations of Zakharov and Shabat[7].]

Recently the apparently quite different class of rational solutions was investigated by Airault, McKean, and Moser[8], and Adler and Moser.[9] In this note we shall show: (i) for KdV how one can recover these rational solutions by taking the long wave limit of the one-dimensional soliton solutions obtained by direct methods. In particular we demonstrate the results for the first few soliton cases, and then we show how performing the above limiting procedure on the Bäck-

lund transformation of KdV[10] yields a recursion formula capable of generating the full class of rational solutions of KdV. This formula was first found by Moser.[9] (ii) We show how these ideas are applied to the first few soliton solutions of the modified KdV, Boussinesq and K–P equation.

In the latter (K–P equation) case, a special solution we find is real, nonsingular, and algebraically decaying in all directions. We refer to this multidimensional soliton as a lump solution. It is in agreement with recently announced results.[11] In our next paper we shall examine the multidimensional problem in more detail, and show how to obtain an N-lump solution for the K–P equation, and, in addition, we will apply these ideas to another physically significant multidimensional nonlinear evolution equation. It is also significant to note that for the KdV equation the rational solutions have poles on the real axis. However, in the case of the modified KdV equation we shall show how real, nonsingular, rational solutions are obtained. These are algebraically decaying solutions. Our results are in agreement with those of Ono.[12]

We feel that the most important consequence of this work is the fact that rational solutions to a broad class of nonlinear evolution equations can be systematically deduced, by a limiting procedure, from the known soliton solutions obtained via direct methods.

2. THE KORTEWEG–DE VRIES EQUATION

The now classical KdV equation,

$$u_t + 6u u_x + u_{xxx} = 0 \tag{2.1}$$

has an N-soliton solution of the form, $u = 2(\log f_N)_{xx}$ (2.2)

where f_N satisfies

$$D_x(D_t + D_x^3)f_N \cdot f_N = 0 \tag{2.3}$$

assuming $u \to 0$, as $|x| \to \infty$, and defining

$$D_x^n D_t^m a \cdot b$$

$$= (\partial x - \partial x')^n (\partial t - \partial t')^m a(x,t)b(x',t')|_{x=x',t=t'}, \tag{2.4}$$

(see for example Refs. 5 and 6). Special solutions f_N can be simply obtained by expanding f_N in the following way,

$$f_N = 1 + \epsilon f_N^{(1)} + \epsilon^2 f_N^{(2)} + \cdots, \tag{2.5}$$

substituting (2.5) into (2.3), and equating coefficients of ϵ. This perturbationlike expansion *truncates* when $f_N^{(1)}$ is chosen to be of the form

$$f_N^{(1)} = \sum_{i=1}^{N} \exp(\eta_i) \qquad (2.6a)$$

where η_i is given by

$$\eta_i = k_i x - k_i^3 t + \eta_i^0. \qquad (2.6b)$$

The first few multisoliton solutions are given by

$$f_1 = 1 + e^{\eta_1}, \qquad (2.7a)$$

$$f_2 = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}}, \qquad (2.7b)$$

$$f_3 = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_1 + \eta_2 + A_{12}} + e^{\eta_2 + \eta_3 + A_{23}},$$
$$+ e^{\eta_3 + \eta_1 + A_{31}} + e^{\eta_1 + \eta_2 + \eta_3 + A_{12} + A_{23} + A_{31}}. \qquad (2.7c)$$

and the general case was proven by Hirota[4] to be given by

$$f_N = \sum_{\mu=0,1} \exp\left(\sum_{i<j}^{N} A_{ij} \mu_i \mu_j + \sum_{i=1}^{N} \mu_i \eta_i \right), \qquad (2.8a)$$

where in the above formula

$$\exp A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2. \qquad (2.8b)$$

The reader can verify (2.7a–c) from (2.8a), and in addition we note that the usual single soliton is easily recovered from (2.7a) to be

$$u = (k_1^2/2) \operatorname{sech}^2 \tfrac{1}{2} (k_1 x_1 - k_1^3 t + \eta_1^{(0)}). \qquad (2.9)$$

The fact that one can recover rational solutions relies on our freedom of choosing the arbitrary phase constants $\eta_i^{(0)}$. For example, in (2.9), if we choose $e^{\eta_1^{(0)}} = -1$, then we have the "singular" soliton

$$u = -(k_1^2/2)\operatorname{cosech}^2 \tfrac{1}{2} (k_1 x - k_1^3 t). \qquad (2.10)$$

Passing to the limit $k_1 \to 0$ (i.e., the "long wave" limit) we find

$$u = -2/x^2. \qquad (2.11)$$

It is significant that by choosing the phase constants appropriately, all the f_N described above, have nontrivial, distinguished limits.

In what follows it is easiest to develop the ideas for f_N. The reader can then easily recover u from (2.2).

In order to best explain the above results we return to formulas (2.7). Letting $\alpha_1 = e^{\eta_1^{(0)}}$, (2.7a) is written as

$$f_1 = 1 + \alpha_1 e^{\xi_1}, \qquad (\xi_1 = k_1(x - k_1^2 t)). \qquad (2.12)$$

As $k_1 \to 0$, we have

$$f_1 = 1 + \alpha_1(1 + \xi_1) + O(k_1^2).$$

If we take $\alpha_1 = -1$, then,

$$f_1 = -k_1(x + O(k_1)).$$

Since we use (2.2) to obtain u, we take $f_1 \backsimeq x + O(k_1)$ (here $f \backsimeq g$ iff $f = e^{gx+b}g$ where a, b are independent of x). Then in the limit $k_1 \to 0$, $f_1 \backsimeq \theta_1$, where

$$\theta_1 = x. \qquad (2.13)$$

Hence in the limit $k_1 \to 0$ we have recovered the rational solution (2.11). The same idea applies to f_2 (as well as the higher f_N). In what follows we consider all the k_i to be of the same asymptotic order as $k_i \to 0$[i.e., $k_i = \epsilon \bar{k}_i, \bar{k}_i = O(1)$]. From (2.7b) we have

$$f_2 = 1 + \alpha_1 e^{\xi_1} + \alpha_2 e^{\xi_2} + \alpha_1 \alpha_2 e^{\xi_1 + \xi_2 + A_{12}}. \qquad (2.14)$$

As $k_1, k_2 \to 0$, we require that the coefficients of the $O(1)$, $O(\mathbf{k})$ terms vanish:

$$O(1): 1 + \alpha_1 + \alpha_2 + \alpha_1 \alpha_2 e^{A_{12}} = 0,$$

$$O(\mathbf{k}): k_1 \alpha_1 + k_2 \alpha_2 + (k_1 + k_2)\alpha_1 \alpha_2 e^{A_{12}} = 0.$$

The solution to these equations is

$$\alpha_1 = -\alpha_2 = (k_1 + k_2)/(k_1 - k_2).$$

As it turns out, this solution also satisfies the equation at $O(\mathbf{k}^2)$, and we find

$$f_2 = -(1/6)k_1 k_2(k_1 + k_2)[(x^3 + 12t) + O(\mathbf{k})]. \qquad (2.15a)$$

Thus as $\mathbf{k} \to 0$, f_2 is equivalent to θ_2:

$$\theta_2 = x^3 + 12t. \qquad (2.15b)$$

f_2 has *three* zero's and so u has *three* poles. In the three soliton case, if we choose,

$$\alpha_1 = \frac{k_1 + k_2}{k_1 - k_2} \cdot \frac{k_3 + k_1}{k_3 - k_1},$$

$$\alpha_2 = \frac{k_2 + k_3}{k_2 - k_3} \cdot \frac{k_1 + k_2}{k_1 - k_2},$$

$$\alpha_3 = \frac{k_3 + k_1}{k_3 - k_1} \cdot \frac{k_2 + k_3}{k_2 - k_3},$$

we find a *six* zero solution for f_3:

$$f_3 = -(1/360)k_1 k_2 k_3(k_1 + k_2)(k_2 + k_3)(k_3 + k_1)$$

$$\times [(x^6 + 60x^3 t - 720t^2) + O(\mathbf{k})], \qquad (2.16a)$$

or simply in the limit $\mathbf{k} \to 0$, $f_3 \backsimeq \theta_3$,

$$\theta_3 = x^6 - 60x^3 t - 720t^2. \qquad (2.16b)$$

In principle this technique applies to any number of solitons. However the calculations are tedious, and here we shall instead use the Bäcklund transformations for KdV to generate a recursion relation for the polynomials.

Before doing this, for convenience, we shall employ a slightly different formula for the N-soliton solution (2.8a). It may be shown that (see Appendix A),

$$f_N \backsimeq \hat{f}_N = \sum_{\epsilon = \pm 1} \prod_{i<j}^{N} \frac{(\epsilon_i k_i - \epsilon_j k_j)}{(k_i^2 - k_j^2)} \cdot \frac{\exp(\sum_{i=1}^{N} \tfrac{1}{2}\epsilon_i \eta_i)}{\prod_{i=1}^{N} \epsilon_i k_i}, \qquad (2.17)$$

so for example

$$\hat{f}_1 = \frac{1}{k_1}(e^{\eta_1/2} - e^{-\eta_1/2}) = -\frac{1}{k_1}e^{-\eta_1/2}(1 - e^{\eta_1}),$$

which is seen to be equivalent to (2.7a). One of the advantages of (2.17) is that the limit as $k_1 \to 0$ directly yields the polynomial in x. We may rewrite $\hat{f}_N$ in the following way. (Hereafter we shall drop the $\wedge$ on $\hat{f}_N$.)

$$f_N = \frac{g_N}{\prod_{i<j}^{N}(k_i^2 - k_j^2) \prod_{i=1}^{N} k_i}, \qquad (2.18)$$

where

$$g_N = \sum_{\epsilon=\pm1} \prod_{i<j}^{N} (\epsilon_i k_i - \epsilon_j k_j) \prod_{i=1}^{N} \epsilon_i \exp(\sum_{i=1}^{N} \tfrac{1}{2}\epsilon_i \eta_i).$$
$$(2.19)$$

It is important to note that g has the following properties:

  (i) $g_N(k_1,k_2,\cdots,k_i,\cdots,k_j,\cdots,k_N) = -g_N(k_1,k_2,\cdots,k_j,\cdots,$
$k_i,\cdots,k_N)$ (for $i<j$) (g_N is antisymmetric in the k_i's),

  (ii) $g_N(k_1=0,k_2,\cdots,k_N)=0$,

  (iii)$g_N(k_1=k_2,k_3,\cdots,\ k_N)=g_N(k_1=-k_2,k_3,\cdots,\ k_N)=0$.

This means the g_N is factorized by

$$\prod_{i<j}^{N}(k_i^2-k_j^2)\prod_{i=1}^{N}k_i.$$

Thus, the first nonzero term as $k_i\to0$, in f_N is at least order one, and we write,

$$f_N = a_N\theta_N(x) + O(\mathbf{k}). \qquad (2.20)$$

We shall later show that $a_N\neq0$. Moreover, since every k_i in the phase factor of (2.17) has an x multiplying it, in order for f_N to be at least order one, the polynomial, $\theta_N(x)=x^P+\cdots$, must have its highest power satisfying

$$P = N + N\left(\frac{N-1}{2}\right) = \frac{N(N+1)}{2}.$$

Next, we derive a recursion relation for θ_N. From Ref. 10 we take the following equations (these formulas are to be regarded as Bäcklund transformations, between an N soliton solution and an $N+1$ soliton solution, see Appendix B for a derivation):

$$(D_x^2 - \tfrac{1}{4}k^2)f_N f_{N+1} = 1, \qquad (2.21a)$$

$$(D_t + \tfrac{3}{4}k^2 D_x + D_x^3)f_N f_{N+1} = 0. \qquad (2.21b)$$

Here $f_N=f_N(k_1,\cdots,k_N)$ (i.e., N parameters). Equations (2.21a,b) yield an $(N+1)$ soliton solution, $f_{N+1}=f_{N+1}$ $(k_1,\cdots,k_N,k)$, from an N soliton solution. It is well known that there are soliton superposition formulae between four soliton solutions. In Appendix C, we outline a derivation following Ref. 13. If we call

$$f_{N-1}=f_{N-1}(k_1,\cdots,k_{N-1}),$$
$$f_N=f_N(k_1,\cdots,k_{N-1},k_N),$$
$$\tilde{f}_N=\tilde{f}_N(k_1,\cdots,k_{N-1},k_{N+1}),$$
$$f_{N+1}=f_{N+1}(k_1,\cdots,k_{N-1},k_N,k_{N+1}),$$

then we have,

$$D_x f_{N+1}\cdot f_{N-1} = \tfrac{1}{2}f_N\tilde{f}_N. \qquad (2.22)$$

Using (2.20) and (2.22) we can obtain a recursion formula for a_N and θ_N. Inserting (2.20) into (2.22) yields

$$a_{N+1}a_{N-1}D_x\theta_{N+1}\cdot\theta_{N-1} = \tfrac{1}{2}a_N^2\theta_N^2. \qquad (2.23)$$

Since $\theta_N(x)$ is a polynomial in x, (2.23) must hold separately for each power, and in particular for the highest power, $x^{N(N+1)/2}$. Hence θ_N satisfies the recursion formula

$$a_{N+1}a_{N-1}(4N+2)=a_N^2 \qquad (2.24)$$

and θ_N satsifies

$$D_x\theta_{N+1}\cdot\theta_{N-1}=(2N+1)\theta_N^2. \qquad (2.25)$$

From our earlier calculations involving the first few solitons,

we have already deduced

$$a_0=1, \quad a_1=1, \quad a_2=\tfrac{1}{6}, \quad a_3=\tfrac{1}{360};$$

$$\theta_0=1, \quad \theta_1=x, \quad \theta_2=x^3+12t, \quad \theta_3=x^6+60x^3t-720t^2.$$

From the recursion relations using a_0,a_1 we see that a_2,a_3 above are recovered, and that $a_N\neq0$, for all $N\geqslant0$. Similarly we can use (2.25) with θ_0,θ_1, to deduce θ_2,θ_3 and the higher θ_N, if we supplement (2.25) with a time evolution equation. For this purpose we can use either the original p.d.e. (KdV in this case) or the time dependent equation obtained from the Bäcklund transformation (2.21b),

$$(D_t+D_x^3)\theta_N\cdot\theta_{N+1}=0. \qquad (2.26)$$

This still allows an arbitrary multiple of θ_{N-1} to be added onto a particular solution θ_{N+1}. In this problem we can take this arbitrary multiple to be zero. In the limiting procedure described above, one can see that to obtain a given asymptotic power of $\mathbf{k}$, each x^3 term corresponds with each power of t. As an example, we know the leading order term in θ_3 is x^6. Thus the general form of this polynomial must be $x^6+\alpha x^3 t+\beta t^2$. We determine $\alpha=60, \beta=-720$ from (2.25)–(2.26). Although we can add the term Cx to this result, and still satisfy (2.25)–(2.26), we take $C=0$ since it cannot arise via the limiting procedure on f_3.

Recent studies by Airault[14] on the second Painlevé transcendent has suggested that (2.25) also yields rational function similarity solutions $f_N(z), f_N(z)=2(\log\theta_N(z))''$ where $u=f_N(z)/(3t)^{2/3}, z=x/(3t)^{1/3}$ in (2.1).

3. OTHER EXAMPLES

It is an important aspect of the present work, that the methods we have employed can be readily adapted to the other nonlinear evolution equations that possess the isospectral property and/or have soliton solutions obtainable by the direct approach. In this section we shall simply list the results of the long wave asymptotic calculations on the first two soliton formulas of the (i) K–P (two-dimensional KdV) equation, (ii) Boussinesq equation, and (iii) modified KdV equation with nonzero background state. Regarding the conclusions of these few examples, perhaps the most significant is that for the multidimensional problem. For the K–P equation we find that the long wave limit of a two soliton state is a mode which decays algebraically in all directions. We refer to such a mode here as a lump. Our results agree with those announced in Ref. 11. In a future paper we shall describe how to obtain an N lump solution. Nevertheless all of these solutions are simply special rational solutions of certain nonlinear evolution equations.

The K–P equation is given by

$$\partial_x(u_t+6uu_x+u_{xxx})+\alpha u_{yy}=0, \qquad (3.1)$$

where α is a constant depending on the dispersive property of the system ($\alpha>0$ corresponds to negative dispersion and vice versa). Following Ref. 15 we look for a solution, $u\to0$ as $|x|\to\infty$, of the form

$$u=2(\log f_N)_{xx}. \qquad (3.2)$$

Inserting (3.2) into (3.1) yields

$$(D_x D_t + D_x^4 + \alpha D_y^2) f_N \cdot f_N = 0. \tag{3.3}$$

The N soliton solution can be ascertained by direct methods.[15] Here we will only discuss the $N = 1,2$ cases. The one, and two-soliton solutions are given by

$$f_1 = 1 + e^{\eta_1} \tag{3.4a}$$

$$f_2 = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}} \tag{3.4b}$$

where

$$\eta_i = k_i (x + P_i y - (k_i^2 + \alpha P_i^2)t) + \eta_i^{(0)}, \tag{3.4c}$$

$$\exp A_{ij} = \frac{3(k_i - k_j)^2 - \alpha(P_i - P_j)^2}{3(k_i + k_j)^2 - \alpha(P_i - P_j)^2}, \tag{3.4d}$$

Taking $e^{\eta_i^{(0)}} = -1$, $k_i \to 0$ [with $P_i = O(1)$, $k_1/k_2 = O(1)$], we find

$$f_1 = -k_1 \theta_1 + O(k_1^2), \tag{3.5a}$$

$$f_2 = k_1 k_2 \left(\theta_1 \theta_2 + \frac{12}{\alpha(P_1 - P_2)^2} \right) + O(k^3), \tag{3.5b}$$

where

$$\theta_i = x + P_i y - \alpha P_i^2 t, \tag{3.5c}$$

and we have used

$$\exp A_{12} \sim 1 + \frac{12 k_1 k_2}{\alpha(P_1 - P_2)^2}. \tag{3.5d}$$

Since u is given by (3.2) we have therefore deduced the following rational solutions:

$$\hat{f}_1 = \theta_1, \tag{3.6a}$$

$$\hat{f}_2 = \theta_1 \theta_2 + B_{12}, \quad B_{12} = 12/\alpha(P_1 - P_2)^2. \tag{3.6b}$$

Although $\hat{f}_1, \hat{f}_2$ are generally singular at some position, a nonsingular solution is obtained for $\hat{f}_2$ by taking $\alpha = -1$ (in which case the wave system has so called positive dispersion) and $P_2 = P_1^*$. In this case we have

$$\hat{f}_2 = \theta_1 \theta_1^* - 12/(P_1 - P_1^*)^2. \tag{3.6c}$$

Letting, $P_1 = P_R + iP_I$, we have

$$u = 2\partial_x^2 \log\left[(x' + P_R y')^2 + P_I^2 y'^2 + \frac{3}{P_I^2} \right], \tag{3.7a}$$

where

$$x' = x - (P_R^2 + P_I^2)t,$$

$$y' = y + 2P_R t.$$

Alternatively (3.7a) may be written explicitly as

$$u = \frac{4\left(-(x' + P_R y')^2 + P_I^2 y'^2 + \frac{3}{P_I^2} \right)}{\left((x' + P_R y')^2 + P_I^2 y'^2 + \frac{3}{P_I^2} \right)^2} \tag{3.7b}$$

Hence we have a permanent lump solution decaying as $O(1/x^2, 1/y^2)$ for $|x|,|y| \to \infty$, and moving with the velocity $v_x = P_R^2 + P_I^2, V_y = -2P_R$. We note that when $n = 4$ one may obtain a two-lump solution which leave zero asymptotic phase shift after interaction. In general when $N = 2M$ this method yields formulae for an M lump solution. Of course

the methods used here unfortunately do not give strong evidence about the role of these solutions in the general initial value problem, e.g., genericity, stability, etc.

As mentioned earlier, this technique extends readily to higher order solitons. Although we shall, in a future paper discuss the complete case, here we simply present the results of the calculation for $N = 3,4$. We find

$$f_3 = \theta_1 \theta_2 \theta_3 + B_{12} \theta_3 + B_{23} \theta_1 + B_{31} \theta_2 \tag{3.8a}$$

$$\begin{aligned} f_4 = {} & \theta_1 \theta_2 \theta_3 \theta_4 + B_{12} \theta_3 \theta_4 + B_{13} \theta_2 \theta_4 + B_{14} \theta_2 \theta_3 + B_{23} \theta_1 \theta_4 \\ & + B_{24} \theta_1 \theta_3 + B_{34} \theta_1 \theta_2 + B_{12} B_{34} + B_{13} B_{24} + B_{14} B_{23}, \end{aligned} \tag{3.8b}$$

where θ_i is given by (3.5c), $B_{ij} = 12/\alpha(P_i - P_j)^2$. Taking $\alpha = -1$, $P_3 = P_1^*$, $P_4 = P_2^*$ in (3.8b) yields the two-lump solution. We note that with this proviso f_4 is positive, and yields a solution u via (3.2) which decays as $O(1/x^2, 1/y^2)$ for $|x|,|y| \to \infty$. This two-lump solution yields zero asymptotic phase shift.

Our second example is the Boussinesq equation,

$$u_{tt} - u_{xx} - (3u^2)_{xx} - u_{xxxx} = 0. \tag{3.9}$$

We note that the methods we employ work equally well if the sign of the last term in (3.9) is changed and hence the equation will be well posed on the infinite interval. However even with this change the equation remains an isospectral flow. It should also be noted that (3.9) arises in a physical problem (e.g., water waves) as a long wave equation. Hence, in the context of the physical problem, it is well posed.

Following Ref. 16 we let

$$u = 2(\log f_N)_{xx} \tag{3.10}$$

and find the bilinear equation

$$(D_t^2 - D_x^2 - D_x^4) f_N \cdot f_N = 0. \tag{3.11}$$

The first two soliton solutions are (as usual) given by

$$f_1 = 1 + e^{\eta_1} \tag{3.12a}$$

$$f_2 = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}}, \tag{3.12b}$$

where

$$\eta_i = k_i x + \epsilon_i k_i \sqrt{1 + k_i^2}\, t + \eta_i^{(0)}, \quad \epsilon_i = +1 \text{ or } -1. \tag{3.12c}$$

and

$$e^{A_{12}} = \frac{3(k_1 - k_2)^2 + (\epsilon_1 \sqrt{1 + k_1^2} - \epsilon_2 \sqrt{1 + k_2^2})^2}{3(k_1 + k_2)^2 + (\epsilon_1 \sqrt{1 + k_1^2} - \epsilon_2 \sqrt{1 + k_2^2})^2}. \tag{3.12d}$$

For $N = 1$, taking $e^{\eta_1^{(0)}} = -1$ and $k_1 \to 0$, we have

$$f_1 \sim -k_1 (x \pm t). \tag{3.13}$$

For the two-soliton solution, we note

$$e^{A_{12}} \sim \begin{cases} \left(\dfrac{k_1 - k_2}{k_1 + k_2} \right)^2 (1 + \frac{1}{3} k_1 k_2), & \text{for } \epsilon_1 \epsilon_2 = 1, \tag{3.14a} \\ 1 - 3k_1 k_2, & \text{for } \epsilon_1 \epsilon_2 = -1. \tag{3.14b} \end{cases}$$

In the case of $\epsilon_1 \epsilon_2 = 1$, we take

$$e^{\eta_1^{(0)}} = \frac{k_1 + k_2}{k_1 - k_2} + \frac{1}{6} k_1 k_2, \tag{3.15a}$$

$$e^{\eta_2^{(0)}} = -\frac{k_1+k_2}{k_1-k_2} + \frac{1}{6}k_1k_2, \qquad (3.15b)$$

and find

$$f_2 \sim -\frac{1}{6}k_1k_2(k_1+k_2)\{(x \pm t)^3 + (x \pm t) \mp 6t\}. \quad (3.16)$$

It is interesting to notice that f_2 gives another rational solution for the case, $\epsilon_1\epsilon_2 = -1$. In this case, taking $e^{\eta_1^{(0)}} = e^{\eta_2^{(0)}} = -1$ and using (3.14b), we obtain

$$f_2 \sim k_1k_2(x^2 - t^2 - 3). \qquad (3.17)$$

Thus, the first three polynomial solutions of the Boussinesq equation are given by (3.10) with the following f_N;

$$x \pm t,$$
$$x^2 - t^2 - 3,$$
$$(x \pm t)^3 + (x \pm t) \mp 6t.$$

Higher order polynomials can be obtained in this manner and presumably the Bäcklund transformation will yield a recursion relation between rational solutions.

Finally, let us consider the modified KdV equation,

$$v_t + 6v^2v_x + v_{xxx} = 0, \qquad (3.18)$$

with the nonzero asymptotic condition, $v \to v_0$ as $|x| \to \infty$. Following Refs. 5 and 6, we look for a solution v of the form

$$v = v_0 + i(\log f'_N/f_N)_x. \qquad (3.19)$$

Substituting (3.19) into (3.18) and decoupling the resulting equation, we get

$$(D_t + 6v_0^2 D_x + D_x^3)f'_N \cdot f_N = 0, \qquad (3.20a)$$

$$(D_x^2 - 2iv_0 D_x)f'_N \cdot f_N = 0. \qquad (3.20b)$$

In order to obtain soliton solutions, we expand

$$f_N = 1 + \epsilon f_{N,1} + \epsilon^2 f_{N,2} + \cdots, \qquad (3.21a)$$

$$f'_N = 1 + \epsilon f'_{N,1} + \epsilon^2 f'_{N,2} + \cdots, \qquad (3.21b)$$

substitute (3.21) into (3.20), and equate coefficient of ϵ. Starting with $f_{N,1} = e^{\eta_1+\phi_1}$ and $f'_{N,1} = e^{\eta_1+\phi'_1}$, we obtain a one-soliton solution

$$f_1 = 1 + e^{\eta_1+\phi_1}, \qquad (3.22a)$$

$$f'_1 = 1 + e^{\eta_1+\phi'_1}, \qquad (3.22b)$$

where

$$\eta_i = k_i x - (6v_0^2 k_i + k_i^3)t + \eta_i^{(0)}, \qquad (3.22c)$$

$$e^{\phi_j} = 1 + ik_j/2v_0, \qquad (3.22d)$$

$$e^{\phi'_j} = 1 - ik_j/2v_0. \qquad (3.22e)$$

Inserting (3.22) into (3.19), we have an explicit form of the one-soliton solution,

$$v = v_0 + k_1^2/(\sqrt{4v_0^2 + k_1^2}\cosh\eta_1 + 2v_0), \qquad (3.23)$$

which is essentially the same as Eq. (2) in Ref. 12. To get a two-soliton solution, we start with

$$f_{N,1} = \sum_{i=1}^2 e^{\eta_i+\phi_i}, f'_{N,1} = \sum_{i=1}^2 e^{\eta_i+\phi'_i}, \text{ and find}$$

$$f_2 = 1 + e^{\eta_1+\phi_1} + e^{\eta_2+\phi_2} + e^{\eta_1+\eta_2+\phi_1+\phi_2+A_{12}}, \quad (3.24a)$$

$$f'_2 = 1 + e^{\eta_1+\phi'_1} + e^{\eta_2+\phi'_2} + e^{\eta_1+\eta_2+\phi'_1+\phi'_2+A_{12}}, \quad (3.24b)$$

where

$$e^{A_{12}} = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2. \qquad (3.24c)$$

As before, rational solutions are deduced by taking the limit $k_i \to 0$ and choosing the phase constant adequately. For $N = 1$, choosing $e^{\eta_1^{(0)}} = -1$, we get

$$f_1 \sim -k_1\left(x - 6v_0^2 t + \frac{i}{2v_0}\right) \qquad (3.25a)$$

$$f'_1 \sim -k_1\left(x - 6v_0^2 t - \frac{i}{2v_0}\right) \qquad (3.25)$$

which gives the rational solution

$$v = v_0 - \frac{4v_0}{4v_0^2(x - 6v_0^2 t)^2 + 1}. \qquad (3.26)$$

This solution was recently found by Ono,[12] and is a nonsingular one-dimensional algebraic soliton. For $N = 2$, taking

$$e^{\eta_1^{(0)}} = -e^{\eta_2^{(0)}} = \frac{k_1+k_2}{k_1-k_2}\left(1 + \frac{k_1k_2}{8v_0^2}\right),$$

and $k_i \to 0$, we find

$$f_2, f'_2 \sim -\frac{1}{6}k_1k_2(k_1+k_2)$$

$$\times \left[\xi^3 + 12t - \frac{3}{4v_0^2}\xi \pm \frac{3i}{2v_0}\left(\xi^2 + \frac{1}{4v_0^2}\right)\right],$$

$$\qquad (3.27a)$$

where

$$\xi = x - 6v_0^2 t \qquad (3.27b)$$

and the upper (lower) sign stands for $f_2(f'_2)$. Substituting (3.27a) into (3.19), we see that the solution is also a nonsingular algebraic solution, given by

$$v = v_0 - \frac{12v_0(\xi^4 + (3/2v_0^2)\xi^2 - 3/16v_0^4 - 24\xi t)}{4v_0^2(\xi^3 + 12t - (3/4v_0^2)\xi)^2 + 3(\xi^2 + 1/4v_0^2)^2}. \qquad (3.28)$$

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APPENDIX A: DERIVATION OF (2.17) FROM (2.8a)

By taking $\epsilon_i = 2\mu_i - 1$ for $i = 1,2,\ldots,N$, (2.8a) reduces to

$$f_N = \sum_{\epsilon=\pm 1} \exp\left[\sum_{i<j}^N \frac{1}{4}(\epsilon_i+1)A_{ij} + \sum_{i=1}^N \frac{1}{2}(\epsilon_i+1)\eta_i\right]. \quad (A1)$$

We make a convenient choice of phase factors: $\eta_i^{(0)} = \eta_i^{(1)} + \eta_i^{(2)}$, and take,

$$\exp\eta_i^{(1)} = -\prod_{\substack{j=1 \\ j\neq i}}^N \left(\frac{k_j+k_i}{k_j-k_i}\right). \qquad (A2)$$

Defining $\xi_i = k_i x - k_i^3 t + \eta_i^{(2)}$, then (A1) becomes

$$f_N = \sum_{\epsilon=\pm 1}(-1)^{N(N+1)/2}\prod_{i=1}^N \epsilon_i$$

$$\times \prod_{i<j}^{N} \frac{\epsilon_i k_i - \epsilon_j k_j}{k_i - k_j} \exp \sum_{i=1}^{N} \tfrac{1}{2}(\epsilon_i + 1)\xi_i, \qquad \text{(A3)}$$

where we have used the relations

$$(-1)^{\sum_{i=1}^{N}(1/2)(\epsilon_i+1)} = (-1)^N \prod_{i=1}^{N} \epsilon_i, \qquad \text{(A4)}$$

$$\prod_{i<k}^{N} \left(\frac{k_i - k_j}{k_i + k_j} \right)^{(1/2)(\epsilon_i+1)(\epsilon_j+1)} \times \prod_{i=1}^{N}\prod_{\substack{j=1 \\ j\neq i}}^{N} \left(\frac{k_i + k_j}{k_i - k_j} \right)^{(1/2)(\epsilon_i+1)}$$

$$= (-1)^{N(N-1)/2} \prod_{i<j}^{N} \frac{\epsilon_i k_i - \epsilon_j k_j}{k_i - k_j}, \qquad \text{(A5)}$$

and,

$$(-1)^{(1/2)(\epsilon_i+1)} \left(\frac{k_i - k_j}{k_i + k_j} \right)^{[(1/2)(\epsilon_i+1)(\epsilon_j+1) - \frac{1}{2}(\epsilon_i+1) - \frac{1}{2}(\epsilon_j+1)]}$$

$$= -\frac{\epsilon_i k_i - \epsilon_j k_j}{k_i - k_j}, \qquad \text{(A6)}$$

for each i,j. Using, $f \leftrightarrow g$ iff $f = e^{ax+b}g$ we have

$$f_N \leftrightarrow \sum_{\epsilon=\pm 1} \prod_{i<j}^{N} \frac{\epsilon_i k_i - \epsilon_j k_j}{k_i^2 - k_j^2} \frac{1}{\prod_{i=1}^{N} \epsilon_i k_i} \exp\left(\sum_{i=1}^{N} \tfrac{1}{2}\epsilon_i \eta_i \right). \qquad \text{(2.17)}$$

APPENDIX B: DERIVATION OF (2.21 a,b) FROM THE INVERSE SCATTERING TRANSFORM OF THE KdV EQUATION

It is well known that the inverse scattering transform for the KdV equation (2.1) may be written as

$$\psi_{xx} + u\psi = \tfrac{1}{4}k^2 \psi, \qquad \text{(B1)}$$

$$\psi_t + 6u\,\psi_x + 3u_x\psi + 4\psi_{xxx} = 0. \qquad \text{(B2)}$$

Taking

$$\psi = f_{N+1}/f_N, \qquad \text{(B3)}$$

and using (2.2), we have from (B1)

$$\frac{1}{f_N^2} D_x^2 f_{N+1} \cdot f_N - \frac{f_{N+1}}{f_N^3} D_x^2 f_N \cdot f_N + \frac{f_{N+1}}{f_N^3} D_x^2 f_N \cdot f_N$$

$$= \tfrac{1}{4}k^2 \frac{f_{N+1}}{f_N}$$

which turns out to be (2.21a). By means of (B1), (B2) may be rewritten as

$$\psi_t - \tfrac{3}{4}k^2\psi_x + \psi_{xxx} + 3u\psi_x = 0. \qquad \text{(B4)}$$

Substituting (2.2) and (B3) into (B4) yields

$$D_t f_{N+1} \cdot f_N + \tfrac{3}{4}k^2 D_x f_{N+1} \cdot f_N + D_x^3 f_{N+1} \cdot f_N = 0. \qquad \text{(2.21b)}$$

From this derivation of the Bäcklund transformation [it may be shown that an $N+1$ soliton solution is produced from an N soliton solution via (2.21a,b)] we see that the eigenfunction ψ in (B1) and (B2) is regarded as a generating function of a new soliton solution (with parameter k) from a previous soliton solution of the KdV equation.

APPENDIX C: DERIVATION OF (2.22) FROM (2.21a)

For four solutions, $f_{N-1}, f_N, \tilde{f}_N$, and f_{N+1} [with parameters as defined following (2.21)], (2.21a) gives

$$(D_x^2 - \tfrac{1}{4}k_N^2)f_{N-1} \cdot f_N = 0, \qquad \text{(C1)}$$

$$(D_x^2 - \tfrac{1}{4}k_{N+1}^2)f_{N-1} \cdot \tilde{f}_N = 0, \qquad \text{(C2)}$$

$$(D_x^2 - \tfrac{1}{4}k_{N+1}^2)f_N \cdot f_{N+1} = 0, \qquad \text{(C3)}$$

$$(D_x^2 - \tfrac{1}{4}k_N^2)\tilde{f}_N \cdot f_{N+1} = 0. \qquad \text{(C4)}$$

Multiplying (C1) by $\tilde{f}_N f_{N+1}$ and (C4) by $f_{N-1}f_N$, and subtracting each other, we have

$$\tilde{f}_N f_{N+1}(D_x^2 f_{N-1} \cdot f_N) - f_{N-1} f_N(D_x^2 \tilde{f}_N \cdot f_{N+1}) = 0. \qquad \text{(C5)}$$

There exists an identity among four functions of $x(a, b, c,$ and $d)$,

$$(D_x^2 a \cdot b)cd - ab(D_x^2 c \cdot d)$$

$$= a_{xx}bcd - 2a_x b_x cd + ab_{xx}cd - abc_{xx}d$$

$$+ 2abc_x d_x - abcd_{xx} \qquad \text{(C6)}$$

$$= (a_x d - a d_x)_x bc - (a_x d - a d_x)(bc)_x$$

$$+ (ad)_x(c_x b - c b_x) - ad(c_x b - c b_x)_x$$

$$= D_x[(D_x a \cdot d) \cdot bc + (ad) \cdot (D_x c \cdot b)].$$

Using this identity, (C5) is reduced to

$$D_x[(D_x f_{N-1} \cdot f_{N+1}) \cdot (\tilde{f}_N f_N)$$

$$+ (f_{N-1} f_{N+1}) \cdot (D_x \tilde{f}_N \cdot f_N)] = 0. \qquad \text{(C7)}$$

Similarly, we get from (C2) and (C3),

$$D_x[(D_x f_N \cdot \tilde{f}_N) \cdot (f_{N-1} f_{N+1})$$

$$+ (f_N \tilde{f}_N) \cdot (D_x f_{N-1} \cdot f_{N+1})] = 0. \qquad \text{(C8)}$$

Subtracting (C8) from (C7) and noticing $D_x a \cdot b = -D_x b \cdot a$, we have

$$D_x(D_x f_{N-1} \cdot f_{N+1}) \cdot (f_N \tilde{f}_N) = 0, \qquad \text{(C9)}$$

which means $D_x f_{N-1} \cdot f_{N+1}$ is proportional to $f_N \tilde{f}_N$, i.e.,

$$D_x f_{N-1} \cdot f_{N+1} = C f_N \tilde{f}_N. \qquad \text{(C10)}$$

The constant, C, is determined by any three soliton solutions (at any location) given by (2.17). For example, using

$$f_0 = 1$$

$$f_1 = \frac{1}{k_1}(e^{\eta_1/2} - e^{-\eta_1/2})$$

$$f_2 = \frac{1}{k_1 k_2(k_1^2 - k_2^2)} [(k_1 - k_2)(e^{(\eta_1+\eta_2)/2} - e^{-(\eta_1+\eta_2)/2})$$

$$+ (k_1 + k_2)(e^{-(\eta_1-\eta_2)/2} - e^{(\eta_1-\eta_2)/2})],$$

we find $C = -\tfrac{1}{2}$, whereupon the superposition formula is given by

$$D_x f_{N+1} \cdot f_{N-1} = \tfrac{1}{2} f_N \tilde{f}_N. \qquad \text{(2.22)}$$

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Solutions of the three-magnon bound state equation. III. The physical eigenstate

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The physical eigenvalue and eigenfunction are found analytically by directly solving the three-magnon bound state equation in one dimension. The structure of the equation suggests a simple ansatz for the eigenfunction, and the calculation is reduced to the solution of a finite number of linear algebraic equations. The method is elementary, but the explicit solution is found after a long calculation. The connection of the wavefunction with previous results is also established.

I. INTRODUCTION

In previous publications several unphysical solutions of the three-magnon bound state equation were found.[1,2] In this paper we shall give a complete analytic solution of that equation to derive the physical bound state eigenfunction as well as the eigenvalue in one dimension.

The three-magnon bound state equation was derived by using Faddeev's general method[3] of the quantum mechanical three-body problem in the Heisenberg linear chain.[4] Consider the Hamiltonian of the isotropic Heisenberg spin $-1/2$ linear chain.

$$H=-\frac{1}{2}J\sum_{i,\delta}\mathbf{S}_i\cdot\mathbf{S}_{i+\delta}. \tag{1}$$

$J > 0$, i goes from 1 to N, and δ joins a site to its nearest neighbors (also $N+1\equiv1$). The Dyson transformation[5] leads to the ideal Hamiltonian

$$H=\tfrac{1}{2}J\sum_{j,\delta} \eta_j^*(\eta_j-\eta_{j+\delta})$$

$$+\tfrac{1}{4}J\sum_{j,\delta} \eta_j^*\eta_{j+\delta}^*(\eta^2_j+\eta^2_{j+\delta}-2\eta_j\eta_{j+\delta}) \tag{2}$$

η^*_j and η_j are creation and annihilation operators, respectively, of an ideal spin deviation quantum at site j. The ground state has all the spins aligned. When a single spin-deviation quantum is created, that is, when a spin is reversed, we get the spin wave by the motion of this reversed spin in the chain. When two spins are reversed, we get a continuum of scattering states of spin waves and a branch of the bound two magnon states split off below the continuum. Similarly, when three spins are reversed, the bound state of the three spin deviations are determined by the equation[4]:

$$\Psi(p_1)=\frac{2\cos^2\tfrac{1}{2}p_1}{[3-E-\cos(K-p_1)]}\left(1-\frac{\tfrac{1}{2}(3-E)-\tfrac{1}{2}\cos(K-p_1)-\cos^2\tfrac{1}{2}p_1}{\{[\tfrac{1}{2}(3-E)-\tfrac{1}{2}\cos(K-p_1)]^2-\cos^2\tfrac{1}{2}p_1\}^{1/2}}\right)^{-1}$$

$$\times\frac{1}{\pi}\int_{-\pi}^{\pi}dp_2\frac{[\cos\tfrac{1}{2}p_1-\cos(K-\tfrac{1}{2}p_1-p_2)]\cos(K-p_1-\tfrac{1}{2}p_2)}{\tfrac{1}{2}[E-3+\cos(K-p_1)+\cos(K-p_2)+\cos(K-p_1-p_2)]}\Psi(p_2). \tag{3}$$

E is the eigenvalue of the three-particle bound state in units of J. K is the total momentum of three spin deviations. Equation (3) is derived from the homogeneous part of the exact Faddeev equations for the problem of three spin deviations.

The physical eigenvalue of the three-magnon bound state

$$E=\tfrac{1}{3}(1-\cos K) \tag{4}$$

was found by Bethe[6] by using his ansatz for the wavefunction of the Hamiltonian (1). The value (4) was numerically verified for Eq. (3) by Majumdar.[4] Van Himbergen and Tjon[1] constructed the wavefunction of the physical bound state by adopting the Bethe ansatz for a linear chain of the transformed Hamiltonian, Eq. (2), but because of the vast amount of algebra they were forced to verify (3) numerically. Gochev[7] has applied a special ansatz to the anisotropic linear chain and found a generalization of Bethe's recursion relation for the eigenvalue of multimagnon bound states. In fact his calculation has furnished recursion relations to cal-culate all multimagnon bound states starting from two-magnon bound states. More recently Van Himbergen[8] has constructed another wavefunction for the linear chain (1) projecting out unphysical parts that come from the use of the Dyson transformation.

A complete analytical derivation of both the eigenvalue and the eigenfunction of Eq. (3) has so far been lacking; it will be supplied in this paper. The method is completely elementary, involving nothing more than simple contour integration and partial fraction decomposition of rational functions. But the algebra is indeed enormous, as remarked by Van Himbergen and Tjon.[1] Before going into the details of the general solution, we shall work out in Sec. III the special case of $K=\pi$ where the algebra is presentable and the arguments can be followed in all its details. Section IV gives the details of the general solution. Finally Sec. V explores the still imperfectly understood connection between the general method and Bethe's ansatz. Finally we discuss some problems that still remain to be cleared up in the solution of the three magnon bound state equation.

II. TRANSFORMATION OF Eq. (3)

Introducing the variables

$$x=\tan\tfrac{1}{2}p_1, y=\tan\tfrac{1}{2}p_2, \qquad (5)$$

we transform Eq. (3) into the following equation:

$$\Psi(x)=4[\pi(a+\cos K)(x^2+1)^{1/2}$$

$$\times\left(x^2-2x\frac{\sin K}{a+\cos K}+\frac{a-\cos K}{a+\cos K}\right)\left(1-\frac{f}{d}\right)]^{-1}$$

$$\times\int_{-\infty}^{\infty}\frac{dy}{(y^2+1)^{3/2}}\left(1-\frac{(y^2+1)f}{(x^2+1)(y-z)(y-\bar{z})}\right)$$

$$\times(yg+h)\Psi(y) \qquad (6)$$

where we use the abbreviations

$$a=3-E, \qquad (7)$$

$$f=x^2-2x\frac{\sin K}{a+\cos K}+\frac{a-2-\cos K}{a+\cos K}, \qquad (8)$$

$$d^2=x^4-4x^3\frac{\sin K}{a+\cos K}+2x^2\frac{a^2-3\cos^2 K}{(a+\cos K)^2}$$

$$-4x\frac{\sin K(a-\cos K)}{(a+\cos K)^2}+\frac{(a-\cos K)^2-4}{(a+\cos K)^2}, \qquad (9)$$

$$z=\frac{2(\sin K-x\cos K)}{(x^2+1)(a+\cos K)}+\frac{id}{x^2+1}, \qquad (10)$$

$$g=x^2\sin K+2x\cos K-\sin K, \qquad (11)$$

$$h=x^2\cos K-2x\sin K-\cos K. \qquad (12)$$

$\bar{z}$ is the complex conjugate of z and the pole $y=z$ lies in the upper half plane if $d>0$, which we take to be the case. The eigenvalue E occurs in a and for the physical problem E is real. An important identity that simplifies the algebra is[9]

$$(z^2+1)(\bar{z}^2+1)=16/[(x^2+1)(a+\cos K)^2]. \qquad (13)$$

Notice the branch points $y=\pm i$ apart from the poles $y=z, \bar{z}$ under the integral. The branch point is reduced to a pole if the solution has a factor $(x^2+1)^{-1/2}$ which occurs actually as a prefactor of the integral. These are two more x dependent prefactors; one is an explicit quadratic in the denominator.

But we have also from (8) and (9)

$$d^2-f^2=\frac{4(a-1+\cos K)}{(a+\cos K)^2}$$

$$\times\left(x^2-2x\frac{\sin K}{a-1+\cos K}+\frac{a-2-\cos K}{a-1+\cos K}\right) \qquad (14)$$

Thus if we rationalize the denominator, we get another quadratic x-dependent prefactor. As an ansatz for the physical solution, we assume that all the terms in the denominator of $\Psi(x)$ are given by these prefactors so that we write

$$\Psi(x)=(x^2+1)^{-1/2}$$

$$\times\left(x^2-2x\frac{\sin K}{a+\cos K}+\frac{a-\cos K}{a+\cos K}\right)^{-1}$$

$$\times\left(x^2-2x\frac{\sin K}{a-1+\cos K}+\frac{a-2-\cos K}{a-1+\cos K}\right)^{-1}F(x).$$

$$(15)$$

F now satisfies

$$F(x)\Big/\left(x^2-2x\frac{\sin K}{a-1+\cos K}+\frac{a-2-\cos K}{a-1+\cos K}\right)$$

$$=\frac{4}{\pi(a+\cos K)[1-(f/d)]}$$

$$\times\int_{-\infty}^{\infty}\frac{dy}{(y^2+1)^2}\left[1-\frac{(y^2+1)f}{(x^2+1)(y-z)(y-\bar{z})}\right]$$

$$\times(yg+h)F(y)\left(y^2-2y\frac{\sin K}{a+\cos K}+\frac{a-\cos K}{a+\cos K}\right)^{-1}$$

$$\times\left(y^2-2y\frac{\sin K}{a-1+\cos K}+\frac{a-2-\cos K}{a-1+\cos K}\right)^{-1}. \qquad (16)$$

We keep the factor $[1-(f/d)]$ to exploit its cancellation against the contribution of the $y=z$ pole later. Equation (15) exhibits, by hypothesis, all the denominators of $\Psi(x)$. So it follows, by counting of powers in the integral, that $F(y)$ can have only a finite number of powers of y. In fact F can be at most a fifth-degree polynomial

$$F(x)=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5. \qquad (17)$$

Equations (15) and (17) complete the ansatz for our solution. Apart from the eigenvalue, we have to determine six quantities, c_0 to c_5. As the integral equation is linear and homogeneous, we have actually to fix five of these six constants. Notice now that the left-hand side of Eq. (16) is a simple rational function in x.

III. SOLUTION FOR THE SPECIAL CASE $K=\pi$

For $K=\pi$, Eq. (16) is

$$\frac{F(x)}{x^2+[(a-1)/(a-2)]}$$

$$=\frac{4}{\pi(a-1)[1-(f/d)]}\int_{-\infty}^{\infty}\frac{dy}{(y^2+1)^2}$$

$$\times\left[1-\frac{(y^2+1)f}{(x^2+1)(y-z)(y-\bar{z})}\right]$$

$$\times(-2xy-x^2+1)F(y)$$

$$\times\left(y^2+\frac{a-1}{a-2}\right)^{-1}\left(y^2+\frac{a+1}{a-1}\right)^{-1}. \qquad (18)$$

f is simply (x^2+1) and can be calculated from (9) and (10) by putting $K=\pi$. F is even in x, so we write

$$F(x)=c_0+c_2x^2+c_4x^4. \qquad (19)$$

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The eigenvalue condition is arrived at as follows. Substituting (19) into (18), and carrying out the integral, we find some terms in front of which stands the factor $[1-(f/d)]^{-1}$ containing the square root branch cut in d. However, in the contribution from the pole $y=z$ we get a factor f/d because $z-\bar{z}=2id/(x^2+1)$. This term can thus give a contribution which is free of the branch cut $[1-(f/d)]^{-1}$ and is in fact a rational function in x. The left-hand side of (18) is a rational function. We now equate these two rational contributions on two sides, hoping that by suitable choice of the constants c_i and the correct eigenvalue a, the remaining terms involving the factor $[1-(f/d)]^{-1}$ vanish. If two rational functions are to be equal, by the partial fraction decomposition, the poles on the two sides must be the same. The denominator of the left side of Eq. (18) must be a factor of the denominator of the right-hand side. This provides the eigenvalue condition.

Carrying out the integral in (18), we get

$$\frac{c_0+c_2x^2+c_4x^4}{x^2+[(a-1)/(a-2)]}$$

$$=\frac{1}{1-(f/d)}\left[-\frac{(a-1)^{5/2}(a-2)}{(a+1)^{1/2}(a-3)}(1-x^2)\left\{c_0-c_2\frac{a+1}{a-1}+c_4\left(\frac{a+1}{a-1}\right)^2\right\}\right.$$

$$+\frac{4(a-2)^{7/2}}{(a-1)^{1/2}(a-3)}(1-x^2)\left\{c_0-c_2\frac{a-1}{a-2}+c_4\left(\frac{a-1}{a-2}\right)^2\right\}+(a-2)(1-x^2)\{-3(a-2)c_0+(3a-4)c_2-(3a-2)c_4\}\right]$$

$$-\frac{x^2+1}{1-(f/d)}\left[\tfrac{1}{2}(a-1)(a-2)(c_0-c_2+c_4)+\frac{(a-1)^{5/2}(a-2)}{(a+1)^{1/2}(a-3)}\left\{c_0-c_2\frac{a+1}{a-1}+c_4\left(\frac{a+1}{a-1}\right)^2\right\}-\frac{4(a-2)^{7/2}}{(a-1)^{1/2}(a-3)}\right.$$

$$\times\left\{c_0-c_2\frac{a-1}{a-2}+c_4\left(\frac{a-1}{a-2}\right)^2\right\}+\frac{8(a-2)^{7/2}}{(a-1)^{3/2}}\left\{c_0-c_2\frac{a-1}{a-2}+c_4\left(\frac{a-1}{a-2}\right)^2\right\}\frac{3x^2+[(3a-7)/(a-1)]}{x^4+2x^2(5a-9)/(a-1)+[(3a-7)/(a-1)]^2}\right]$$

$$+\frac{f/d}{1-(f/d)}\left[x^2\left\{-c_2(a-3)(a-1)+c_4\frac{(a-1)(2a^2-5a+1)}{a-2}\right\}-c_2(a-1)^2+\frac{c_4(a-1)(2a^2-a-5)}{(a-2)}\right]$$

$$-\frac{f/d}{2[1-(f/d)]}(a-1)(a-2)\left\{c_0-c_2\frac{a-1}{a-2}+c_4\left(\frac{a-1}{a-2}\right)^2\right\}\left(x^6\frac{5a-11}{a-1}+x^4\frac{7a^3+3a^2-91a+113}{(a-1)^3}\right.$$

$$\left.-x^2\frac{a^2-36a+67}{(a-1)^2}-\frac{3a-7}{a-1}\right)\left[x^4+2x^2\frac{5a-9}{a-1}+\left(\frac{3a-7}{a-1}\right)^2\right]^{-1}. \tag{20}$$

The last two terms come from the pole $y=z$. From these a term free of the square-root branch cut can be extracted by writing (f/d) in the numerator as $1-[1-(f/d)]$. This part is a rational function of x. The denominator on the right is a quartic:

$$x^4+2x^2\frac{5a-9}{a-1}+\left(\frac{3a-7}{a-1}\right)^2, \tag{21}$$

while that on the left side is a quadratic

$$x^2+[(a-1)/(a-2)]. \tag{22}$$

Demanding that (22) be a factor of (21), we get the eigenvalue condition

$$3a-7=0 \text{ or } a=7/3. \tag{23}$$

The other factor of the denominator on the right cancels now with the numerator, and equating the coefficients of equal powers of x, with (23), we get the equation

$$c_0-7c_2+4c_4=0. \tag{24}$$

All the powers of x give the same equation. The term with the prefactor $[1-(f/d)]^{-1}$ now can be made to vanish if we choose

$$(16\sqrt{10}/5-11)c_0+(29-8\sqrt{10})c_2+(-65+20\sqrt{10})c_4=0. \tag{25}$$

Solving (24) and (25) we get

$$c_0/(-113+36\sqrt{10})=c_2/(-7+\frac{12}{5}\sqrt{10})=c_4/(16-\frac{24}{5}\sqrt{10}). \tag{26}$$

Equations (23), (24), and (25) are the three crucial features and will have their counterparts in the more general case described below.

IV. SOLUTION FOR ARBITRARY K

Substituting (17) into (16), we can easily carry out the integration by closing the contour in the upper half plane. Only the pole $y=z$ will give a contribution having f/d, because $z-\bar{z}=2id/(x^2+1)$. From this term we extract the rational part by writing $f/d=1-(1-f/d)$ and canceling the branch cut in $[1-(f/d)]^{-1}$. The denominator on the right side is again a quartic

$$(5+4\cos K)x^4 - 8x^3\frac{\sin K(a+1+2\cos K)}{a+\cos K}$$

$$+2x^2\frac{(5a^2-4a-2a\cos K-12a\cos^2 K+8+12\cos K+\cos^2 K-12\cos^3 K)}{(a+\cos K)^2}+8x\frac{\sin K(-a^2+3a+a\cos K-4-\cos K+2\cos^2 K)}{(a+\cos K)^2}$$

$$+\frac{5a^2-4a^2\cos K-24a+18a\cos K+32-24\cos K-3\cos^2 K+4\cos^3 K}{(a+\cos K)^2}. \tag{27}$$

The denominator on the left side is the quadratic

$$x^2-2x\frac{\sin K}{a-1+\cos K}+\frac{a-2-\cos K}{a-1+\cos K}. \tag{28}$$

Requiring that (28) be a factor of (27), we get the eigenvalue condition

$$3a-8-\cos K=0 \text{ or } E=\frac{1}{3}(1-\cos K). \tag{29}$$

Now equate the rational parts of the two sides, and demand that all the terms with the prefactor $[1-(f/d)]^{-1}$ vanish. For $K=\pi$, we get exactly the requisite number of equations for determining the three constants. Here we get more than six homogeneous linear equations for determining c_0 to c_5. But it is easy to check that they are not linearly independent. In fact, having solved for the six quantitic c_0 to c_5 from six homogeneous linear equations all the remaining equations are shown to be identically satisfied. The calculations are elementary, but extremely long and tedious, and not in any way illuminating. We shall simply write down the final solutions for the constants c_0 to c_5.

$$c_0(1-\cos K)^{-1}[(11+\cos K)^{1/2}(-7\cos^3 K-98\tfrac{1}{4}\cos^2 K-48\cos K+1469\tfrac{1}{2})$$

$$+(5+\cos K)^{1/2}(7\cos^3 K+120\cos^2 K+189\cos K-2179)]^{-1}$$

$$=c_1(\sin K)^{-1}[(11+\cos K)^{1/2}(35\cos^3 K+558\tfrac{3}{4}\cos^2 K+951\cos K-5493\tfrac{1}{2})]$$

$$+(5+\cos K)^{1/2}(-35\cos^3 K-669\cos^2 K-1851\cos K+8144)]^{-1}$$

$$=c_2(1-\cos K)^{-1}[(11+\cos K)^{1/2}(49\cos^3 K+1124\tfrac{1}{4}\cos^2 K+5443\tfrac{1}{2}\cos K+4460)$$

$$+(5+\cos K)^{1/2}(-49\cos^3 K-1285\tfrac{1}{2}\cos^2 K-7695\cos K-6603\tfrac{1}{2})]^{-1}$$

$$=c_3(\sin K)^{-1}[(11+\cos K)^{1/2}(7\cos^3 K-422\tfrac{1}{4}\cos^2 K-3618\cos K-6112)$$

$$+(5+\cos K)^{1/2}(-7\cos^3 K+420\cos^2 K+4864\tfrac{1}{2}\cos K+9059\tfrac{1}{2})]^{-1}$$

$$=c_4(1-\cos K)^{-1}[(11+\cos K)^{1/2}(56\cos^3 K+637\tfrac{1}{2}\cos^2 K+2026\tfrac{1}{2}\cos K+1249)$$

$$+(5+\cos K)^{1/2}(-56\cos^3 K-793\tfrac{1}{2}\cos^2 K-2893\tfrac{1}{2}\cos K-1846)]^{-1}$$

$$=c_5(\sin K)^{-1}(2+\cos K)^{-1}[(11+\cos K)^{1/2}(-28\cos^2 K-340\cos K-1009)$$

$$+(5+\cos K)^{1/2}(28\cos^2 K+421\cos K+1495)]^{-1}. \tag{30}$$

c_1, c_3, and c_5 are proportional to $\sin K$ and vanish for $K=\pi$; $\Psi(x)$ then becomes even in x as discussed in Sec. II. When $|x|\to\infty$ for general K, $\Psi(x)$ does not vanish, and is not therefore square integrable in $-\infty<x<\infty$. We have checked the solutions (30) also against our own numerical computations[4] and those of van Himbergen and Tjon.[1]

V. CONTACT WITH OTHER SOLUTIONS

To construct the three spin-deviation spectrum, Bethe[6] wrote the eigenfunctions as

$$\psi=\sum_{m_1 m_2 m_3} a(m_1 m_2 m_3)S^+_{m_1}S^+_{m_2}S^+_{m_3}|0\rangle. \tag{31}$$

$|0\rangle$ is the state with all spins down. The coefficient $a(m_1 m_2 m_3)$ is given by the ansatz

$$a(m_1 m_2 m_3)=\sum_P \exp\left\{i\sum_{l=1}^{3}f_{Pl}m_l+\tfrac{1}{2}i\sum_{l<j}^{1,3}\phi_{Pl,Pj}\right\}. \tag{32}$$

The permutations P constitute the symmetric group S_3 of 1,2,3. The three wave vectors f_i and the three functions ϕ_{ij} are shown to satisfy the equations

$$2\cot\tfrac{1}{2}\phi_{k,l} = \cot\tfrac{1}{2}f_k - \cot\tfrac{1}{2}f_l, \quad -\pi \leqslant \phi_{k,l} \leqslant \pi, \quad (33)$$

$$Nf_i = 2\pi\lambda_i + \sum_k{}' \phi_{i,k}. \quad (34)$$

λ_i are integers $0,1,...,N-1$. For the bound state some of the wave vectors f_i are complex. We write $f_1 = u + iv$, $f_2 = u - iv$, $f_3 = K - 2u$, such that K is the center-of-mass momentum equal to the sum $f_1 + f_2 + f_3$. Neglecting terms of order e^{-N} Bethe showed that

$$u = \cot^{-1}\frac{1}{3}z - \cot^{-1}z, \quad (35)$$

$$e^{-2v} = (z^2 + 1)/(z^2 + 9). \quad (36)$$

$$z = 3\cot\tfrac{1}{2}K. \quad (37)$$

The bound state eigenvalue E determined by Bethe is given by

$$E = 6/(z^2 + 9). \quad (38)$$

The bound state wavefunction is constructed by van Himbergen and Tjon[1] using again the Bethe-type ansatz on the linear chain after Dyson—Maleev transformation. This is written as

$$F(p,k;u,v)/[a + \cos(K - p)][\cos(K - u - p) - \cosh v]. \quad (39)$$

u, v are parameters of the bound state equation and determined again by the equations (35) to (37). The numerator $F(p,K;u,v)$ is expressed as a complicated function of the relevant variables.

The equations (35) to (37) are also the basis of the more recent calculation of Van Himbergen.[8]

To make contact with Eqs. (35) to (37), we rationalize the denominator of the branch cut in the prefactor of Eq. (3) and obtain

$$\Psi(p_1) = \frac{\{[a - \cos(K - p_1)]^2 - 4\cos^2\tfrac{1}{2}p_1\}^{1/2}(\{[a - \cos(K - p_1)]^2 - 4\cos^2\tfrac{1}{2}p_1\}^{1/2} + a - \cos(K - p_1) - 2\cos^2\tfrac{1}{2}p_1\})}{2\cos\tfrac{1}{2}p_1[a - \cos(K - p_1)]\left[a - \dfrac{3}{2} - (\cos K + \tfrac{1}{2})\cos p_1 - \sin K \sin p_1\right]}$$

$$\times \frac{1}{\pi}\int_{-\pi}^{\pi} dp_2 \left[\frac{a - \cos(K - p_1) - 2\cos^2\tfrac{1}{2}p_1}{a - \cos(K - p_1) - \cos(K - p_2) - \cos(K - p_1 - p_2)} - 1\right]\cos(K - p_1 - \tfrac{1}{2}p_2)\Psi(p_2). \quad (40)$$

The two factors that become quadratics in $x = \tan\tfrac{1}{2}p$ in (15) are easily identified. To make the solution look like (39), we put

$$A \cosh v = a - \tfrac{3}{2},$$

$$A \cos(K - u) = \cos K + \tfrac{1}{2},$$

$$A \sin(K - u) = \sin K. \quad (41)$$

Then, with the definition (37),

$$A^2 = \tfrac{1}{4}(5 + 4\cos K) = \tfrac{9}{4}(z^2 + 1)(z^2 + 9)^{-1}. \quad (42)$$

Also,

$$\cot(K - u) = \frac{\cos K + \tfrac{1}{2}}{\sin K}$$

$$= \frac{3\cot^2\tfrac{1}{2}K - 1}{4\cot\tfrac{1}{2}K}.$$

So

$$K - u = \cot^{-1}(\cot\tfrac{1}{2}K) + \cot^{-1}(3\cot\tfrac{1}{2}K).$$

Hence

$$u = \tfrac{1}{2}K - \cot^{-1}(3\cot\tfrac{1}{2}K) = \cot^{-1}(\tfrac{1}{3}z) - \cot^{-1}z.$$

v is expressed in terms of the eigenvalue through (41). Putting the eigenvalue in (41), we get

$$A \cosh v = \tfrac{3}{2}[(z^2 + 5)/(z^2 + 9)]. \quad (43)$$

Combining (42) and (43), we get back Bethe's solution (36).

Gochev[7] obtained the eigenvalue of the bound states in a longitudinally anisotropic chain by a special ansatz. Our general method also works for this case, and we can reproduce his eigenvalue.

VI. DISCUSSION

We have found three types of solutions for the bound state equation. Once we go over from the Heisenberg linear chain to Dyson's ideal spin waves, three types of solutions might have been expected: (a) the physical solution maintaining proper kinematical restrictions that two spin$-1/2$ deviations never occur at the same site; (b) an unphysical solution with two of three spin deviations occupying the same site, and (c) another unphysical solution with all three spin deviations on the same site. The work of van Himbergen and Tjon clearly shows this. However, the purely mathematical problem of showing that we have found all the solutions has not been solved. This is made harder by the fact that the acceptable solutions $\Psi(x)$ do not vanish $x \to \pm\infty$, that is, they are not square integrable but are merely bounded.

An over-all improvement of the calculational technique is desirable. Our method is elementary, but the algebra is too long. The eigenvalue and the simple analyticity properties of the eigenfunction suggest that a shorter method out to be found. We have noticed several transformations different from (5) that might simplify that calculation. We hope to

discuss these transformations in detail in further work on this problem.

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Sequences of $Z_2 \oplus Z_2$ graded Lie algebras and superalgebras[a]

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Applying methods similar to those used for classical Lie superalgebras (Z_2 graded algebras), we construct sequences of $Z_2 \oplus Z_2$ graded Lie superalgebras. In this way one obtains the $\mathrm{spl}(m,n,r,s)$, $\mathrm{osp}(m,n,r,s)$, $P_1(m,r)$, $P_3(m,n)$, $\mathrm{osp}P_3(m,n)$, $P_{1,2}(m)$, and $\tilde{Q}(m)$ series. We also give series of $Z_2 \oplus Z_2$ graded Lie algebras. Closed forms for superdeterminants and determinants of $Z_2 \oplus Z_2$ graded matrices are presented.

I. INTRODUCTION

Generalized Lie algebras and superalgebras can be defined in the following way.[1] Consider a set of generators $X_{\alpha,i}, X_{\beta,j}, X_{\gamma,k}$, where α, β, γ, (called *grading vectors*) are n-dimensional vectors whose components are real or complex numbers and $i,j,k,\cdots$ are supplementary indices. We define a bilinear mapping into the complex numbers (α, β) satisfying the following conditions

$$(\alpha,\beta) + (\beta,\alpha) = 2r, \tag{1.1a}$$

$$(\alpha,\beta+\gamma) = (\alpha,\beta) + (\alpha,\gamma) + 2s, \tag{1.1b}$$

where r and s are integer numbers [note, that in general $(\alpha,\beta) \neq (\beta,\alpha)$]. The Lie product

$$\langle X_{\alpha,i}, X_{\beta,j} \rangle = C^{\alpha+\beta,k}_{\alpha,i;\beta,j} X_{\alpha+\beta,k} \tag{1.2}$$

has the symmetry property

$$\langle X_{\alpha,i}, X_{\beta,j} \rangle = -(-1)^{(\alpha,\beta)} \langle X_{\beta,j}, X_{\alpha,i} \rangle \tag{1.3}$$

and satisfies the generalized Jacobi identities

$$\langle X_{\alpha,i}, \langle X_{\beta,j}, X_{\gamma,k} \rangle \rangle (-1)^{(\gamma,\alpha)}$$
$$+ \langle X_{\beta,j}, \langle X_{\gamma,k}, X_{\alpha,i} \rangle \rangle (-1)^{(\alpha,\beta)}$$
$$+ \langle X_{\gamma,k}, \langle X_{\alpha,i}, X_{\beta,j} \rangle \rangle (-1)^{(\beta,\gamma)} = 0. \tag{1.4}$$

In Eq. (1.2) the coefficients $C^{\alpha+\beta,k}_{\alpha,i;\beta,j}$ are the structure constants. If we have

$$(\alpha,\alpha) = 2r' \tag{1.5}$$

for all α (r' is an integer number) we have a generalized Lie algebra. If for at least one grading vector ω

$$(\omega,\omega) = 2s' + 1 \tag{1.6}$$

(s' is an integer number), we have a generalized Lie superalgebra. In Ref. 1, several examples of generalized Lie algebras and superalgebras are considered and some possible physical applications are discussed.

In the present paper we specialize to $Z_2 \oplus Z_2$ algebras. In this case the grading vectors α are two-dimensional, having components which are integer numbers modulo 2. For $Z_2 \oplus Z_2$ generalized Lie algebras the mapping (α,β) is

$$(\alpha,\beta) = \alpha_1\beta_2 - \alpha_2\beta_1. \tag{1.7}$$

We will denote them by $C(2,a)$ [because the grading vectors are two-dimensional, "a" because the scalar product (1.7) is antisymmetric].

$Z_2 \oplus Z_2$ generalized Lie superalgebras are obtained by taking the scalar product (α,β) to be symmetric,

$$(\alpha,\beta) = \alpha_1\beta_1 + \alpha_2\beta_2. \tag{1.8}$$

We denote them by $C(2,s)$ [2 because the grading vectors are two-dimensional, "s" because the scalar product (1.8) is symmetric].

If we denote by X_α the set of generators $X_{\alpha,i}$ having the same index α, the generalized commutation relations for $C(2,a)$ algebras are

$$[X_{(0,0)},X_{(\alpha_1,\alpha_2)}] \subset X_{(\alpha_1,\alpha_2)}, \quad [X_{(1,1)},X_{(1,1)}] \subset X_{(0,0)},$$

$$\{X_{(1,1)},X_{(1,0)}\} \subset X_{(0,1)}, \quad \{X_{(1,1)},X_{(0,1)}\} \subset X_{(1,0)}, \tag{1.9}$$

$$[X_{(1,0)},X_{(1,0)}] \subset X_{(0,0)}, \quad [X_{(0,1)},X_{(0,1)}] \subset X_{(0,0)},$$

$$\{X_{(1,0)},X_{(0,1)}\} \subset X_{(1,1)}. \tag{1.10}$$

For $C(2,s)$ superalgebras Eq. (1.9) is unchanged but instead of Eqs. (1.10) we have

$$\{X_{(1,0)},X_{(1,0)}\} \subset X_{(0,0)}, \quad \{X_{(0,1)},X_{(0,1)}\} \subset X_{(0,0)},$$

$$[X_{(1,0)},X_{(0,1)}] \subset X_{(1,1)}. \tag{1.11}$$

As one can see from Eqs. (1.9) and (1.10) the generators $X_{(1,0)}$, $X_{(0,1)}$, and $X_{(1,1)}$ of the $C(2,a)$ algebras behave like three-color parabosonic operators.[2] For $C(2,s)$ algebras the generators $X_{(1,0)}$ and $X_{(0,1)}$ behave like parafermionic operators.[2] Since color symmetry was first introduced in particle physics through parafields, we have decided to call the $C(2,a)$ [$C(2,s)$] generalized Lie algebras (superalgebras), *color algebras* (*color superalgebras*).

In this paper we construct sequences of $C(2,a)$ and $C(2,s)$ algebras. We use the same methods as those used to define the classical Lie algebras or the classical superalgebras[3]: We consider groups of linear transformations and we look for subgroups which preserve a certain metric. The *parameters* of these groups (we call them color groups) satisfy the commutation relations

$$\theta_{\alpha,m}\theta_{\beta,n} - (-1)^{(\beta,\alpha)}\theta_{\beta,n}\theta_{\alpha,m} = 0 \tag{1.12}$$

[as in the case of supergroups, if the parameters are defined in different algebras compatible with (1.12), one obtains different color groups].

The classical color groups can be obtained considering $Z_2 \oplus Z_2$ graded matrices having matrix elements verifying commutation relations of the type (1.12). The group multiplication is given by the usual multiplication rule of matrices.

An important operation on such matrices is the determinant. As a byproduct of our investigation we have derived compact expressions for such determinants. They are given in Appendix A for both the $C(2,s)$ and the $C(2,a)$ case. If we specialize the algebra of the parameters to quaternions [the $C(2a)$ case] we obtain the expression of the determinant of certain matrices in which the matrix elements are quaternions. (Those are not the most general matrices over quaternions but the matrices are not self-dual. The last case was already studied in the literature.[4])

The techniques used in this paper can be used for other generalized Lie algebras or superalgebras.[1] We thus consider our study of the $C(2,a)$ and $C(2,s)$ algebras mainly as a "laboratory" in which one can learn how to construct sequences of algebras of a richer structure (like $Z_N \oplus Z_N$, $N > 2$ algebras and superalgebras). We are aware that much of what we found about the $C(2,s)$ superalgebras is known to Kac who briefly discusses them in Ref. 5.

The paper is organized as follows: In Sec. 2 we remind the reader how the classical superalgebras are obtained [the $spl(m,n)$, $osp(m,n)$, $P(m)$, and $\tilde{Q}(m)$ series]. Then, in Sec. 3 we give sequences of $C(2,s)$ algebras. We first define the superdeterminant and several transpose operations of $Z_2 \oplus Z_2$ graded matrices and use them in order to define subsupergroups. Taking infinitesimal transformations we get the $spl(m,n,r,s)$, $osp(m,n,r,s)$, $P_1(m,r)$, $P_3(m,n)$, $ospP_3(m,n)$, $P_{1,2}(m)$, and $\tilde{Q}(m)$ algebras.

In Sec. 4 we derive sequences of $C(2,a)$ algebras applying the same procedure. We present the $sl(m,n,r,s)$, $o(m,n,r,s)$, $P_1(m,n)$, $oP_1(m,n)$, $P_{1,2}(m)$, $oP_{1,2}(m)$, and $\bar{Q}(m)$ algebras.

Compact expressions for superdeterminants and determinants of matrices having matrix elements verifying Eq. (1.12) are given in Appendix A. In Appendix B we show how one can construct a $C(2,s)$ superalgebra from any given superalgebra and how one can get a $C(2,a)$ algebra from any Lie algebra.

2. SUPERGROUPS AND SUPERALGEBRAS

In this section we briefly review the classical supergroups[3] and superalgebras. We consider $(m+n) \times (m+n)$ matrices that we write in the block form

$$M = \begin{pmatrix} A_0 & A_1 \\ B_1 & B_0 \end{pmatrix}, \qquad (2.1)$$

where A_0 is an $m \times m$ matrix, B_0 is an $n \times n$ matrix, etc. The matrix elements of M are Z_2 graded. It is convenient to write each matrix element M as

$$M_{i,j} = M_{\alpha;i,j}, \qquad (2.2)$$

where $\alpha = 0,1$ indicates if we have taken the matrix element from a block with $\alpha = 0$ (A_0 or B_0) or from a block with $\alpha = 1$ (A_1 or B_1). The matrix elements commute or anticommute,

$$M_{\alpha;i,j} M_{\beta;k,l} - (-1)^{\alpha\beta} M_{\beta;k,l} M_{\alpha;i,j} = 0. \qquad (2.3)$$

A product of two Z_2 graded matrices is a Z_2 graded matrix so that the general linear supergroup is defined for matrices (2.1) which have an inverse.

We define the supertrace of a Z_2 graded matrix

$$\mathrm{str}M = \mathrm{tr}A_0 - \mathrm{tr}B_0, \qquad (2.4)$$

the superdeterminant

$$\mathrm{sdet}M = e^{\mathrm{str}\,\ln M} \qquad (2.5)$$

the supertranspose

$$M^{\mathrm{sT}} = \begin{pmatrix} A_0^T & -B_1^T \\ A_1^T & B_0^T \end{pmatrix}, \qquad (2.6)$$

and the P transpose (for $m = n$)

$$M^P = \begin{pmatrix} B_0^T & -A_1^T \\ B_1^T & A_0^T \end{pmatrix}. \qquad (2.7)$$

If $A_0 = B_0$ and $A_1 = B_1$ we can define the ω superdeterminant,

$$\mathrm{str}_\omega M = \omega \mathrm{tr}A_1; \quad \mathrm{sdet}_\omega M = e^{\mathrm{str}_\omega \ln M}, \qquad (2.8)$$

where ω anticommutes with all the matrix elements of A_1. We now define the classical supergroups.[3]

$$\mathrm{SPL}(m,n): \mathrm{sdet}M = 1, \qquad (2.9)$$

$$\mathrm{OSP}(m,n); \ n = 2p: M^{\mathrm{sT}}HM = H, \qquad (2.10a)$$

where

$$H = \begin{pmatrix} I_m & 0 \\ 0 & C_n \end{pmatrix}, \quad C_n = \begin{pmatrix} 0 & I_P \\ -I_P & 0 \end{pmatrix}, \qquad (2.10b)$$

I_m is the $m \times m$ unit matrix.

$$P(m): MM^P = I_{2m}, \quad \mathrm{sdet}M = 1, \qquad (2.11)$$

$$\tilde{Q}(m): \mathrm{sdet}_\omega M = 1. \qquad (2.12)$$

The $spl(m,n)$, $osp(m,n)$, $P(m)$, and $\tilde{Q}(m)$ superalgebras[6] are obtained taking infinitesimal transformations in (2.9)–(2.12). The $Q(m)$ superalgebra is obtained from the $\tilde{Q}(m)$ superalgebra by dividing by its center.

3. C(2,s) COLOR SUPERGROUPS AND SUPERALGEBRAS

We consider $Z_2 \oplus Z_2$ graded matrices that we write in the block form

$$M = \begin{pmatrix} A_{(0,0)} & A_{(1,0)} & A_{(0,1)} & A_{(1,1)} \\ B_{(1,0)} & B_{(0,0)} & B_{(1,1)} & B_{(0,1)} \\ C_{(0,1)} & C_{(1,1)} & C_{(0,0)} & C_{(1,0)} \\ D_{(1,1)} & D_{(0,1)} & D_{(1,0)} & D_{(0,0)} \end{pmatrix}. \qquad (3.1)$$

The matrix M is $(m+n+r+s) \times (m+n+r+s)$-dimensional. Here the matrix $A_{(0,0)}$ is $(m \times m)$-, $A_{(1,0)}$ is $(m \times n)$- $A_{(0,1)}$ is $(m \times r)$- and $A_{(1,1)}$ is $(m \times s)$-dimensional. We write each matrix element M_{pq} of M as

$$M_{i,j} \equiv M_{(\alpha_1,\alpha_2);i,j} \quad (i,j=1,\ldots,m+n+r+s), \qquad (3.2)$$

where (α_1,α_2) corresponds to the denomination of the block. For example, $(\alpha_1,\alpha_2) = (1,0)$ if the matrix element belongs to one of the blocks $A_{(1,0)}$, $B_{(1,0)}$, $C_{(1,0)}$, or $D_{(1,0)}$. The matrix elements have the commutation or anticommutation relations

$$M_{(\alpha_1,\alpha_2);i,j} M_{(\beta_1,\beta_2);k,l}$$
$$-(-1)^{\alpha_1\beta_1+\alpha_2\beta_2} M_{(\beta_1,\beta_2);k,l} M_{(\alpha_1,\alpha_2);i,j} = 0. \qquad (3.3)$$

Since the product of two $Z_2 \oplus Z_2$ graded matrices is a graded matrix, the *general linear $C(2,s)$ color supergroup*, $GL(m,n,r,s)$, is defined by taking matrices (3.1) which have an inverse. In order to define subsupergroups we define the determinant and various transposes for the matrices (3.1).

The *supertrace* is defined as follows,

$$\mathrm{str} M = \mathrm{tr} A_{00} - \mathrm{tr} B_{00} - \mathrm{tr} C_{00} + \mathrm{tr} D_{00}, \qquad (3.4)$$

and the *superdeterminant* is

$$\mathrm{sdet} M = e^{\mathrm{str}\, \ln M}, \qquad (3.5)$$

with

$$\mathrm{sdet}(MN) = \mathrm{sdet} M \cdot \mathrm{sdet} N. \qquad (3.6)$$

The rule for computing superdeterminants derived from the definition (3.5) is given in Appendix A.

The *supertranspose* M^{sT} of the matrix M is

$$M^{\mathrm{sT}} = \begin{pmatrix} A^T_{(0,0)} & \xi\xi B^T_{(1,0)} & -\zeta C^T_{(0,1)} & \xi\eta\zeta D^T_{(1,1)} \\ -\xi\zeta A^T_{(1,0)} & B^T_{(0,0)} & \xi C^T_{(1,1)} & \eta D^T_{(0,1)} \\ \zeta A^T_{(0,1)} & \xi B^T_{(1,1)} & C^T_{(0,0)} & -\xi\eta D^T_{(1,0)} \\ \xi\eta\zeta A^T_{(1,1)} & -\eta B^T_{(0,1)} & \xi\eta C^T_{(1,0)} & D^T_{(0,0)} \end{pmatrix}, \qquad (3.7)$$

where $\xi^2 = \eta^2 = \zeta^2 = 1$.

Note that

$$\mathrm{sdet} M = \mathrm{sdet} M^{\mathrm{sT}}. \qquad (3.8)$$

Similarly to superalgebras, we can define "permutation"-transposed forms that we denote by M^{P_1}, M^{P_2}, and M^{P_3}:

$$M^{P_1} = \begin{pmatrix} B^T_{(0,0)} & \xi\xi A^T_{(1,0)} & -\zeta D^T_{(0,1)} & \xi\eta\zeta C^T_{(1,1)} \\ -\xi\zeta B^T_{(1,0)} & A^T_{(0,0)} & \xi D^T_{(1,1)} & \eta C^T_{(0,1)} \\ \zeta B^T_{(0,1)} & \xi A^T_{(1,1)} & D^T_{(0,0)} & -\xi\eta C^T_{(1,0)} \\ \xi\eta\zeta B^T_{(1,1)} & -\eta A^T_{(0,1)} & \xi\eta D^T_{(1,0)} & C^T_{(0,0)} \end{pmatrix}, \qquad (3.9)$$

$$M^{P_2} = \begin{pmatrix} C^T_{(0,0)} & \xi\xi D^T_{(1,0)} & -\zeta A^T_{(0,1)} & \xi\eta\zeta B^T_{(1,1)} \\ -\xi\xi C^T_{(1,0)} & D^T_{(0,0)} & \xi A^T_{(1,1)} & \eta B^T_{(0,1)} \\ \zeta C^T_{(0,1)} & \xi D^T_{(1,1)} & A^T_{(0,0)} & -\xi\eta B^T_{(1,0)} \\ \xi\eta\zeta C^T_{(1,1)} & -\eta D^T_{(0,1)} & \xi\eta A^T_{(1,0)} & B^T_{(0,0)} \end{pmatrix}, \qquad (3.10)$$

$$M^{P_3} = \begin{pmatrix} D^T_{(0,0)} & \xi\xi C^T_{(1,0)} & -\zeta B^T_{(0,1)} & \xi\eta\zeta A^T_{(1,1)} \\ -\xi\xi D^T_{(1,0)} & C^T_{(0,0)} & \xi B^T_{(1,1)} & \eta A^T_{(0,1)} \\ \zeta D^T_{(0,1)} & \xi C^T_{(1,1)} & B^T_{(0,0)} & -\xi\eta A^T_{(1,0)} \\ \xi\eta\zeta D^T_{(1,1)} & -\eta C^T_{(0,1)} & \xi\eta B^T_{(1,0)} & A^T_{(0,0)} \end{pmatrix}, \qquad (3.11)$$

again $\xi^2 = \eta^2 = \zeta^2 = 1$.

The P_1 operation can be defined if $m=n$, $r=s$; the P_2 operation if $m=r$, $n=s$, and the P_3 operation if $m=s$, $n=r$.

We have the relations

$$(MN)^{\mathrm{sT}} = N^{\mathrm{sT}} M^{\mathrm{sT}}, \quad (MN)^{P_i} = N^{P_i} M^{P_i} \quad (i=1,2,3). \qquad (3.12)$$

We now define various color supergroups imposing different conditions [based on Eqs. (3.6) and (3.12) on the $Z_2 \oplus Z_2$ graded matrices (3.1).

$SPL(m,n,r,s)$:

$$\mathrm{sdet} M = 1. \qquad (3.13)$$

$OSP(m,n,r,s)$ (where n and r are even numbers):

$$M^{\mathrm{sT}} H M = H, \qquad (3.14)$$

where

$$H = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & C_n & 0 & 0 \\ 0 & 0 & C_r & 0 \\ 0 & 0 & 0 & I_s \end{pmatrix}. \qquad (3.15)$$

The matrices I_m and C_n are defined by Eq. (2.10b).

$P_1(m,r)$:

$$M^{P_1} M = 1, \quad \mathrm{sdet} M = 1. \qquad (3.16)$$

$P_3(m,n)$:

$$M^{P_3} M = 1, \qquad (3.17)$$

$OSPP_3(m,n)$ (where n is even):

$$M^{\mathrm{sT}} H M = H, \quad M^{P_3} M = 1. \qquad (3.18)$$

$P_{1,2}(m)$:

$$M^{P_1} M = 1, \quad M^{P_2} M = 1, \quad \mathrm{sdet} M = 1. \qquad (3.19)$$

The $C(2,s)$ superalgebras are obtained taking infinitesimal transformations for the matrices M in (3.12)–(3.18). In this way we obtain the $\mathrm{spl}(m,n,r,s)$, $\mathrm{osp}(m,n,r,s)$, $P_1(m,r)$, $P_3(m,n)$, $\mathrm{osp}P_3(m,n)$, and $P_{1,2}(m)$ $C(2,s)$ superalgebras. For example the condition (3.12) defines the $\mathrm{spl}(m,n,r,s)$ algebra given by the matrices X which have the property

$$\mathrm{str} X = 0. \qquad (3.20)$$

In order to define the $\tilde{Q}(m)$ algebras it is instructive to

give another method instead of generalizing the operation sdet_ω used for superalgebras. This method is based on the properties of the SU(m) $m \times m - \lambda_i$ matrices which together with the unit matrix close under both commutation and anticommutation relations:

$$[\lambda_i, \lambda_j] = f_{ijk} \lambda_k, \quad \{\lambda_i, \lambda_j\} = d_{ijk} \lambda_k + \frac{1}{m} \delta_{ij} \mathbb{1} \tag{3.21}$$

$$[i,j,k = 1,2,\ldots,(m^2-1)].$$

Let us consider now four 4×4 matrices:

$$\omega_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \omega_{(1,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\omega_{(0,1)} = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \quad \omega_{(1,1)} = \begin{pmatrix} 0 & \tau_1 \\ \tau_1 & 0 \end{pmatrix}, \tag{3.22}$$

where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrices (3.22) satisfy the indentities

$$\omega_{(\alpha_1,\alpha_2)} \omega_{(\beta_1,\beta_2)} = \omega_{(\beta_1,\beta_2)} \omega_{(\alpha_1,\alpha_2)} = \omega_{(\alpha_1+\beta_1, \alpha_2+\beta_2)}. \tag{3.23}$$

We consider now the $4m \times 4m$ matrices

$$X_{(\alpha_1,\alpha_2),i} = \lambda_i \omega_{(\alpha_1,\alpha_2)}, \quad X_{(\alpha_1,\alpha_2),0} = \mathbb{1}\omega_{(\alpha_1,\alpha_2)}$$

$$[i = 1,2,\ldots(m^2-1)]. \tag{3.24}$$

Using Eqs. (3.21), (3.23), and (3.24), we derive the $\tilde{Q}(n)$ algebra (which is obtained taking $X_{(1,1),0} = 0$),

$$[X_{(0,0),i}, X_{(\alpha_1,\alpha_2),j}] = f_{ijk} X_{(0,0),k}, \quad [X_{(0,0),i}, X_{(\alpha_1,\alpha_2),0}] = 0, \tag{3.25a}$$

$$[X_{(0,0),0}, X_{(\alpha_1,\alpha_2),i}] = [X_{(0,0),0}, X_{(\alpha_1,\alpha_2),0}] = 0.$$

$$\{X_{(\alpha_1,\alpha_2),i}, X_{(\alpha_1,\alpha_2),j}\} = d_{ijk} X_{(0,0),k} + \frac{\delta_{ij}}{m} X_{(0,0),0}; \tag{3.25b}$$

$$\{X_{(\alpha_1,\alpha_2),i}, X_{(\alpha_1,\alpha_2),0}\} = 2X_{(0,0),i}; \quad \{X_{(\alpha_1,\alpha_2),0}, X_{(\alpha_1,\alpha_2),0}\} = 0;$$

$$((\alpha_1,\alpha_2) = (1,0), (0,1)).$$

$$[X_{(1,1),i}, X_{(1,1),j}] = f_{ijk} X_{(0,0),k};$$

$$[X_{(1,0),i}, X_{(0,1),j}] = f_{ijk} X_{(1,1),k}; \tag{3.25c}$$

$$[X_{(1,0),0}, X_{(0,1),i}] = [X_{(1,0),i}, X_{(0,1),0}] = 0$$

$$[X_{(1,0),0}, X_{(0,1),0}] = 0.$$

$$\{X_{(1,0),i}, X_{(1,1),j}\} = d_{ijk} X_{(0,1),k} + \frac{1}{m} X_{(0,1),0} \delta_{ij};$$

$$\{X_{(0,1),i}, X_{(1,1),j}\} = d_{ijk} X_{(1,0),k} + \frac{1}{m} X_{(1,0),0} \delta_{ij}; \tag{3.25d}$$

$$\{X_{(1,0),0}, X_{(1,1),i}\} = 2X_{(0,1),i};$$

$$\{X_{(0,1),0}, X_{(1,1),i}\} = 2X_{(1,0),i}.$$

4. $C(2,a)$ COLOR GROUPS AND ALGEBRAS

We consider again matrices of the form (3.1)–(3.2) but now instead of Eq. (3.3) we have

$$M_{(\alpha_1,\alpha_2);i,j} M_{(\beta_1,\beta_2);k,l}$$

$$- (-1)^{\alpha_1\beta_2 + \alpha_2\beta_1} M_{(\beta_1,\beta_2);k,l} M_{(\alpha_1,\alpha_2);i,j} = 0. \tag{4.1}$$

The trace operation is now defined as for ordinary matrices,

$$\mathrm{tr}M = \mathrm{tr}A_{(0,0)} + \mathrm{tr}B_{(0,0)} + \mathrm{tr}C_{(0,0)} + \mathrm{tr}D_{(0,0)}, \tag{4.2}$$

and the determinant is

$$\det M = e^{\mathrm{tr}\,\ln M}. \tag{4.3}$$

A compact expression of the determinant in terms of the matrix elements is given in Appendix A.

We also have

$$\det(MN) = \det M \det N. \tag{4.4}$$

We now define the transpose and the "permutation" transpose operations,

$$M^T = \begin{pmatrix} A^T_{(0,0)} & -\xi\zeta B^T_{(1,0)} & \zeta C^T_{(0,1)} & \xi\eta\zeta D^T_{(1,1)} \\ -\xi\zeta A^T_{(1,0)} & B^T_{(0,0)} & \xi C^T_{(1,1)} & \eta D^T_{(0,1)} \\ \zeta A^T_{(0,1)} & \xi B^T_{(1,1)} & C^T_{(0,0)} & -\xi\eta D^T_{(1,0)} \\ \xi\eta\zeta A^T_{(1,1)} & \eta B^T_{(0,1)} & -\xi\eta C^T_{(1,0)} & D^T_{(0,0)} \end{pmatrix}, \tag{4.5}$$

$$M^{P_1} = \begin{pmatrix} B^T_{(0,0)} & -\xi\zeta A^T_{(1,0)} & \zeta D^T_{(0,1)} & \xi\eta\zeta C^T_{(1,1)} \\ -\xi\zeta B^T_{(1,0)} & A^T_{(0,0)} & \xi D^T_{(1,1)} & \eta C^T_{(0,1)} \\ \zeta B^T_{(0,1)} & \xi A^T_{(1,1)} & D^T_{(0,0)} & -\xi\eta C^T_{(1,0)} \\ \xi\eta\zeta B^T_{(1,1)} & \eta A^T_{(0,1)} & -\xi\eta D^T_{(1,0)} & C^T_{(0,0)} \end{pmatrix}, \tag{4.6}$$

$$M^{P_2} = \begin{pmatrix} C^T_{(0,0)} & -\xi\zeta D^T_{(1,0)} & \zeta A^T_{(0,1)} & \xi\eta\zeta B^T_{(1,1)} \\ -\xi\zeta C^T_{(1,0)} & D^T_{(0,0)} & \xi A^T_{(1,1)} & \eta B^T_{(0,1)} \\ \zeta C^T_{(0,1)} & \xi D^T_{(1,1)} & A^T_{(0,0)} & -\xi\eta B^T_{(1,0)} \\ \xi\eta\zeta C^T_{(1,1)} & \eta D^T_{(0,1)} & -\xi\eta A^T_{(1,0)} & B^T_{(0,0)} \end{pmatrix}, \tag{4.7}$$

where $\xi^2 = \eta^2 = \zeta^2 = 1$.

The P_1 transpose exists if $m = n$, $r = s$; the P_2 transpose can be defined if $m = r$, $n = s$.

The definitions (4.5)–(4.7) have been chosen such that

$$(MN)^T = N^T M^T, \quad (MN)^{P_i} = N^{P_i} M^{P_i} \quad (i = 1,2). \tag{4.8}$$

We also have

$$\det M^T = \det M^{P_i} = \det M \quad (i = 1,2). \tag{4.9}$$

We now list some color groups:

SL(m,n,r,s):

$$\det M = 1. \tag{4.10}$$

$O(m,n,r,s$):

$$MM^T = \mathbb{1}. \tag{4.11}$$

$P_1(m,r$):

$$MM^{P_1} = \mathbb{1}. \tag{4.12}$$

$OP_1(m,r$):

$$MM^{P_1} = \mathbb{1}, \quad MM^T = \mathbb{1}. \tag{4.13}$$

$P_{1,2}(m$):

$$MM^{P_1} = \mathbb{1}, \quad MM^{P_2} = \mathbb{1}. \tag{4.14}$$

$OP_{1,2}(m$):

$$MM^{P_1} = \mathbb{1}, \quad MM^{P_2} = \mathbb{1}, \quad MM^T = \mathbb{1}. \tag{4.15}$$

The $C(2,a)$ color algebras are obtained taking infinitesimal transformations in (4.10)–(4.15) and in this way we obtained the $sl(m,n,r,s)$, $o(m,n,r,s)$, $P_1(m,r)$, $oP_1(m,r)$, $P_{1,2}(m)$, and $oP_{1,2}(m)$ series.

In order to obtain the $\widetilde{Q}(m)$ series, we again use the Eqs. (3.21)–(3.24), and obtain:

$$[X_{(\alpha_1,\alpha_2),i}, X_{(\alpha_1,\alpha_2),j}] = f_{ijk} X_{(0,0),k},$$

$$[X_{(0,0),i}, X_{(\alpha_1,\alpha_2),j}] = f_{ijk} X_{(\alpha_1,\alpha_2),k}, \tag{4.16a}$$

$$[X_{(0,0),0}, X_{(\alpha_1,\alpha_2),i}] = [X_{(\alpha_1,\alpha_2),0}, X_{(\alpha_1,\alpha_2),i}] = 0,$$

$$\{X_{(\alpha_1,\alpha_2),i}, X_{(\beta_1,\beta_2),j}\} = d_{ijk} X_{(\alpha_1+\beta_1,\alpha_2+\beta_2),k}$$

$$+ \frac{\delta_{ij}}{m} X_{(\alpha_1+\beta_1,\alpha_2+\beta_2),0}$$

$$\{X_{(\alpha_1,\alpha_2),i}, X_{(\beta_1,\beta_2),0}\} = 2X_{(\alpha_1+\beta_1,\alpha_2+\beta_2),i},$$

$$\{X_{(\alpha_1,\alpha_2),0}, X_{(\beta_1,\beta_2),0}\} = 2X_{(\alpha_1+\beta_1,\alpha_2+\beta_2),0},$$

$$[(\alpha_1,\alpha_2) \neq (\beta_1,\beta_2) \neq (0,0)]. \tag{4.16b}$$

ACKNOWLEDGMENT

APPENDIX A: DETERMINANTS OF $Z_2 \oplus Z_2$ GRADED MATRICES

We consider the $(m+n+r+s) \times (m+n+r+s)$ matrix

$$M = \begin{pmatrix} A_{(0,0)} & A_{(1,0)} & A_{(0,1)} & A_{(1,1)} \\ B_{(1,0)} & B_{(0,0)} & B_{(1,1)} & B_{(0,1)} \\ C_{(0,1)} & C_{(1,1)} & C_{(0,0)} & C_{(1,0)} \\ D_{(1,1)} & D_{(0,1)} & D_{(1,0)} & D_{(0,0)} \end{pmatrix}, \tag{A.1}$$

where $A_{(0,0)}$ is an $m \times m$ matrix, $B_{(0,0)}$ is $n \times n$, $C_{(0,0)}$ is $r \times r$, $D_{(0,0)}$ is $s \times s$, $A_{(1,0)}$ is $m \times n$, etc. The elements of M are denoted by

$$M_{i,j} = M_{(\alpha_1,\alpha_2);i,j}, \quad i,j = 1,2...,m+n+r+s, \tag{A2}$$

where (α_1,α_2) corresponds to the type of block matrix of which M_{ij} is an element. For example, $(\alpha_1,\alpha_2) = (1,0)$, if the

matrix element belongs to one of the $A_{(1,0)}, B_{(1,0)}, C_{(1,0)}, D_{(1,0)}$. It will be also convenient to use for M the form

$$M = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}, \tag{A3}$$

where A is an $m \times (m+n+r+s)$ matrix, etc., with element $A_{i,a}$ ($i = 1,..., m$ and $a = 1,..., m+r+r+s$); similarly for B,C,D.

In the following we will consider the determinants of the two possible $Z_2 \oplus Z_2$ graded matrices of the above form.

1. Determinants of C(2,s) matrices (superdeterminants)

$C(2,s)$ matrices are matrices M of the form (A1) whose elements satisfy

$$M_{(\alpha_1,\alpha_2);i,j} M_{(\beta_1,\beta_2);k,l}$$

$$-(-1)^{\alpha_1\beta_1 + \alpha_2\beta_2} M_{(\beta_1,\beta_2);k,l} M_{(\alpha_1,\alpha_2);i,j} = 0. \tag{A4}$$

The *superdeterminant* (sdet) of M is defined to be [recall that $\mathrm{str} M = \mathrm{tr}(A_{(0,0)} + D_{(0,0)} - B_{(0,0)} - C_{(0,0)})$, see Eq. (3.4)]

$$\mathrm{sdet} M = e^{\mathrm{str} \ln M}. \tag{A5}$$

(A5) can be brought into a more convenient form. If M^{-1} is the inverse of M, defined by the usual matrix multiplication, $M^{-1}M = MM^{-1}$, and is denoted by

$$M^{-1} = \begin{pmatrix} A'_{(0,0)} & A'_{(1,0)} & A'_{(0,1)} & A'_{(1,1)} \\ B'_{(1,0)} & B'_{(0,0)} & B'_{(1,1)} & B'_{(0,1)} \\ C'_{(0,1)} & C'_{(1,1)} & C'_{(0,0)} & C'_{(1,0)} \\ D'_{(1,1)} & D'_{(0,1)} & D'_{(1,0)} & D'_{(0,0)} \end{pmatrix}, \tag{A6}$$

then

$$\mathrm{sdet} M = \det \begin{pmatrix} A_{(0,0)} & A_{(1,1)} \\ D_{(1,1)} & D_{(0,0)} \end{pmatrix} \det \begin{pmatrix} B'_{(0,0)} & B'_{(1,1)} \\ C'_{(1,1)} & C'_{(0,0)} \end{pmatrix}. \tag{A7}$$

The determinants in (A7) are usual determinants.

It is straightforward to check that (A7) satisfies

$$\mathrm{sdet}(MN) = \mathrm{sdet} M \, \mathrm{sdet} N. \tag{A8}$$

Let us give two simple examples for $m = n = r = s = 1$.

Example 1.

$$M = \begin{pmatrix} 1 & \theta_1 & \theta_2 & 0 \\ \theta_1 & 1 & 0 & 0 \\ \theta_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\theta_i \theta'_i + \theta'_i \theta_i = 0, \quad \theta_i^2 = \theta'^2_i = 0 \quad (i = 1,2) \tag{A9}$$

$$[\theta_1, \theta_2] = [\theta'_1, \theta'_2] = [\theta_1, \theta'_2] = [\theta'_1, \theta_2] = 0.$$

Using (A6), we obtain

$$\mathrm{sdet}M = 1 + \theta'_1\theta_1 + \theta'_2\theta_2.$$

Example 2.

$$M = \begin{pmatrix} X_{(0,0)} & \theta_{(1,0)} & \theta_{(0,1)} & \theta_{(1,1)} \\ \theta_{(1,0)} & Y_{(0,0)} & \theta_{(1,1)} & \theta_{(0,1)} \\ \theta_{(0,1)} & \theta_{(1,1)} & Y_{(0,0)} & \theta_{(1,0)} \\ \theta_{(1,1)} & \theta_{(0,1)} & \theta_{(1,0)} & X_{(0,0)} \end{pmatrix} \qquad (A10)$$

For this matrix we get

$$\mathrm{sdet}M = \frac{X^2_{(0,0)} - \theta^2_{(1,1)}}{Y^2_{(0,0)} - \theta^2_{(1,1)}}$$

$$+ \frac{2\theta_{(1,1)}(Y_{(0,0)} - X_{(0,0)})\{\theta_{(1,0)}, \theta_{(0,1)}\}}{(Y^2_{(0,0)} - \theta^2_{(1,1)})^2}. \qquad (A11)$$

Let us compare the superdeterminants of $Z_2 \oplus Z_2$ graded matrices and of Z_2 graded matrices [see Eq. (2.5) and Ref. 3]: we associate with (0,0) and (1,1) the number 0, and with (1,0) and (0,1) the number 1. We now demand

$$M_{(\alpha_1,\alpha_2);i,j}M_{(\beta_1,\beta_2);k,l}$$
$$- (-1)^{\alpha\beta}M_{(\beta_1,\beta_2);k,l}M_{(\alpha_1,\alpha_2);i,j} = 0. \qquad (A12)$$

where α and β are the numbers associated with (α_1, α_2) and (β_1, β_2) in the above way. Then, M becomes a Z_2 graded matrix [see Eqs. (2.1) and (2.2)], for which the determinant has the same form (A7). However, the different commutation relations will in general give a different result when calculating sdet. In the above example (A12) would imply

$$\mathrm{sdet}M = \frac{X^2_{(0,0)} - \theta^2_{(1,1)}}{Y^2_{(0,0)} - \theta^2_{(1,1)}},$$

since $\{\theta_{(1,0)}, \theta_{(0,1)}\} = 0$.

2. Determinants of C(2,a) matrices

$C(2,a)$ matrices are matrices M of the form (A1) whose elements obey

$$M_{(\alpha_1,\alpha_2);i,j}M_{(\beta_1,\beta_2);k,l}$$
$$- (-1)^{\alpha_1\beta_2 - \alpha_2\beta_1}M_{(\beta_1,\beta_2);k,l}M_{(\alpha_1,\alpha_2);i,j} = 0. \qquad (A13)$$

The determinant is defined by

$$\det M = e^{\mathrm{tr}\,\ln M} \qquad (A14)$$

In this case it is possible to write the determinant in a polynomial form similar to the one of ordinary matrices. We have

$$\det M = \sum_{\substack{a_i, b_i, \\ c_i, d_i = 1}}^{m+n+r+s} \epsilon_{a_1,a_2,\dots,a_m;b_1,\dots,b_n;c_1,\dots,c_r;d_1,\dots,d_s}$$

$$\cdot A_{1a_1}A_{2a_2}A_{3a_3}\cdots A_{ma_m}B_{1b_1}\cdots B_{nb_n}$$

$$\cdot C_{1c_1}\cdots C_{rc_r}\cdot D_{1d_1}\cdots D_{sd_s}. \qquad (A15)$$

To each index a_i, b_i, c_i, d_i we relate a grading vector α which gives the grading property of the corresponding index. We have [see (A1)]

$$\alpha(a_i) = \begin{cases} (0,0) & \text{for} & 1 \leqslant a_i \leqslant m, \\ (1,0) & \text{for} & m < a_i \leqslant m+n, \\ (0,1) & \text{for} & m+n < a_i \leqslant m+n+r, \\ (1,1) & \text{for} & m+n+r < a_i \leqslant m+n+r+s, \end{cases}$$
$$(A16)$$

$$\alpha(b_i) = \begin{cases} (1,0) & \text{for} & 1 \leqslant b_i \leqslant m, \\ (0,0) & \text{for} & m < b_i \leqslant m+n, \\ (1,1) & \text{for} & m+n < b_i \leqslant m+n+r, \\ (0,1) & \text{for} & m+n+r < b_i \leqslant m+n+r+s, \end{cases}$$

etc.

The properties of the ϵ under exchanges of the nearest indices are as follows:

(i) $\epsilon_{1,2,3,\dots,m;m+1,\dots,m+n;m+n+1,\dots,m+n+r;m+n+r+1,\dots,m+n+r+s}$
$$= 1. \qquad (A17)$$

(ii) permutation of a_i indices,

$$\epsilon_{\dots a_i a_j \dots;\dots} = -(-1)^{(\alpha(a_i),\alpha(a_j))}\epsilon_{\dots a_j a_i \dots;\dots}, \qquad (A18)$$

where

$$(\alpha,\beta) = \alpha_1\beta_2 - \alpha_2\beta_1.$$

The same rule applies for permutations of the b_i, c_i, and d_i indices, respectively.

(iii) permutation of a_i indices with b_j indices,

$$\epsilon_{\dots a_m;b_1\dots b_n;c_1\dots c_r;d_1\dots d_s}$$
$$= -(-1)^{(\alpha(a_m),\alpha(b_1)) + (\alpha(a_m),\gamma(A) + \gamma(B))}$$
$$\times \epsilon_{\dots b_1;a_m\dots b_n;c_1\dots c_r;d_1\dots d_s}, \qquad (A19a)$$

where γ is a vector associated with each row in (A3) as follows:

$$A \rightarrow (0,0), \quad B \rightarrow (1,0), \quad C \rightarrow (0,1), \quad D \rightarrow (1,1). \qquad (A20)$$

The analogous rules hold for permutations of the b_i with C_j indices and the C_i with d_j indices, for example,

$$\epsilon_{\dots a_m;b_1\dots b_n;c_1\dots c_r;d_1\dots d_s}$$
$$= -(-1)^{(\alpha(b_n),\alpha(c_1)) + (\alpha(b_n),\gamma(B) + \gamma(C))}$$
$$\times \epsilon_{\dots a_m;b_1\dots c_1;b_n\dots c_r;d_1\dots d_s}, \qquad (A19b)$$

One can show that the determinant (A15) with

(A16)–(A20) satisfies

$$\det(MN) = \det M \cdot \det N, \qquad (A21)$$

where M, N are matrices of the form (A3) satisfying (A13).

As an example we take the 4×4 matrix

$$M = \begin{pmatrix} X_{(0,0)} & \theta_{(1,0)} & \sigma_{(0,1)} & \rho_{(1,1)} \\ \theta_{(1,0)} & X_{(0,0)} & \rho_{(1,1)} & \sigma_{(0,1)} \\ \sigma_{(0,1)} & \rho_{(1,1)} & X_{(0,0)} & \theta_{(1,0)} \\ \rho_{(1,1)} & \sigma_{(0,1)} & \theta_{(1,0)} & X_{(0,0)} \end{pmatrix}. \qquad (A22)$$

Then we obtain, using (A15)–(A20)

$$\det M = [X_{(0,0)}^2 - (\theta_{(1,0)}^2 + \sigma_{(0,1)}^2 + \rho_{(1,1)}^2)]^2. \qquad (A23)$$

In contrast the usual determinant of an ordinary 4×4 matrix (with commuting elements) of the above form is

$$\det M = [X_{(0,0)}^2 - (\theta_{(1,0)}^2 + \sigma_{(0,1)}^2 + \rho_{(1,1)}^2)]^2$$

$$+ 4[2\sigma_{(0,1)}\rho_{(1,1)}\theta_{(1,0)}X_{(0,0)}$$

$$- (\sigma_{(0,1)}^2\rho_{(1,1)}^2 + \sigma_{(0,1)}^2\theta_{(1,0)}^2 + \theta_{(1,0)}^2\rho_{(1,1)}^2)].$$

We close with a comment on determinants over quaternions. Quaternions are described by four units, $1, \tau_i (i=1,2,3)$ which obey

$$[1,\tau_i] = 0, \quad \{\tau_i \tau_j\} = 0 \quad (i \neq j), \quad \tau_i^2 = -1 \quad (i=1,2,3), \qquad (A24)$$

$$\tau_1 \tau_2 = \tau_3, \quad \tau_2 \tau_3 = \tau_1, \quad \tau_3 \tau_1 = \tau_2.$$

A special choice for the matrix elements $M_{(\alpha_1,\alpha_2);i,j}$ (A2) of the block matrices in (A1) is

$$M_{(\alpha_1,\alpha_2);i,j} = \sigma_{(\alpha_1,\alpha_2)} f_{i,j} \quad (i,j = 1,2,\ldots,m,\text{etc.}) \qquad (A25)$$

where $f_{i,j}$ are complex numbers and

$$\sigma_{(0,0)} = 1, \quad \sigma_{(1,0)} = \tau_1, \quad \sigma_{(0,1)} = \tau_2, \quad \sigma_{(1,1)} = \tau_1 \tau_2. \qquad (A26)$$

Then, using (A24) we find that the matrix element $M_{(\alpha_1,\alpha_2);i,j}$ in (A25) satisfies Eq. (A13). Therefore, we can calculate its determinant using our formulas (A15)–(A20). Thus, these rules give a prescription how to evaluate certain determinants over quaternions. Determinants of matrices with quaternions have been given by Dyson in closed form for self-dual matrices.[7] In contrast, the applicability of (A15)–(A20) does not depend on whether the matrix is self-dual or not.

APPENDIX B: CONSTRUCTION OF $C(2,s)$ COLOR SUPERALGEBRAS OUT OF SUPERALGEBRAS AND $C(2,a)$ COLOR ALGEBRAS OUT OF LIE ALGEBRAS

We define the following set of 4×4 matrices:

$$\rho_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_{(1,0)} = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \quad \rho_{(0,1)} = \begin{pmatrix} 0 & \tau_3 \\ \tau_3 & 0 \end{pmatrix},$$

$$\rho_{(1,1)} = \begin{pmatrix} 0 & \tau_2 \\ \tau_2 & 0 \end{pmatrix}, \qquad (B1)$$

where the τ_i are the 2×2 Pauli matrices.

Let us consider first the $C(2,s)$ algebras. Take *any* superalgebra with generators Q_m, V_μ which satisfy

$$[Q_m, Q_n] = f_{mn}^p Q_p,$$
$$[Q_m, V_\mu] = F_{m\mu}^\nu V_\nu. \qquad (B2)$$
$$\{V_\mu, V_\nu\} = A_{\mu\nu}^m Q_m.$$

Now we set

$$X_{(0,0),m} = \rho_{(0,0)} Q_m, \quad X_{(1,1),m} = \rho_{(1,1)} Q_m$$

$$X_{(1,0),\mu} = \rho_{(1,0)} V_\mu, \quad X_{(0,1),\mu} = \rho_{(0,1)} V_\mu. \qquad (B3)$$

Using the commutation rules for the $\rho_{(\alpha_1,\alpha_2)}$ which are easily derived from Eq. (B1) and applying Eqs. (B2) we get

$$[X_{(0,0),m}, X_{(0,0),n}] = f_{mn}^p X_{(0,0),p},$$

$$[X_{(0,0),m}, X_{(\alpha_1,\alpha_2),\mu}] = F_{m\mu}^\nu X_{(\alpha_1,\alpha_2),\nu}(\alpha_1,\alpha_2)$$

$$= (1,0) \text{ or } (0,1),$$

$$[X_{(0,0),m}, X_{(1,1),n}] = f_{mn}^p X_{(1,1),p},$$

$$\{X_{(1,0),\mu}, X_{(1,0),\nu}\} = A_{\mu\nu}^m X_{(0,0),m},$$

$$\{X_{(0,1),\mu}, X_{(0,1),\nu}\} = A_{\mu\nu}^m X_{(0,0),m},$$

$$[X_{(1,1),m}, X_{(1,1),n}] = f_{mn}^p X_{(0,0),p}, \qquad (B4)$$

$$[X_{(1,0),\mu}, X_{(0,1),\nu}] = i A_{\mu\nu}^m X_{(1,1),m},$$

$$\{X_{(1,1),m}, X_{(1,0),\mu}\} = i F_{m\mu}^\nu X_{(0,1),\nu},$$

$$\{X_{(1,1),m}, X_{(1,0),\mu}\} = -i F_{m\mu}^\nu X_{(0,1),\nu}.$$

Equation (B4) thus tells us that starting from an arbitrary superalgebra (B2) we can construct a $C(2,s)$ algebra with the *same* structure constant and which satisfy the generalized Jacobi identity of the $C(2,s)$ algebra.

The case of the $C(2,a)$ algebras is treated in the same fashion. We start from *any* Lie algebra,

$$[Q_m, Q_n] = C_{mn}^p Q_p, \qquad (B5)$$

and set

$$X_{(0,0),m} = \rho_{(0,0)} Q_m, \quad X_{(1,1),m} = \rho_{(1,1)} Q_m,$$

$$X_{(1,0),m} = \rho_{(1,0)} Q_m, \quad X_{(0,1),m} = \rho_{(0,1)} Q_m, \qquad (B6)$$

It is straightforward and to show that the $X_{(\alpha_1,\alpha_2),m}$ defined this way obey

$$[X_{(0,0),m}, X_{(\alpha_1,\alpha_2),n}] = C_{mn}^p X_{(\alpha_1,\alpha_2),p},$$

$$[X_{(\alpha_1,\alpha_2),m}, X_{(\alpha_1,\alpha_2),n}] = C_{mn}^p X_{(0,0),p},$$

$$\{X_{(1,0),m}, X_{(0,1),n}\} = i C_{mn}^p X_{(1,1),p}, \qquad (B7)$$

$$\{X_{(1,1),m}, X_{(\alpha_1,\alpha_2),n}\} = i C_{mn}^p X_{(\alpha_1,\alpha_2),p},$$

$$(\alpha_1,\alpha_2) = (1,0) \text{ or } (0,1).$$

Also in this case we see that an arbitrary Lie algebra (B5) gives rise to a $C(2,a)$ color algebra with the *same* structure constants which satisfy the Jacobi identiy of the $C(2,a)$ algebra.

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[7]A quaternion matrix M is self-dual, if $M_{lk} = \bar{M}_{kl}$, where $\bar{M}_{kl}$ is the quaternion complex conjugate: If $M_{kl} = a_0 + a_i \tau_i (a_0, a_i$ real), then
$$\bar{M}_{kl} = a_0 - a_i \tau_i.$$

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$$\bar{M}_{kl} = a_0 - a_i \tau_i.$$

On the inversion of the Weyl transform

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An inversion theorem for the Weyl operators of the Hilbert–Schmidt class is proved by using the polar decomposition of an operator.

1. INTRODUCTION

While the Weyl transform assigns phase space functions to linear operators in a Hilbert space,[1,2] the Wigner method[3] defines the quasiprobability distribution function corresponding to the density operator of a given physical system in equilibrium at temperature T. It is generally known[4] that the Weyl transform and the Wigner method are "equivalent." Segal[5] gave, in the language of a locally compact group, a rigorous inversion formula for the case when the Weyl operators were of trace class. Peetre,[6] in his interest to point out a connection between the Weyl transform and the Laguerre polynomials, derived explicitly the inverse Weyl transform of an orthogonal projection operator. In the present paper, we make use of the polar decomposition of an operator and give an inversion theorem for the Weyl operators of Hilbert–Schmidt class. Our method of proof is uncomplicated and may point the way to a much more general inversion theorem, since the domain of the Weyl correspondence has been recently extended to a setting which includes a large class of tempered distributions.[2]

2. WIGNER QUASIPROBABILITY DISTRIBUTION FUNCTIONS

Let π be the irreducible Weyl system[7] on the Hilbert space of square-integrable function on R, $L^2(R)$ and for each ϕ,ψ in $L^2(R)$, define $K_{\phi,\psi}(s,t)=\langle\pi(s,t)\phi,\psi\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(R)$. Since $[\pi(s,t)f]=e^{is(x-t/2)}f(x-t)$, (for the sake of simplicity, we let the parameter c corresponding to the Planck constant equal to 1). $K_{\phi,\psi}$ has the following integral expression:

$$K_{\phi,\psi}(s,t)=\int e^{is(x-t/2)}\phi(x-t)\bar{\psi}(x)dx$$

and $K_{\phi,\psi}$ can be shown[1] to be square-integrable or, an element of $L^2(R^2)$. The Fourier transform of $K_{\phi,\psi}$ is given by

$$\tilde{K}_{\phi,\psi}(p,q)=\int e^{-ipt}\phi(q-\tfrac{1}{2}t)\bar{\psi}(q+\tfrac{1}{2}t)dt,$$

where $\bar{\psi}$ denotes the complex conjugate of ψ. When $\phi=\psi$, $\tilde{K}_{\phi,\phi}$ is the Wigner quasiprobability distribution function corresponding to the wavefunction ϕ. Also, Pool[1] showed that $\tilde{K}_{\phi,\psi}=U(\phi\times\bar{\psi})$, where U is a unitary operator from $L^2(R^2)$ to itself and $(\phi\times\bar{\psi})(p,q)=\phi(p)\psi(q)$. If $\{\phi_l\}$ is a complete orthonormal basis in $L^2(R)$ and let

$$\tilde{K}_{lm}(p,q)=\int e^{-ipt}\phi_l(q-\tfrac{1}{2}t)\bar{\psi}_m(q+\tfrac{1}{2}t)dt,$$

$\{\tilde{K}_{lm}\}$ is a complete orthonormal basis in $L^2(R^2)$ since $\{\phi_l\times\bar{\phi}_m\}$ is. It follows then that $\{K_{lm}\}$ is also a complete orthonormal basis.

3. AN INVERSION THEOREM FOR THE WEYL OPERATORS OF HILBERT–SCHMIDT CLASS

For the properties of the Weyl correspondence, the reader is referred to Sec. III of Ref. 2, where other related references are cited. When the domain of the Weyl correspondence is restricted to $L^2(R^2)$, we have the following inversion theorem:

Theorem: Let A be a linear operator on $L^2(R)$ of Hilbert–Schmidt type. Then there is a unique function g in $L^2(R^2)$ such that $W(g)=A$, where W denotes the Weyl correspondence.

To give a proof to the above theorem, the following lemma is needed. Let P_{lm} be the linear operator on $L^2(R)$ to itself given by

$$P_{lm}x=\langle x,\phi_m\rangle\phi_l,$$

$$x\in L^2(R), \quad l,m,=0,1,2,\cdots.$$

Lemma: $W(\tilde{\tilde{K}}_{lm})=P_{lm}$.

Proof: By definition,

$$\langle W(\tilde{K}_{lm})\phi_j,\phi_k\rangle$$

$$=\int\int\langle\pi(s,t)\phi_j,\phi_k\rangle\bar{K}_{lm}(s,t)ds\,dt$$

$$=\int\int K_{j,k}(s,t)\bar{K}_{lm}(s,t)ds\,dt$$

$$=\left(\int\phi_j(x)\bar{\phi}_l(x)dx\right)\left(\int\phi_m(y)\bar{\phi}_k(y)dy\right)$$

$$=\langle\phi_j,\phi_l\rangle\langle\phi_m,\phi_k\rangle.$$

In the above we have made use of the fact that

$$\tilde{K}_{lm}=U(\phi_l\times\bar{\phi}_m).$$

Now let $x=\Sigma\,a_j\phi_j,\,y=\Sigma\,b_k\phi_k$ so that

$$\langle\pi(s,t)x,y\rangle=\sum_{j,k}a_j\bar{b}_k\langle\pi(s,t)\phi_j,\phi_k\rangle$$

and we have for each l,m,

$$\langle W(\tilde{\tilde{K}}_{lm})x,y\rangle=\langle x,\phi_l\rangle\langle\phi_m,y\rangle$$

$$=\sum_{j,k}a_j\bar{b}_k\langle\phi_j,\phi_j\rangle\langle\phi_m,\phi_k\rangle$$

$$=a_l\bar{b}_m,$$

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for all x,y, in $L^2(R)$. The above interchange of summation and integration can be justified since $\pi(s,t)$ is unitary and W is an isometry. On the other hand,

$$\langle P_{lm}, x, y \rangle = \langle \langle x, \phi_l \rangle \phi_m, y \rangle$$
$$= \sum_{j,k} a_j \bar{b}_k \langle \phi_j, \phi_l \rangle \langle \phi_m, \phi_k \rangle$$
$$= a_l \bar{b}_m,$$

for all x,y and $L^2(R)$. Hence

$$W(\widetilde{\widetilde{K}}_{lm}) = P_{lm} \quad \text{for } l,m = 0,1,2,\cdots.$$

Proof of Theorem: Since A us if Hilbert–Schmidt class, A has a unique polar representation[8] $A = V[A]$, where V is a partial isometry and $[A]$ is the positive part of A and

$$[A] = \sum \lambda_l P_{ll},$$

where $\lambda_l \geqslant 0$ and $\Sigma \lambda_l^2 < \infty$. Let $g_0 = \Sigma \lambda_l \widetilde{\widetilde{K}}_{ll}$. g_0 is in $L^2(R^2)$ since $\Sigma \lambda_l^2 < \infty$. If

$$g_{0n} = \sum_{l=0}^{n} \lambda_l \widetilde{\widetilde{K}}_{ll},$$

then $g_{0n} \to g_0$ in the L^2 sense. Since W is an isometry from $L^2(R^2)$ to the Banach algebra of Hilbert–Schmidt class operators, we have, using our lemma,

$$W(g_{0n}) \to W(g_0) \quad \text{as } n \to \infty.$$

But

$$W(g_{0n}) = \sum_{l=0}^{n} \lambda_l W(\widetilde{\widetilde{K}}_{ll})$$
$$= \sum_{l=0}^{n} \lambda_l P_{ll} \to [A] = \sum_{l=0}^{\infty} \lambda_l P_{ll},$$

so we have

$$W(g_0) = [A].$$

Now

$$A = V[A] = V \sum \lambda_l P_{ll}$$
$$= \sum \lambda_l V P_{ll} = \sum \lambda_l P_{lVl},$$

where $P_{lVl} x = \langle x, \phi_l \rangle V \phi_l$ and $Vl = V\phi_l$. If we let

$$g = \sum \lambda_l \widetilde{\widetilde{K}}_{lVl},$$

we have

$$W(g) = \sum \lambda_l P_{lVl}$$
$$= V \sum \lambda_l P_{ll} = A.$$

The uniqueness of g follows from the unique polar decomposition of A and that W is an injection.

The following corollary is immediate:

Corollary: Suppose A is of trace class, then

$$g = \sum \lambda_l P_{lVl} \quad \text{with} \quad \sum \lambda_l < \infty$$

and

$$\text{tr} A = \text{tr}[W(g)] = \sum \lambda_l,$$

where $\text{tr} A$ is the trace of A.

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Killing vectors in gauge supersymmetry

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The definition of a Killing vector for spaces with both Bose and Fermi dimensions is developed. Some properties of these vectors are studied, and they are used to define spaces of maximal supersymmetry. Certain theorems on symmetric spaces are shown to have supersymmetric analogs.

Gauge supersymmetry[1-3] and geometrized supergravity theories[4-8] exhibit many formal similarities to general relativity.[9] It is therefore profitable to transcribe as much as possible of the mathematical formalism developed for general relativity to the language of the more inclusive supersymmetry space theories. Killing vectors are particularly interesting as a convenient mechanism for classifying the invariance of given solutions of the supersymmetry field equations (although of course they remain mute on the mechanism producing the invariances).

This note extends the definition of Killing vectors to supersymmetry space and illustrates how some of the theorems on symmetric spaces of Bose coordinates carry over to spaces with Fermi coordinates as well. The algebraic manipulations are patterned as closely as possible after those of Weinberg.[10] The notation is that of Arnowitt and Nath,[2] except that elements of "left covariant" vectors, transforming as left derivatives of scalars, appear here as $(_A u)$, that is with the supersymmetry index as prescript.

Consider a supersymmetry space equipped with a connection $_A\Gamma^B{}_C$ and a metric tensor $_A g_B$. The connection may include a torsion tensor $_A T^B{}_c$ possessing the antiaffine symmetry

$$_A T^B{}_C = -(-1)^{a+c+ab+ca+bc}{}_C T^B{}_A \qquad (1)$$

and the metric tensor has vanishing covariant derivative,

$$_A g_{B;C} = 0 \qquad (2)$$

defined on the full connection $_A\Gamma^B{}_C$. If the metric is form invariant, to first order, relative to transformations

$$z'^A = z^A + \xi^A(z), \qquad (3)$$

then to this order the transformation functions ξ^A obey

$$0 = \xi_{A;B} + (-1)^{ab}\xi_{B;A} + 2(T_{ABC} + (-1)^{ab}T_{BAC})\xi^C, \quad (4)$$

where

$$T_{ABC} \equiv (-1)^{a+b+a(b+d)}{}_B g_D{}_A T^D{}_C \qquad (5)$$

defines the right covariant tensor T_{ABC}, which has the symmetry

$$T_{ABC} = -(-1)^{ab+ca+bc}T_{CBA}. \qquad (6)$$

Equation (4) is the natural definition of a Killing vector in this space as it reduces, in torsion-free spaces of Bose coordinates, to the usual Killing condition

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \qquad (7)$$

If the space has N_b Bose and N_f Fermi dimensions, then upon decomposing Eq. (4) into its Bose–Bose, Bose–Fermi, and Fermi–Fermi sectors one finds that

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} + 2(T_{\mu\nu A} + T_{\nu\mu A})\xi^A = 0 \qquad (8)$$

represents $\frac{1}{2}N_b(N_b + 1)$ conditions on $\xi_{\mu;\nu}$;

$$\xi_{\mu;\alpha} + \xi_{\alpha;\mu} + 2(T_{\mu\alpha A} + T_{\alpha\mu A})\xi^A = 0 \qquad (9)$$

represents $N_b N_f$ conditions on $\xi_{\mu;\alpha}$ and $\xi_{\alpha;\mu}$; and

$$\xi_{\alpha;\beta} - \xi_{\beta;\alpha} + 2(T_{\alpha\beta A} - T_{\beta\alpha A})\xi^A = 0 \qquad (10)$$

represents $\frac{1}{2}N_f(N_f - 1)$ conditions on $\xi_{\alpha;\beta}$, since it imposes no constraints on the "diagonal" elements of $\xi_{\alpha;\beta}$. The total number of constraints is thus $\frac{1}{2}(N_b + N_f)(N_b + N_f + 1) - N_f$, leaving $\frac{1}{2}(N_b + N_f)(N_b + N_f - 1) + N_f$ linearly independent elements $\xi_{A;B}$. It is advantageous to use the symmetry of Eq. (4) to reduce it to the form

$$0 = \xi_{A|B} + (-1)^{ab}\xi_{B|A} + 2(T_{ABC} + (-1)^{ab}T_{BAC})\xi^C, \qquad (11)$$

where the slash indicates covariant differentiation on the affine-symmetric part only of the connection. Using the identity

$$u_{A|B|C} - (-1)^{bc}u_{A|C|B} = -(-1)^{a+ad}u_{DA}R^D{}_{BC}, \qquad (12)$$

where $_A R^D{}_{BC}$ is reduced curvature tensor constructed from the affine-symmetric part of the connection, one finds upon exhausting the symmetry represented by the cyclic identity

$$(-1)^{a+a(c+d)}{}_A R^D{}_{BC}$$
$$+ (-1)^{b+b(a+d)}{}_B R^D{}_{CA} + (-1)^{c+c(b+d)}{}_C R^D{}_{AB} = 0 \qquad (13)$$

on $_A R^B{}_{CD}$ that $\xi_{A|B|C}$ is given by

$$\xi_{C|B|A} = -(-1)^{bc+ab+ca+a+ad}\xi_{DA}R^D{}_{BC}$$
$$+ \frac{1}{2}((T_{CBD} + (-1)^{bc}T_{BCD})\xi^D)_{|A}$$
$$- \frac{1}{2}(-1)^{ab+ca}((T_{ACD} + (-1)^{ac}T_{CAD})\xi^D)_{|B}$$
$$+ \frac{1}{2}(-1)^{ab+ca+bc}((T_{ABD} + (-1)^{ab}T_{BAD})\xi^D)_{|C}. \qquad (14)$$

From Eq. (14) one concludes that specification on ξ_A and $\xi_{A,B}$ for a given "point" (with coordinates Z^A) in supersymmetry space suffices to specify the Killing vector everywhere as a linear function of these quantities; thus there are at most $\frac{1}{2}(N_b + N_f)(N_b + N_f + 1) + N_f$ independent Killing vectors in a supersymmetry space of $N_b + N_f$ dimensions, corresponding to the $N_b + N_f$ quantities $\xi_A(Z)$ plus the $\frac{1}{2}(N_b + N_f)(N_b + N_f - 1) + N_f$ independent members of $\xi_{A|B}(Z)$. A space admitting this many Killing vectors is maximally supersymmetric.

An obvious example of a maximally supersymmetric space is a "flat supermetric" space admitting global coordinates for which the connection vanishes. In this case the solution of the Killing equations are

$$\xi_\mu = \epsilon_\mu + \epsilon_{\mu\nu} x^\mu + \epsilon_{\mu\alpha}\theta^\alpha \tag{15}$$

and

$$\xi_\alpha = \lambda_\alpha + \lambda_{\alpha\beta}\theta^\beta - \epsilon_{\mu\alpha} x^\mu \tag{16}$$

$$\epsilon_{(\mu\nu)} = 0 = \lambda_{[\alpha\beta]}. \tag{17}$$

This space has been studied in detail by Friedman and Srivastava.[11] The space of global supersymmetry,[5] of four Bose and four (Majorana) Fermi dimensions, admits fourteen Killing vectors. Thus the metric for this space is[4]

$$_\mu g_\nu = \eta_{\mu\nu},$$
$$_\mu g_\alpha = -i(\bar\theta\gamma_\mu)_\alpha,$$
$$_\alpha g_\beta = k\eta_{\alpha\beta} + (\bar\theta\gamma^\lambda)_\alpha(\bar\theta\gamma_\lambda)_\beta, \tag{18}$$

from which one obtains he Bose–Bose sector Killing equations

$$\xi_{\mu,\nu} + \xi_{\nu,\mu} = 0; \tag{19}$$

the Bose–Fermi sector equations

$$\xi_{\alpha,\mu} + \xi_{\mu,\alpha}$$
$$+ \frac{2}{k}(i(\bar\theta\sigma_\mu{}^\nu)_\alpha$$
$$- \bar\theta_\alpha\delta_\mu{}^\nu)\xi_\nu - \frac{2i}{k}(\xi\gamma_\mu)_\alpha = 0 \tag{20}$$

and the Fermi–Fermi sector equations

$$\xi_{\beta,\alpha} - \xi_{\alpha,\beta} + 2((\bar\theta\gamma^\mu)_\alpha\bar\theta_\beta - (\bar\theta\gamma_\mu)_\beta\bar\theta_\alpha)\xi_\mu$$
$$- 2i((\bar\theta\gamma_\mu)_\alpha(\bar\theta\sigma^{\mu\nu})_\beta - (\bar\theta\gamma_\mu)_\beta(\bar\theta\sigma^{\mu\nu})_\alpha)\xi_\nu$$
$$+ \frac{2}{k}((\bar\theta\gamma_\mu)_\alpha(\xi\gamma^\mu)_\beta(\bar\theta\gamma_\mu)_\beta(\xi\gamma^\mu)_\alpha) = 0 \tag{21}$$

These equations have the unique solutions[12]

$$\xi_\mu = \epsilon_\mu + \epsilon_{\mu\nu} x^\nu + 2i\bar\lambda\gamma_\mu\theta$$
$$+ \tfrac{1}{2}\bar\theta\gamma_\mu\sigma^{\nu\rho}\theta\epsilon_{\nu\rho} \tag{22}$$

and

$$\xi_\mu = k\bar\lambda_\alpha - i(\bar\theta\gamma_\mu)_\alpha\epsilon^\mu$$
$$- \frac{ik}{2}(\bar\theta\sigma^{\mu\nu})_\alpha\epsilon_{\mu\nu} - i(\bar\theta\gamma^\mu)_\alpha\epsilon_{\mu\nu} x^\nu$$
$$+ (2\bar\lambda\gamma^\mu\theta)(\bar\theta\gamma_\mu)_\alpha - \frac{i}{2}(\bar\theta\gamma^\mu\sigma^{\nu\rho}\theta)(\bar\theta\gamma_\mu)_\alpha\epsilon_{\nu\rho}. \tag{23}$$

These correspond to the transformations of global supersymmetry,

$$\xi^\mu = \epsilon^\mu + \epsilon^\mu{}_{,\nu} x^\nu + i\bar\lambda\gamma^\mu\theta \tag{24}$$

and

$$\xi^\alpha = \lambda^\alpha + \frac{i}{2}\epsilon_{\mu\nu}(\sigma^{\mu\nu}\theta)^\alpha \tag{25}$$

as one verifies by raising indices with the inverse of the global supersymmetry metric,

$$g^{\mu\nu} = \eta^{\mu\nu}\left(1 - \frac{1}{k}\,\bar\theta\theta\right),$$

$$g^{\mu\alpha} = -\frac{i}{k}\,(\gamma^\mu\theta)^\alpha,$$

$$g^{\alpha\beta} = \frac{1}{k}\eta^{\alpha\beta}. \tag{26}$$

Global supersymmetry on deSitter space[13-14] also possesses fourteen Killing vectors,[15] corresponding to Lorentz rotations, pseudotranslations, and four Fermi-like transformations, the analog of the λ^α appearing in Eq. (25). It is perhaps worth emphasizing that global supersymmetry on deSitter space is *not* maximally supersymmetric in the sense used here. In fact it even possesses no maximally supersymmetric subspaces, if one adopts as the definition of a maximally supersymmetric subspace the following. Let a space of $N_b + N'_b$ Bose and $N_f + N'_f$ Fermi dimensions be characterized by coordinates u^K, $K = 1,2,...,N_b + N_f$, and v^A, $A = 1,2,...,N'_b + N'_f$, where N_b of the u^K are Bose coordinates (the convention being that v-coordinate labels are chosen from the beginning and u-coordinate labels from the middle of the alphabet). The subspaces of fixed v^A are maximally supersymmetric if the metric of the whole space is form-invariant relative to $\frac{1}{2}(N_b + N_f)(N_b + N_f + 1) + N_f$ independent transformations of the form

$$u'^K = u^K + \xi^K(u,v),\quad v'^A = v^A. \tag{27}$$

As such spaces may have some physical interest it is worthwhile to show that the metric of a space with a maximally supersymmetric subspace can always be cast into the form

$$ds^2 = dv^A{}_A g_B(v)dv^B$$
$$+ f(v)du^K{}_K\tilde{g}_L(u)du^L, \tag{28}$$

where $_K\tilde{g}_L$ serves as the metric of the u^K-subspace. (This is the analog of a theorem in a Riemann space of Bose dimensions.) The proof hinges on finding supercoordinates for which $_A g_K = 0$; for if this can be done the relevant Killing equations reduce to

$$_K g_M \xi^M{}_{,A} = 0 \tag{29}$$

and

$$_A g_{B,M}\xi^M = 0 \tag{30}$$

plus

$$(-1)^{k+km}\xi^M{}_{,KM} g_L + +_K g_{L,M}\xi^M = 0. \tag{31}$$

Equation (29) implies that ξ^K is a function only of the u-coordinates and Eq. (30), which must hold for the entire suite of Killing vectors, implies that $_A g_B$ is a function only of the v-coordinates. As a consequence, Eq. (31), for possible *fixed* choices v_0^A for the v-coordinates, shows that $_K g_L(u,v_0) = _K\tilde{g}_L(u)$ is acceptable as a metric for the u-subspace, as it satisfies the Killing condition on the subspace for each of the Killing vectors. Furthermore, even for arbitrary choices of the v-coordinate, Eq. (31) states that the Lie derivative $_K g_L(u,v)$ with respect to each Killing vector vanishes, and hence

$$_K g_L(u,v) = f(v)\,_K\tilde{g}_L(u). \tag{32}$$

To see this, observe that since the $N_b + N_f \, \xi^A$ and the $\frac{1}{2}(N_b + N_f)(N_b + N_f - 1) + N_f$,

$$\xi^{(A;B)} \equiv \xi^{A;B} + (-1)^{ab}\xi^{B;A} , \tag{33}$$

are independent, where

$$\xi^{A;B} \equiv \xi^A{}_{;C} g^{CB} ; \tag{34}$$

one has that the coefficient of $\xi^{(A;B)}$ must itself vanish, where

$$_K \tilde{g}_{LM} \tilde{g}_N + (-1)^{k+l+n(k+m)+k(l+m)} {}_L \tilde{g}_N {}_M \tilde{g}_K$$
$$= (-1)^{l+m+lm} {}_K \tilde{g}_M {}_L \tilde{g}_N$$
$$+ (-1)^{k+l+k(l+m+n)+ln} {}_M \tilde{g}_N {}_L \tilde{g}_K . \tag{35}$$

Contracting with $\tilde{g}^{NM}$ produces

$$(N_b + N_f) {}_K g_L = (-1)^m ({}_M g^M) {}_K \tilde{g}_L \equiv f {}_K \tilde{g}_L . \tag{36}$$

Applying this result to Eq. (31) shows that $f_{,K} = 0$, and therefore it is only a function of the v-coordinates, as required. To verify that supercoordinates can in fact be found for which $_A g_K$ vanishes, consider a transformation

$$u^K = U^K(u', v'), \quad v^A = v'^A \tag{37}$$

such that

$$_A g'_K(u', v') = {}_A g_L u^L{}_{,K}$$
$$+ (-1)^{a+al} u^L{}_{,A} {}_L g_M u^M{}_{,K} = 0 \tag{38}$$

The condition on the functions U are thus

$$_A g_K(U, v') + (-1)^{a+al} U^L{}_{,A}(u', v') {}_L g_K(U, v') = 0 . \tag{39}$$

Upon defining $\bar{g}^{KM}$ via

$$_L g_K \bar{g}^{KM} = \delta_M^L , \tag{40}$$

Eq. (39) becomes

$$U^L{}_{,A} = -F^L{}_A(U, v) , \tag{41}$$
where

$$F^L{}_A(U, v) \equiv (-1)^{a+al} {}_A g_K(U, v) \bar{g}^{KL}(U, v) . \tag{42}$$

For a solution to exist, U^L must satisfy the "integrability conditions"

$$U^L{}_{,AB} = (-1)^{ab} U^L{}_{,BA} \tag{43}$$

which translate into the following condition on $F^L{}_A$,
$$F^L{}_{A,K}(u, v) F^K{}_B(u, v) - F^L{}_{A,B}(u, v) = (-1)^{ab}(F^L{}_{B,K}(u, v) F^K{}_A(u, v)$$
$$- F^L{}_{B,A}(u, v)). \tag{44}$$

That this conditions is satisfied becomes apparent by referring back to the Killing equations; after multiplication by suitable factors $\bar{g}^{MK}$ two of them become

$$0 = (-1)^{a+ak}\xi^K{}_{,A} + {}_A g_L \xi^L{}_{,M} \bar{g}^{MK}$$
$$+ {}_A g_{M,L} \xi^L \bar{g}^{MK} \tag{45}$$
and

$$0 = \xi^N{}_{,K} \bar{g}^{KI}$$
$$+ (-1)^{k+kn} {}_K g_M \xi^M{}_{,L} \bar{g}^{LN} \bar{g}^{KP}$$
$$+ (-1)^{k+kn} {}_K g_{L,M} \xi^M \bar{g}^{LN} \bar{g}^{KP} . \tag{46}$$

By combining Eqs. (45) and (46) the derivative of ξ^K with respect to the v-coordinates appears as

$$\xi^K{}_{,A} = \xi^K{}_{,M} F^M{}_A F^K{}_{A,M} \xi^M . \tag{47}$$

A second derivative with respect to the v-coordinates produces the factor $F^L{}_{A,B} + (-1)^{ab}F^L{}_{B,M}F^M{}_A$ as the coefficient of $\xi^K{}_{,L}$. Thus, since the Killing vectors all satisfy the integrability condition by assumption and the coefficient of $\xi^K{}_{,L}$ in $\xi^K{}_{,AB} - (-1)^{ab}\xi^K{}_{,BA} = 0$ must vanish by itself, $F^L{}_A$ must satisfy Eq. (46).

The theorem that a maximally symmetric space without torsion has constant curvature extends directly to supersymmetry spaces. Thus for a torsion-free supersymmetry space one has for any vector the identity

$$u_{A|B|C|D} - (-1)^{cd} u_{A|B|D|C}$$
$$= -(-1)^{b+be} u_{A|E} {}_B R^E{}_{CD}$$
$$- (-1)^{ab+ea+be+a} u_{E|B} {}_A R^E{}_{CD} . \tag{48}$$

In particular for the Killing vectors the torsion-free version of Eq. (14) applied to this equation produces
$$-(-1)^{ab+ca+bc+c+ce}(\xi_E {}_C R^E{}_{BA})_D$$
$$+ (-1)^{ab+da+bd+d+de}(\xi_E {}_D R^E{}_{BA})_C$$
$$= -(-1)^{b+be}\xi_{A|E} {}_B R^E{}_{CD}$$
$$- (-1)^{ab+ea+be+a}\xi_{E|B} {}_A R^E{}_{CD}. \tag{49}$$

Applying the (torsion-free) Killing condition on $\xi_{A|B}$ to this expression and setting the Killing-symmetric coefficient of $\xi_{A|B}$ to zero, one obtains the following rather intricate identity on the curvature:

$$-(-1)^{b+(a+b)e-[(a+b)(c+d)+cd+ab+ef]/2}\delta^F{}_A {}_B R^E{}_{CD}$$
$$-(-1)^{b+(a+b)f-[(a+b)(c+d)+cd+ab-ef]/2}\delta^E{}_A {}_B R^F{}_{CD}$$
$$-(-1)^{a+(a+b)e-[(a+b)(c+d)+cd-ab+ef]/2}\delta^F{}_B {}_A R^E{}_{CD}$$
$$+(-1)^{a+(a+b)f-[(a+b)(c+d)+cd-ab-ef]/2}\delta^E{}_B {}_A R^F{}_{CD}$$
$$-(-1)^{c+(c+d)e+[(a+b)(c+d)+cd-ab-ef]/2}\delta^F{}_D {}_C R^E{}_{AB}$$
$$+(-1)^{c+(c+d)f+[(a+b)(c+d)+cd-ab+ef]/2}\delta^E{}_D {}_C R^F{}_{AB}$$
$$+(-1)^{d+(c+d)e+[(a+b)(c+d)-cd-ab-ef]/2}\delta^F{}_C {}_D R^E{}_{AB}$$
$$-(-1)^{d+(c+d)f+[(a+b)(c+d)-cd-ab+ef]/2}\delta^E{}_C {}_D R^F{}_{AB}$$

$$= 0 \tag{50}$$

For a torsion-free space the curvature tensor exhibits the further symmetry

$$_A R_{DBC} = -(-1)^{a+d+ad} {}_D R_{ABC} . \tag{51}$$

Lowering the remaining contravariant index in Eq. (50) to exhaust the symmetries of the curvature, one finds upon contraction that

$$(N_b + N_f)(N_b + N_f + 1)_A R_{BCD}$$
$$= R((-1)^{b(c+d)+c+cd}{}_A g_D c g_B$$
$$-(-1)^{b(c+d)+d}{}_A g_C D g_B) ,$$
(52)

where the curvature scalar is defined as

$$R \equiv (-1)^a (_A R^A)$$
(53)

and the Ricci tensor is

$$_A R_B \equiv (-1)^c {}_A R^C{}_{CB} .$$
(54)

As a consequence of the contracted Bianchi identity for a torsion-free space the curvature scalar is constant, $R_{,A} = 0$, and hence upon setting $R \equiv -(N_b + N_f)(N_b + N_f + 1)K$, one has that a torsion-free maximally supersymmetry space is uniquely characterized by its curvature constant K, in the sense that two metrics for which Eq. (52) is valid with the same numerical constant K are necessarily related by a coordinate transformation.

A similar analysis on spaces with torsion can be carried out; first one must separate out the reduced curvature tensor $_A R^D{}_{BC}$ to exhaust the cyclic symmetry of Eq. (13) and then this must be further decomposed into a "metric curvature" plus the residuum left over from the torsion,
$$_A R^D{}_{BC} = _A R^D{}_{BC}(\text{metric})$$

$$+ (-1)^{bd}{}_{(A}T_{B)}{}^D{}_{|C}$$

$$- (-1)^{c(b+d)}{}_{(A}T_{C)}{}^D{}_{|B}$$

$$+ (-1)^{d(b+c)}{}_{(A}T_{B)}{}^E{}_{(E}T_{C)}{}^D$$

$$(-1)^{bc+db+cd}{}_{(A}T_{C)}{}^B{}_{(E}T_{B)}{}^D ,$$
(55)

where for convenience the abbreviation

$$_{(A}T_{B)}{}^C \equiv (-1)^a (T_{ABD}) + (-1)^{ab} T_{BAD}) g^{DC}$$
(56)

was used. This step is required since only the metric curvature exhibits the additional symmetry of Eq. (51). The ultimate result is that a maximally supersymmetric space with torsion is *not* a space of constant curvature, but rather the curvature scalar is a complicated function of the torsion and its derivative.

One verifies by explicit construction that the metric of a maximally supersymmetric (torsion-free) space of curvature constant K takes the form

$$_A g_B = _A \eta_B + K(1 - Kz^C{}_C \eta_D z^D)^{-1}{}_A \eta_C z^C z^D{}_D \eta_B , \quad (57)$$

where $_A \eta_B$ are constants exhibiting the symmetry.

$$_A \eta_B = (-1)^{a+b+ab}{}_B \eta_A$$
(58)

and that such a space of N_b Bose and N_f Fermi dimensions can be embedded in a flat supermetric space of $N_b + 1$ Bose and N_f Fermi dimensions. The general question of how to embed a curved supersymmetry space into a flat space of higher dimension remains open at present.

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General expressions for divergence relations and multipole expansions for arbitrary scalar functions

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In theories of the cohesion of a system whose matter density is constant (e.g., classical liquid drops) or nearly constant except in a thin surface region (e.g., nuclei or neutron stars), there occur computationally difficult single and double volume integrals of energy densities. Such integrals have been important in recent dynamical calculations of fission and heavy-ion reactions. Even if the matter density is diffuse, the integrals can be written as integrals over finite volumes by modeling the density as the convolution of a step function and a diffuseness function. If the integrands can be written as divergences of tensor fields, the integrals over finite volumes can be reduced to surface integrals by Gauss' theorem. We have found general expressions, which we call divergence relations, for a vector field whose divergence is a given scalar function and for a second-rank tensor field whose double divergence is the scalar. The equations derived are much easier to use and apply to a wider class of functions than formulas previously obtained in the literature. The generalization to nth order for application to many-body forces is included, and for all orders the dimensionality is arbitrary. The interaction energy of two nonoverlapping systems is often most simply evaluated by using the generalized Slater multipole expansion of the two-body interaction. A new expression is derived for the radial factor $G_i(r_1, r_2)$ appearing in the multipole expansion of an arbitrary scalar, two-body function. This G_l is expressed as an integral involving the product of the Fourier transform of the interaction and two Bessel functions. For some cases this integral can easily be evaluated by contour integration.

I. INTRODUCTION

In the liquid-drop model of the nucleus and in its generalizations to describe fission and the collision of two heavy nuclei[1-7] there occur single and double volume integrals of energy densities, which are difficult to evaluate. If the integrand is the divergence of a vector or tensor field, the calculation of the integral is simplified considerably by reducing it to a single or double surface integral by use of Gauss' theorem. We address the problem of expressing a general scalar density as the divergence of a vector field or the double divergence of a second-rank tensor field. Expressions of this kind, which we refer to as "divergence relations," have been obtained and generalized to N dimensions and to nth order. These relations could be used also in the theories of other leptodermous[1] (thin-skinned) systems such as neutron stars or in the limiting case of classical liquid drops, whose surface is macroscopically "sharp."

In the nuclear applications, one finds that, for static and dynamical studies of fission and heavy-ion reactions using the macroscopic-microscopic method,[1,2,6,7] it is necessary to calculate various volume integrals of the Coulomb and nuclear two-body potentials. For computing the microscopic Strutinsky shell corrections,[1,2,8,9] one needs to evaluate one-body potentials which are obtained from single volume integrals, whereas the macroscopic Coulomb and nuclear energies are obtained by computing double volume integrals.[3,4] If the energy densities appearing in these single and double integrals satisfy appropriate divergence relations, then the volume integrals can be converted into surface integrals,[1-5] thereby reducing the number of integration variables, which considerably simplifies the calculations. For axially-symmetric shapes these surface integrals are especially easy to

evaluate numerically, e.g., by Gaussian–Legendre quadrature.[2,4]

Also, in contrast to the previously mentioned models of fission[6] in which the radius of the neck goes to zero at scission, recent dynamical calculations have allowed the neck to rupture with a nonzero radius.[10] As the nuclear system descends dynamically from the saddle point to scission, there is a delicate balance between the Coulomb and nuclear *interaction* forces between the two separating parts of the nucleus. These interfragment forces, which determine the configuration at which the neck ruptures, can be calculated from Coulomb and nuclear interaction energies. The integrands in these double volume integrals satisfy double divergence relations,[10] which fact again by Gauss' theorem allows one to convert them to surface integrals.

Whereas in many liquid-drop model calculations[1-4,6,7,10] the nucleus is assumed to have a (finite or sharp) surface, in other studies[5,11-15] the nuclear density is allowed to decrease roughly exponentially at large radii. Then the nucleus is said to have a "diffuse surface." One model often used[11] is the parametrization of the density with a Fermi function. Unfortunately the resulting expressions for one-body potentials and energies are very cumbersome to evaluate. In particular, the volume integrals extend over all space and, thus, it is impossible to simplify them by using Gauss' theorem. A second model[5,12-15] is to write the nuclear density distribution as a convolution of a three-dimensional step function having a surface of arbitrary shape and a Gaussian or Yukawa diffuseness function. Volume integrals of the energy densities obtained from such nuclear densities can be reexpressed[5,12-15] as integrals over the finite volume where the step function is nonzero. Thus, these volume integrals in principle can be converted into surface integrals if one can demonstrate that the integrands satisfy divergence relations. However, due to the folding process the final integrands can be considerably

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more complicated than the Coulomb or nuclear functions appearing in the original diffuse-surface volume integrals. Therefore, it is very important to develop a general method for finding single- and double-divergence relations for arbitrary two-body functions.

In Appendices A and B of Ref. 5 general methods were presented for expressing an arbitrary two-body function as a single or double divergence of an appropriate tensor field. Two important restrictions were made in that paper.

(i) First of all, it was assumed that the two-body function f is a *scalar* quantity, i.e.,

$$f(\mathbf{p}) = f(\rho), \tag{1.1}$$

where

$$\mathbf{p} = \mathbf{r}_1 - \mathbf{r}_2 \tag{1.2}$$

is the vector separation of the two particles, with $\rho = |\mathbf{p}|$ its magnitude.

(ii) Next, it was assumed that the three-dimensional Fourier transform $\hat{f}(k)$ of $f(\rho)$ exists, so that $f(\rho)$ can be expressed as

$$f(\rho) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty dk \, \frac{\sin(k\rho)}{\rho} \, k \hat{f}(k)$$

$$= \frac{1}{\sqrt{2\pi i}} \int_0^\infty dk \left[\frac{e^{ik\rho}}{\rho} - \frac{e^{-ik\rho}}{\rho}\right] k\hat{f}(k). \tag{1.3}$$

From Eq. (1.1), $\hat{f}(k)$ is also a scalar function. Both assumptions (i) and (ii) are easily satisfied for most functions of physical interest. Note that the terms $e^{\pm ik\rho}/\rho$ in Eq. (1.3) correspond to Yukawa factors $e^{-\rho/a_\pm}/\rho$, where $1/a_\pm = \mp ik$, respectively. The "trick" used in Ref. 5 was to observe that *if* a particular expression exists for the *Yukawa function*, then from Eq. (1.3) one can derive a corresponding general expression for an *arbitrary* function $f(\rho)$ satisfying assumptions (i) and (ii). For the Yukawa two-body potential, single-divergence[1,2] and double-divergence[3] relations are known. When these are substituted into Eq. (1.3), one obtains general Fourier transform expressions for single- and double-divergence relations for an arbitrary scalar function.[5] As a check these were used to derive the previously known single-divergence[2] and double-divergence[4] relations for $1/\rho$, the Coulomb potential. In Appendices A and B of Ref. 5 tabulations were made of the single- and double-divergence relations appropriate to the following functions: The expotential potential $(e^{-\rho/a})$, the function $\rho e^{-\rho/a}$, the Gaussian potential $(e^{-\rho^2/a^2})$, and the function $1/\rho^2$. However, as the form of $f(\rho)$ or $\hat{f}(k)$ becomes more complicated, it becomes increasingly difficult to construct divergence relations by using the Fourier transform method.

In Sec. II of this paper we present completely new methods for generating single- and double-divergence relations for arbitrary, scalar, two-body functions. The formulas obtained apply to an arbitrary number (N) of spatial dimensions. The derivations are very simple and the expressions obtained are much easier to use than those of the Fourier transform method.[5] As an example, the single and double divergence relations in three-dimensional space are evaluat-

ed for the function $f(\rho) = \rho \, e^{-\rho^2/a^2}$, whose Fourier transform is very complicated. In contrast to the method of Ref. 5, we do not explicitly impose any condition requiring the existence of the Fourier transform of $f(\rho)$. Rather, a condition is imposed on the behavior of $f(\rho)$ only at $\rho = 0$. No condition is required for the behavior of $f(\rho)$ as $\rho \to \infty$. Thus the condition obtained here is *less* restrictive than that of Ref. 5 and our methods are applicable to a wider class of functions; e.g., some functions whose Fourier transforms do not exist, such as the r^2, r^4,... functions used to calculate radial moments of a nuclear shape.[5,12] A radial moment involves a single integral of a one-body function $f(r)$ [rather than of a two-body function $f(\rho)$]. If $\mathbf{r}_1$ (or $\mathbf{r}_2$) is set to zero in Eq. (1.2), then the single-divergence relation applies to the calculation of such moments and in fact to a general integral of a one-body function $f(r)$.

In Sec. II, we also generalize the divergence relations to an arbitrary order n (which specifies the number of divergence derivatives) so as to be applicable to systems with n-body forces. For $n \geqslant 3$ one must use a more general definition of $\mathbf{p}$ than Eq. (1.2). Thus, we let $\mathbf{p}$ be expressed as a general linear combination of K single-particle vectors $\mathbf{r}_\alpha$, $1 \leqslant \alpha \leqslant K$, where $K \geqslant n$.

Finally, in Sec. III of this paper we derive a new general expression for the radial factors in the multipole expansion of an arbitrary scalar two-body function.[16] The known expression for the multipole expansion of a Yukawa function[16,17] is substituted into Eq. (1.3) to obtain a general expression for any function. Thus, we find another application of the "Fourier transform, Yukawa trick" previously mentioned. As an example we give the multipole expansion of the function $f(\rho) = 1/\rho^2$, which is not found in standard references.[16]

II. GENERAL DIVERGENCE RELATIONS

Consider a differential equation of the form

$$f(\rho) = - \sum_{i_1, i_2, \dots, i_n = 1}^{N} \frac{\partial}{\partial \rho_{i_1}} \frac{\partial}{\partial \rho_{i_2}} \cdots \frac{\partial}{\partial \rho_{i_n}}$$

$$\times [\rho_{i_1} \rho_{i_2} \cdots \rho_{i_n} F_n(\rho)], \tag{2.1}$$

where both $f(\rho)$ and $F_n(\rho)$ are assumed to be scalar functions as defined by Eq. (1.1) and ρ_i is the ith Cartesian component in an N-dimensional space of the vector ρ, defined in Eq. (1.2) for $n = 1$ or 2 and in Eq. (2.34) for general n. The number n is the *order* of the divergence relation being studied, i.e., the number of derivatives occurring in Eq. (2.1). For a given function $f(\rho)$ and with specified values of n and N, we derive a general formula for constructing $F_n(\rho)$. The quantity in brackets in Eq. (2.1) is a special tensor field of order n. We shall sometimes refer to $F_n(\rho)$ as the nth-order "surface function" corresponding to $f(\rho)$. The minus sign multiplying the sum in Eq. (2.1) occurs in order to be consistent with notations previously adopted.[5]

In the next two subsections we will obtain expressions for $n = 1$ and $n = 2$, which are the single- and double-diver-

gence relations, respectively. These will then be specialized to three-dimensional space and compared with expressions previously derived.[5] In the third subsection we will prove by induction the general nth-order divergence relation.

A. The single-divergence relation

For $n = 1$, Eq. (2.1) becomes

$$f(\rho) = -\sum_{i=1}^{N} \frac{\partial}{\partial \rho_i} \rho_i F_1(\rho) = \nabla_{2} \cdot (\rho F_1(\rho)), \tag{2.2}$$

where the integration is over the $(N-1)$-dimensional surface S_2. The integral in Eq. (2.3a) converges if $f(\rho)$ is finite, except possibly at the coordinate singularity $\rho = 0$, for which we require

$$I(\mathbf{r}_1) = \int_{V_2} f(\rho) d^N r_2, \tag{2.3a}$$

and the integration is over a finite volume V_2. From Gauss' theorem Eq. (2.3a) becomes

$$I(\mathbf{r}_1) = \oint_{S_2} (d\mathbf{S}_2 \cdot \boldsymbol{\rho}) F_1(\rho), \tag{2.3b}$$

where the integration is over the $(N-1)$-dimensional surface S_2. The integral in Eq. (2.3a) converges if $f(\rho)$ is finite, except possibly at the coordinate singularity $\rho = 0$, for which we require

$$\lim_{\rho \to 0} [\rho^N f(\rho)] = 0. \tag{2.4}$$

This condition also assures that the integral vanishes as $V_2 \to 0$. The vanishing of the integral in Eq. (2.3b) as the surface S_2 shrinks arbitrarily close to $\mathbf{r}_1$ requires

$$\lim_{\rho \to 0} [\rho^N F_1(\rho)] = 0. \tag{2.5}$$

Now let the volume be an N-dimensional sphere of radius a and centered at $\mathbf{r}_2 = 0$. If we then let $\mathbf{r}_1 = 0$ in Eqs. (2.3a) and (2.3b), we find that

$$\omega_N \int_0^a dr_2 \, r_2^{(N-1)} f(r_2) = -a^{N-1} \int_{S_2} d\Omega_2 \left(\frac{\mathbf{r}_2}{r_2}\right) \cdot \mathbf{r}_2 F_1(r_2)$$
$$= -\omega_N a^N F_1(a), \tag{2.6}$$

where ω_N is the area of a unit sphere in N dimensions.[18] Equation (2.6) can be rewritten as

$$F_1(\rho) = -\frac{1}{\rho^N} \int_0^\rho d\rho' (\rho')^{N-1} f(\rho'), \tag{2.7}$$

which is a general expression for $F_1(\rho)$, the single-divergence "surface" function.

This is a very simple derivation of the $n = 1$ relation. However, it is helpful to present another slightly more involved derivation in order to gain insight into the methods to be used in the following subsections for the higher-order divergence formulas. These methods involve solving directly the original differential equation (2.1). Since $F_1(\rho)$ is a scalar function, we can reexpress Eq. (2.2) as

$$f(\rho) = -\nabla_\rho \cdot (\rho F_1(\rho))$$
$$= -\left[N F_1(\rho) + \rho \frac{dF_1}{d\rho}\right], \tag{2.8a}$$

or

$$\frac{d}{d\rho} [\rho^N F_1(\rho)] = -\rho^{(N-1)} f(\rho). \tag{2.8b}$$

Equation (2.8b) is a differential equation for $F_1(\rho)$ which, with the boundary condition Eq. (2.5), can be easily solved to obtain again Eq. (2.7).

In *three*-dimensional space Eqs. (2.4), (2.5), and (2.7) specialize respectively to

$$\lim_{\rho \to 0} [\rho^3 f(\rho)] = 0, \tag{2.9}$$

$$\lim_{\rho \to 0} [\rho^3 F_1(\rho)] = 0, \tag{2.10}$$

and

$$F_1(\rho) = -\frac{1}{\rho^3} \int_0^\rho d\rho' (\rho')^2 f(\rho'). \tag{2.11}$$

Substituting Eq. (1.3) into Eq. (2.11) and reversing the order-of-integration, we find that

$$F_1(\rho) = -\frac{1}{\rho^3} \sqrt{\frac{2}{\pi}} \int_0^\infty dk \, k \hat{f}(k) \int_0^\infty d\rho' \, \rho' \sin(k\rho'),$$

which becomes

$$F_1(\rho) = -\frac{1}{\rho^3} \sqrt{\frac{2}{\pi}}$$
$$\times \int_0^\infty \frac{dk}{k} \hat{f}(k) \, [\sin(k\rho) - k\rho \cos(k\rho)]. \tag{2.12}$$

Equation (2.12) is the Fourier transform expression[5] for the $N = 3$ single-divergence relation. It is much easier to calculate $F_1(\rho)$ from Eq. (2.11) than from Eq. (2.12). In Appendix A of Ref. 5 three-dimensional "surface" functions $F_1(\rho)$ are presented for each of the following functions[19]: $1/\rho$, $1/\rho^2$, $e^{-\rho/a}/\rho$, $e^{-\rho/a}$, $\rho e^{-\rho/a}$, and $e^{-\rho^2/a^2}$.

Equation (2.12) can be reexpressed as

$$F_1(\rho) = -\frac{1}{\rho^3} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{dx}{x} \hat{f}\left(\frac{x}{\rho}\right) (\sin x - x \cos x),$$

which demonstrates that Eq. (2.10) is satisfied if

$$\lim_{k \to \infty} [\hat{f}(k)] = 0. \tag{2.13a}$$

In addition, by examining the behavior of the integrand of Eq. (2.12) for small values of k,[5] we find that convergence of the integral requires

$$\lim_{k \to 0} [k^3 \hat{f}(k)] = 0. \tag{2.13b}$$

Equation (2.13b) states that at the origin in k-space $\hat{f}(k)$ is no more singular than $k^{-3+\epsilon}$, where $\epsilon > 0$. Equations (2.13) are the conditions which guarantee that the three-dimensional Fourier transform of $f(\rho)$ *exists*.[5] It should be emphasized that Eq. (2.13a) follows from Eq. (2.10), which is equivalent to Eq. (2.9); i.e., a condition on $f(\rho)$ at $\rho = 0$ imposes a condition on its Fourier transform $\hat{f}(k)$ as $k \to \infty$. Similarly, Eq. (2.13b) implies that $f(\rho) \to 0$ as $\rho \to \infty$. Thus, the Fourier transform method[5] can be used only for func-

tions $f(\rho)$ which are well behaved at both $\rho=0$ and $\rho\to\infty$. In contrast, the validity of Eq. (2.11) depends only on Eq. (2.9), the condition at $\rho=0$, and thus the present method applies to a wider class of functions than the Fourier transform method. In particular, Eq. (2.11) can be used to find divergence relations for some functions whose Fourier transforms do not exist.

Also, our present method is quite useful for application to a function whose Fourier transform exists but is a very cumbersome mathematical expression. For example, the Fourier transform of $\hat{f}(\rho)=\rho e^{-\rho^2/a^2}$ can be expressed as[20]

$$\hat{f}(k)=\sqrt{\frac{2}{\pi}}\frac{a^4}{8}\left[2+(ka)(2-k^2a^2)\sum_{j=0}^{\infty}\frac{(-)^jj!}{(2j+1)!}(ka)^{2j}\right],$$

which involves an infinite sum. Therefore, it is not trivial to obtain a simple closed form expression for $F_1(\rho)$ by using the Fourier transform equation (2.12). However, by using Eq. (2.11), we easily find that

$$F_1(\rho)=-\frac{1}{2}\left(\frac{\rho}{a}\right)^{-3}a\left(1-\left(1+\frac{\rho^2}{a^2}\right)e^{-\rho^2/a^2}\right).$$

There are many applications of these single-divergence relations. For example, Eq. (2.3a) represents for $N=3$ a one-body potential generated from an integral of a two-body potential over a finite volume. The two-body potential is usually either a Coulomb potential $(1/\rho)$ or some type of nuclear interaction, e.g., Yukawa $(e^{-\rho/a}/\rho)$ or Gaussian $(e^{-\rho^2/a^2})$ functions. Such potentials are used to calculate microscopic Strutinsky shell corrections for use in studies of fission or heavy-ion reactions.[1,2] If $f(\rho)$ satisfies the divergence relation (2.2), then the one-body potential (2.3a) can be converted into a surface integral of the form (2.3b). These integrals can easily be evaluated numerically, especially for axially-symmetric shapes.[2]

Now consider a one-body potential calculated from a "diffuse-surface" integral of the form

$$I(\mathbf{r}_1)=\int_{\infty}f(\rho)\Gamma(\mathbf{r}_2)d^3r_2 \tag{2.14}$$

where the integration is over all space. The density function $\Gamma(\mathbf{r}_2)$ is of the form [5,12–15]

$$\Gamma(\mathbf{r}_2)=\Gamma_0\int_{\infty}g(|\mathbf{r}_2-\mathbf{r}_3|)\theta_V(\mathbf{r}_3)d^3r_3$$

$$=\Gamma_0\int_V g(|\mathbf{r}_2-\mathbf{r}_3|)d^3r_3, \tag{2.15}$$

where Γ_0 is a normalization constant[5] and the step function

$$\theta(\mathbf{r}_3)=\begin{cases}1 & \text{for} \quad \mathbf{r}_3 \text{ within } V \\ 0 & \text{for} \quad \mathbf{r}_3 \text{ outside of } V\end{cases}$$

is the "sharp-surface" density used in most liquid-drop-model calculations. The scalar function $g(\rho)$ is normalized by[5]

$$\int_{\infty}g(\rho)d^3\rho=1,$$

so that

$$\int_{\infty}\Gamma(\mathbf{r}_2)d^3r_2=\Gamma_0\int_{\infty}\theta_V(\mathbf{r}_3)d^3r_3=\Gamma_0 V.$$

In Eq. (2.15) we have folded a function g with a step function θ having a finite surface of arbitrary shape. The diffuseness function g is usually taken to be either a Yukawa or a Gaussian. It can be shown that[5]

$$I(\mathbf{r}_1)=\Gamma_0\int_V y(\rho)d^3r_2, \tag{2.16}$$

where

$$y(\rho)=\int_{\infty}d^3\rho' f(|\boldsymbol{\rho}-\boldsymbol{\rho}'|)g(\rho')$$

$$=\int_{\infty}d^3k\,\hat{f}(k)\hat{g}(k)e^{i\mathbf{k}\cdot\boldsymbol{\rho}}, \tag{2.17}$$

and $\hat{g}(k)$ is the Fourier transform of $g(\rho)$. The integrand of Eq. (2.16) is usually a more complicated function of ρ than the original energy density f. Notice that Eq. (2.16) is a finite volume integral to which Gauss' theorem is directly applicable. [In contrast, if the diffuse density distribution is generated from a Fermi function,[11] then the one-body potential, Eq. (2.14), remains an infinite integral, which cannot be easily handled.] Equation (2.16) is especially easy to convert to a surface integral using Eqs. (2.2) and (2.11).

If we set $\mathbf{r}_1$ (or $\mathbf{r}_2$) to zero in Eq. (1.2), then the single-divergence relation specializes to the evaluation of an integral of a one-body function $f(r)$. Suppose that we have an integral of the form[5,12,13]

$$I=\int_{\infty}f(r)\Gamma(\mathbf{r})d^3r, \tag{2.18}$$

where $\Gamma(\mathbf{r})$ is the diffuse density defined by convolution in Eq. (2.15). There are two cases of special interest.

(i) If the Fourier transform of $f(r)$ exists, it can easily be shown that

$$I=\int_V J(r)d^3r, \tag{2.19}$$

where

$$J(r)=\frac{4\pi}{r}\Gamma_0\int_0^{\infty}k\sin(kr)\hat{f}(k)\hat{g}(k)\,dk. \tag{2.20}$$

(ii) For the calculation of radial moments of a nuclear density,[5,12,13] one uses

$$f(r)=r^{2m}, \quad m=1,2,3,\cdots. \tag{2.21}$$

For this type of function, it can be shown[5,12] that Eq. (2.18) transforms to

$$I=\sum_{j=0}^{m}b_j\int_V r^{2j}d^3r, \tag{2.22}$$

where the b_j's are constants. Gauss' theorem can be applied to both Eqs. (2.19) and (2.22) since all integrals are over the finite volume V. The single-divergence relations Eqs. (2.2) and (2.11) can be used to convert Eqs. (2.19) and (2.22) to surface integrals. However, Eq. (2.12) is not applicable to Eq. (2.22) since the Fourier transforms of the r^{2j} functions do not exist.

B. The double divergence relation

For $n = 2$, Eq. (2.1) reduces to

$$f(\rho) = - \sum_{i,j=1}^{N} \frac{\partial}{\partial \rho_i} \frac{\partial}{\partial \rho_j} [\rho_i \rho_j F_2(\rho)] \qquad (2.23a)$$

or from Eq. (1.2)

$$f(\rho) = \sum_{i,j=1}^{N} \frac{\partial}{(\partial r_1)_i} \frac{\partial}{(\partial r_2)_j} [\rho_i \rho_j F_2(\rho)], \qquad (2.23b)$$

where $(r_1)_i$ is the ith Cartesian component of r_1. Consider now a double volume integral

$$I = \int_{V_1} d^N r_1 \int_{V_2} d^N r_2 f(\rho), \qquad (2.24a)$$

where the N-dimensional volumes V_1 and V_2 are finite. By using Eq. (2.23b) and applying Gauss' theorem twice, we have

$$I = \int_{S_1} \int_{S_2} (d\mathbf{S}_1 \cdot \hat{\rho})(d\mathbf{S}_2 \cdot \hat{\rho}) \rho^2 F_2(\rho), \qquad (2.24b)$$

where $\hat{\rho}$ is ρ/ρ. The double volume integral converges if $f(\rho)$ is finite except possibly at the coordinate singularity $\rho = 0$, where $f(\rho)$ satisfies Eq. (2.4). Consider V_2 to be a small N-dimensional hypersphere of radius a, centered at a point r_1 contained in V_1. The volume integral over V_2 vanishes as $a \to 0$, so the surface integral over S_2 must vanish also. But the surface integral over S_2 in this special case is

$$a^{N+1} F_2(a) \int_{S_1} \hat{\rho}(\hat{\rho} \cdot d\mathbf{S}_2),$$

so we find that

$$\lim_{\rho \to 0} \rho^{N+1} F_2(\rho) = 0. \qquad (2.25)$$

By comparison of Eqs. (2.2) and (2.23a), we see that

$$\nabla_\rho \cdot [\rho_i \rho F_2(\rho)] = \rho_i F_1(\rho), \qquad (2.26a)$$

or

$$\rho \frac{dF_2(\rho)}{d\rho} + (N+1)F_2(\rho) = F_1(\rho), \qquad (2.26b)$$

or from Eq. (2.7)

$$\frac{d}{d\rho} [\rho^{N+1} F_2(\rho)] = \rho^N F_1(\rho) = - \int_0^\rho d\rho' (\rho')^{N-1} f(\rho'). \qquad (2.26c)$$

The solution of Eq. (2.26c) which satisfies the boundary condition (2.25) is

$$F_2(\rho) = \frac{1}{\rho^{N+1}} \int_0^\rho d\rho' (\rho')^N F_1(\rho') \qquad (2.27a)$$

$$= \frac{-1}{\rho^{N+1}} \int_0^\rho d\rho'' \int_0^{\rho''} d\rho'(\rho')^{N-1} f(\rho'), \qquad (2.27b)$$

or, by interchange of order of integration,

$$F_2(\rho) = \frac{1}{\rho^{N+1}} \int_0^\rho d\rho' (\rho')^N f(\rho') - \frac{1}{\rho^N}$$

$$\times \int_0^\rho d\rho'(\rho')^{N-1} f(\rho') \qquad (2.28a)$$

$$= \frac{1}{\rho^{N+1}} \int_0^\rho d\rho' (\rho')^N f(\rho') + F_1(\rho). \qquad (2.28b)$$

Note that Eqs. (2.26b), (2.27a), and (2.28b) are recursion relations between $F_1(\rho)$ and $F_2(\rho)$. Equations (2.27a) and (2.28b) are useful for generating $F_2(\rho)$ from $F_1(\rho)$. On the other hand, Eq. (2.26b) is not particularly helpful as a recursion relation since we need to know $F_2(\rho)$ in order to construct the lower-order $F_1(\rho)$.

We now present some results and applications for $N = 3$. Equation (2.28a) becomes

$$F_2(\rho) = \frac{1}{\rho^3} \int_0^\rho d\rho' (\rho')^3 f(\rho') - \frac{1}{\rho^4} \int_0^\rho d\rho' (\rho')^2 f(\rho'). \qquad (2.29)$$

By substituting Eq. (1.3) into Eq. (2.29), we obtain

$$F_2(\rho) = \sqrt{\frac{2}{\pi}} \frac{1}{\rho^4} \int_0^\infty \frac{dk}{k^2} \hat{f}(k)$$

$$\times [-2 + 2\cos(k\rho) + k\rho \sin(k\rho)], \qquad (2.30)$$

which is the Fourier transform expression derived in Ref. 5 for $N = 3$. In Appendix B of Ref. 5, the $N = 3$ "surface" functions $F_2(\rho)$ are presented for each of the following $f(\rho)$ functions: $1/\rho$, $1/\rho^2$, $e^{-\rho/a}/\rho$, $e^{-\rho/a}$, $\rho e^{-\rho/a}$, and $e^{-\rho^2/a^2}$. It is much easier to calculate $F_2(\rho)$ from Eq. (2.29) than from Eq. (2.30). From Eq. (2.25) for $N = 3$ and Eq. (2.30) it can be shown that Eq. (2.13a) is satisfied. By examining the behavior of the integrand of Eq. (2.30) at $k = 0$,[5] we find that Eq. (2.13b) must also be satisfied. Thus, just as in the previous subsection, the validity of Eq. (2.30) depends on spatial conditions on $f(\rho)$ at $\rho = 0$ and as $\rho \to \infty$ whereas Eq. (2.29) depends only on Eq. (2.25) [or equivalently on Eq. (2.9)], a condition at $\rho = 0$. Thus, Eq. (2.29) applies to a wider class of functions than Eq. (2.30).

Also, Eq. (2.29) can be useful for finding a divergence expression for a function whose Fourier transform exists but is much too complicated to be easily handled by Eq. (2.30). For example, for $f(\rho) = \rho e^{-\rho^2/a^2}$, we easily find from Eq. (2.29) that

$$F_2(\rho) = \frac{1}{2}\left(\frac{\rho}{a}\right)^{-4} a \left[-\frac{\rho}{a} - \frac{1}{2}\frac{\rho}{a} e^{-\rho^2/a^2} \right.$$

$$\left. + \frac{3}{4}\sqrt{\pi} \, \text{erf}\left(\frac{\rho}{a}\right) \right].$$

For $N = 3$, Eq. (2.24a) is the Coulomb or nuclear energy of the system if $f(\rho)$ is the appropriate energy density. Such energies are the macroscopic energies used in static and

dynamical calculations of fission and heavy-ion reactions.[1-7] Also, Eq. (2.24a) has the form of the Coulomb and nuclear *interaction forces* used in recent dynamical studies of fission[10] to determine scission configurations at which the neck of the nucleus ruptures. If $f(\rho)$ satisfies the divergence relation (2.23b) for $N = 3$,[3,4,5,10] then the sixfold volume integral in Eq. (2.24a) can be converted into the simpler fourfold surface integral in Eq. (2.24b). For axially symmetric nuclear shapes, the surface integrals further simplify to three-dimensional integrals which can be evaluated by Gaussian–Legendre quadrature.[4]

Now consider Coulomb and nuclear energies calculated from a convoluted "diffuse surface" density distribution,[5,14,15] with

$$I = \int_\infty \int_\infty f(\rho) \Gamma(\mathbf{r}_1) \Gamma(\mathbf{r}_2) \, d^3r_1 \, d^3r_2, \qquad (2.31)$$

where the integrations are over all space and $\Gamma(\mathbf{r})$ is given by Eq. (2.15). It can be shown[5] that Eq. (2.31) can be reexpressed as

$$I = \Gamma_0^2 \int_{V_1} d^3r_1 \int_{V_2} d^3r_2 \, w(|\mathbf{r}_1 - \mathbf{r}_2|), \qquad (2.32)$$

where

$$w(\rho) = \int_\infty d^3\rho' \, y(|\boldsymbol{\rho} + \boldsymbol{\rho}'|) g(\rho')$$

$$= (2\pi)^{3/2} \int_\infty d^3k \, \hat{f}(k) [\hat{g}(k)]^2 e^{i\mathbf{k}\cdot\boldsymbol{\rho}}, \qquad (2.33)$$

and $y(\rho)$ is given by Eq. (2.17). The integrand of Eq. (2.32) is generally a more complicated function than $f(\rho)$. Equation (2.32) is of the form of Eq. (2.24a) so that we can directly apply Gauss' theorem to both terms.[5] In contrast, the energies calculated using a "diffuse surface" density generated from a Fermi function[11] involve infinite integrals, which are difficult to handle and cannot be converted to simpler surface integrals.

C. The divergence relation of arbitrary order, n

A generalization of the previously described divergence relations to third or higher order may be of interest for the evaluation of many-body interaction energies in nuclei or other leptodermous[1] systems. Many-body forces usually will depend on several variables, but it may be possible to reexpress or approximate the energy density so that certain terms depend on only the magnitude ρ of a single linear combination of the coordinate vectors,

$$\boldsymbol{\rho} = \sum_{\alpha=1}^{K} c_\alpha \mathbf{r}_\alpha. \qquad (2.34)$$

For example, in a three-body system there might be terms depending only on the magnitude of one of the Jacobi vectors

$$\boldsymbol{\rho}_\alpha = \mathbf{r}_\alpha - \frac{m_\beta \mathbf{r}_\beta + m_\gamma \mathbf{r}_\gamma}{m_\beta + m_\gamma},$$

$$(\alpha\beta\gamma) = (123), (231), \text{ or } (312). \qquad (2.35)$$

Given the energy density or other scalar function, $f(\rho)$, we wish to simplify the calculation of the n-fold volume integral in N dimensions

$$I = \int_{V_1} d^N r_1 \cdots \int_{V_n} d^N r_n \, f(\rho), \quad 1 \leqslant n \leqslant K, \qquad (2.36)$$

for arbitrary finite volumes $V_1, \ldots, V_n$. Since

$$\sum_{i_1, \ldots, i_n = 1}^{N} \frac{\partial^n}{\partial (r_1)_{i_1} \cdots \partial (r_n)_{i_n}} [\rho_{i_1} \cdots \rho_{i_n} F(\rho)]$$

$$= \left(\prod_{\alpha=1}^{n} c_\alpha \right) \sum_{i_1 \cdots i_n = 1}^{N} \frac{\partial^n}{\partial \rho_{i_1} \cdots \partial \rho_{i_n}} [\rho_{i_1} \cdots \rho_{i_n} F(\rho)], \qquad (2.37)$$

it follows that

$$I = -\left(\prod_{\alpha=1}^{n} c_\alpha \right)^{-1} \int_{S_1} \cdots \int_{S_n} (d\mathbf{S}_1 \cdot \hat{\rho}) \cdots (d\mathbf{S}_n \cdot \hat{\rho}) \rho^n F(\rho) \qquad (2.38)$$

provided that Eq. (2.1) is satisfied and that the surface integral converges. (For the single and double surface integrals discussed earlier we had $c_1 = 1 = -c_2$.) Assume that V_n is an infinitesimal sphere centered at

$$-\left(\sum_{\alpha=1}^{n-1} c_\alpha \mathbf{r}_\alpha \right) / c_n.$$

Then the convergence of the single volume integral over V_n requires $f(\rho)$ to satisfy Eq. (2.4) and the convergence of the single surface integral over S_n requires a boundary condition of F_n namely

$$\lim_{\rho \to 0} \rho^{n+N-1} F_n(\rho) = 0. \qquad (2.39)$$

Equation (2.1) implies that F_{n+1} is related to F_n by

$$\sum_{i_{n+1}=1}^{N} \frac{\partial}{\partial \rho_{i_{n+1}}} [\rho_{i_1} \cdots \rho_{i_n} \rho_{i_{n+1}} F_{n+1}(\rho)]$$

$$= \rho_{i_1} \cdots \rho_{i_n} F_n(\rho), \quad 1 \leqslant n \leqslant K-1, \qquad (2.40a)$$

or

$$\rho \frac{dF_{n+1}(\rho)}{d\rho} + (N+n) F_{n+1}(\rho) = F_n(\rho), \quad 1 \leqslant n \leqslant K-1, \qquad (2.40b)$$

or

$$\frac{d}{d\rho} [\rho^{N+n} F_{n+1}(\rho)] = \rho^{n+N-1} F_n(\rho), \quad 1 \leqslant n \leqslant K-1. \qquad (2.40c)$$

The solution which satisfies Eq. (2.39) is

$$F_{n+1}(\rho) = \rho^{-N-n} \int_0^\rho (\rho')^{N+n-1} F_n(\rho') \, d\rho'. \qquad (2.41)$$

Iteration of Eq. (2.41) gives

$$F_{n+1}(\rho) = \rho^{-N-n} \int_0^\rho d\rho_1 \int_0^{\rho_1} d\rho_2 \cdots$$

$$\times \int_0^{\rho_{k-1}} d\rho_k (\rho_k)^{N+n-k} F_{n+1-k}(\rho_k),$$

$$k = 1, 2, \ldots, n, \qquad (2.42)$$

where ρ_j now refers to an integration variable rather than the component of a vector. Substitution of Eq. (2.7) into (2.42) with $k = n$ then yields

$$F_{n+1}(\rho) = -\rho^{-N-n} \int_0^\rho d\rho_1 \int_0^{\rho_1} d\rho_2 \cdots$$

$$\times \int_0^{\rho_n} d\rho_{n+1} (\rho_{n+1})^{N-1} f(\rho_{n+1}). \qquad (2.43)$$

An alternative expression in terms of a single integral over f can be obtained as follows. Interchange of the order of integrations in Eq. (2.42) for $k = 2$ gives

$$\rho^{N+n} F_{n+1}(\rho) = \int_0^\rho d\rho' (\rho')^{N+n-2} F_{n-1}(\rho')(\rho-\rho'),$$
$$\qquad (2.44)$$

or, by use of Eq. (2.41) with $n \to n-1$,

$$\rho^{N+n}[F_{n+1}(\rho) - F_n(\rho)]$$
$$= -\int_0^\rho d\rho' \rho' (\rho')^{N+n-2} F_{n-1}(\rho').$$

By alternating substitutions from Eq. (2.41) and interchanges of orders of double integration, one finds

$$\rho^{N+n}\left[F_{n+1}(\rho) - \sum_{j=0}^{m} \frac{(-)^j}{(j+1)!} F_{n-j}(\rho)\right]$$
$$= \frac{(-)^{m+1}}{(m+1)!} \int_0^\rho d\rho' (\rho')^{m+1} \cdot (\rho')^{N+n-m-2} F_{n-m-1}(\rho'),$$

$$m = 0,1,...,n-2. \qquad (2.45)$$

Substitution of Eq. (2.7) into Eq. (2.45) with $m = n-2$ gives

$$\rho^{N+n}\left[F_{n+1}(\rho) - \sum_{j=0}^{n-2} \frac{(-)^j}{(j+1)!} F_{n-j}(\rho)\right]$$
$$= \frac{(-)^n}{(n-1)!} \int_0^\rho d\rho'(\rho')^{n-1} \int_0^{\rho'} d\rho''(\rho'')^{N-1} f(\rho'')$$
$$= \frac{(-)^n}{(n-1)!} \int_0^\rho d\rho''(\rho'')^{N-1} f(\rho'')\left[\frac{\rho^n - (\rho'')^n}{n}\right]$$
$$= \frac{(-)^{n-1}}{n!}\left[\rho^{N+n}F_1(\rho) + \int_0^\rho d\rho'(\rho')^{N+n-1} f(\rho')\right],$$

or

$$F_{n+1}(\rho) = \sum_{j=0}^{n-1} \frac{(-)^j}{(j+1)!} F_{n-j}(\rho)$$

$$+ \frac{(-)^{n-1}}{n!}\rho^{-N-n} \int_0^\rho d\rho'(\rho')^{N+n-1} f(\rho'), \quad n = 1,2,\cdots.$$
$$\qquad (2.46)$$

For $n = 1$ this is just Eq. (2.28b). Thus, one can calculate F_{n+1} for successively larger values of n by single integrations.

This leads also to a third form of the solution for F_{n+1}, as a linear combination of single integrals over f. We shall show by induction that for order n

$$F_n(\rho) = \frac{1}{\rho^{N-1}} \sum_{j=1}^{n} \frac{(-)^j}{(n-j)!(j-1)!} \frac{1}{\rho^j}$$

$$\times \int_0^\rho d\rho' (\rho')^{j+N-2} f(\rho'). \qquad (2.47)$$

From Eqs. (2.7) and (2.28a), we see that Eq. (2.47) is valid for $n = 1$ and $n = 2$. We will now show that Eq. (2.47), if valid for order n, is also valid for order $n+1$. Substituting Eq. (2.47) into Eq. (2.41), we find that

$$F_{n+1}(\rho) = \rho^{-(n+N)} \sum_{j=1}^{n} \frac{(-)^j}{(n-j)!(j-1)!} \int_0^\rho (\rho')^{n-j} d\rho'$$

$$\times \int_0^{\rho'} d\rho'' (\rho'')^{j+N-2} f(\rho''),$$

which can be transformed to

$$F_{n+1}(\rho) = \rho^{-(n+N)} \sum_{j=1}^{n} \frac{(-)^j}{(n-j)!(j-1)!}$$

$$\times \int_0^\rho d\rho'' (\rho'')^{j+N-2} f(\rho'') \int_{\rho''}^\rho (\rho')^{n-j} d\rho'.$$
$$\qquad (2.48)$$

Explicitly performing the second integral, we obtain

$$F_{n+1}(\rho) = \rho^{-(n+N)}$$

$$\times \sum_{j=1}^{n} \frac{(-)^j}{(n-j)!(j-1)!} \int_0^\rho d\rho'' (\rho'')^{j+N-2} f(\rho'')$$

$$\times \frac{1}{(n-j+1)} [\rho^{n-j+1} - (\rho'')^{n-j+1}],$$

which can be rewritten as

$$F_{n+1}(\rho) = \rho^{-(n+N)} \sum_{j=1}^{n} \frac{(-)^j}{(n-j+1)!(j-1)!}$$

$$\times \left\{\rho^{n-j+1} \int_0^\rho d\rho' (\rho')^{j+N-2} f(\rho')\right.$$

$$\left. - \int_0^\rho d\rho' (\rho')^{n+N-1} f(\rho')\right\}. \qquad (2.49)$$

The sum multiplying the last term is easily evaluted:

$$\sum_{j=1}^{n} \frac{(-)^j}{(n-j+1)!(j-1)!} = \frac{(-)^n}{n!},$$

and Eq. (2.49) becomes

$$F_{n+1}(\rho) = \frac{1}{\rho^{N-1}} \sum_{j=1}^{n} \frac{(-)^j}{(n-j+1)!(j-1)!} \frac{1}{\rho^j}$$

$$\times \int_0^\rho d\rho' (\rho')^{j+N-2} f(\rho')$$

$$- \frac{1}{\rho^{(n+N)}} \frac{(-)^n}{n!} \int_0^\rho d\rho' (\rho')^{n+N-1} f(\rho')$$

$$= \frac{1}{\rho^{N-1}} \sum_{j=1}^{n+1} \frac{(-)^j}{(n-j+1)!(j-1)!} \frac{1}{\rho^j}$$

$$\times \int_0^\rho d\rho' (\rho')^{j+N-2} f(\rho').$$
$$\qquad (2.50)$$

Comparing Eqs. (2.47) and (2.50) we see that the desired relation is valid for order $n+1$ if it is valid for order n. Therefore, by induction, Eq. (2.47) is true in general. An alternative proof by induction is obtained by substituting Eq. (2.47) into Eq. (2.46) and interchanging the order of summations.

Equations (2.40), (2.41), (2.42), and (2.46) are recursion relations, all of which can be used to check for algebraic errors in constructing F functions of adjacent orders. How-

ever, Eqs. (2.40) are not very useful recursion relations since one needs to know $F_{n+1}(\rho)$ in order to construct $F_n(\rho)$. On the other hand, Eqs. (2.41), (2.42), and (2.46) are helpful since they allow one to generate $F_{n+1}(\rho)$ from integrals or sums involving the lower-order F functions.

III. MULTIPOLE EXPANSIONS

The generalized Slater[21] expansion of an arbitrary, scalar, two-body function $f(|\mathbf{r}_1 - \mathbf{r}_2|)$ into its multipole components is

$$f(\rho) = \sum_{lm} G_l(r_1, r_2) Y^*_{lm}(\hat{r}_1) Y_{lm}(\hat{r}_2), \qquad (3.1)$$

where $Y_{lm}(\hat{r}_2)$ is a spherical harmonic[22] and ρ, $\mathbf{r}_1$, and $\mathbf{r}_2$ are the three-dimensional vectors defined in Eq. (1.2). From the addition theorem[22]

$$\sum_m Y^*_{lm}(\hat{r}_1) Y_{lm}(\hat{r}_2) = \left(\frac{2l+1}{4\pi}\right) P_l(\cos\theta_{12}), \qquad (3.2)$$

where $P_l(\cos\theta_{12})$ is a Legendre polynomial and θ_{12} is the angle between the two directions $\hat{r}_1$ and $\hat{r}_2$, we rewrite Eq. (3.1) as

$$f(\rho) = \frac{1}{4\pi} \sum_l (2l+1) G_l(r_1, r_2) P_l(\cos\theta_{12}), \qquad (3.3)$$

so that, from the orthonormality relation for the Legendre functions,

$$G_l(r_1, r_2) = 2\pi \int_{-1}^{+1} P_l(\cos\theta_{12}) f(\rho) d(\cos\theta_{12}). \qquad (3.4)$$

G_l is known analytically for the Newtonian or Coulomb potential ρ^{-1} (which is the generating function of the Legendre polynomials), for the Gaussian and Yukawa functions (by using the series expansions of the P_l's[17]), and for a few additional functions of interest.[16] Since $f(\rho)$ can in general be a very complicated function of $\cos\theta_{12}$, it is not always easy to evaluate Eq. (3.4). Therefore, it is desirable to derive another formula for evaluating $G_l(r_1, r_2)$ for an arbitrary two-body function $f(\rho)$.

The multipole expansion of the Yukawa function is well known.[16,17] If the range a of the Yukawa is made purely imaginary, then one obtains the addition theorems for $h_0^{(1)}$[16] and j_0[23]

$$\frac{e^{ik\rho}}{\rho} = ik\, h_0^{(1)}(k\rho)$$

$$= 4\pi ik \sum_{lm} j_l(kr_<) h_l^{(1)}(kr_>) Y^*_{lm}(\hat{r}_1) Y_{lm}(\hat{r}_2), \qquad (3.5)$$

and

$$\frac{\sin k\rho}{\rho} = k j_0(k\rho)$$

$$= 4\pi k \sum_{lm} j_l(kr_<) j_l(kr_>) Y^*_{lm}(\hat{r}_1) Y_{lm}(\hat{r}_2), \qquad (3.6)$$

where

$$h_l^{(1)}(x) = j_l(x) + i n_l(x), \qquad (3.7)$$

in which j_l, n_l, and $h_l^{(1)}$ are the spherical Bessel, Neumann, and Hankel functions[24] of order l, and where $r_>$ and $r_<$ are

the greater and lesser, respectively, of r_1 and r_2. Substituting Eq. (3.5) or (3.6) into Eq. (1.3) and comparing with Eq. (3.1), we find that

$$G_l(r_1, r_2) = 4\sqrt{2\pi} \int_0^\infty k^2 j_l(kr_1) j_l(kr_2) \hat{f}(k) \, dk, \qquad (3.8)$$

or, expressed in terms of ordinary Bessel functions[24]

$$G_l(r_1, r_2) = \frac{(2\pi)^{3/2}}{(r_1 r_2)^{1/2}}$$

$$\times \int_0^\infty k J_{l+1/2}(kr_1) J_{l+1/2}(kr_2) \hat{f}(k) \, dk. \qquad (3.9)$$

Equations (3.8) and (3.9) are general equations for any function $f(\rho)$ whose Fourier transform $\hat{f}(k)$ exists.

We now impose the special condition

$$\hat{f}(-k) = \hat{f}(k), \qquad (3.10)$$

which applies to most functions of physical interest.[5] From the relation[24]

$$j_l(-x) = (-)^l j_l(x), \qquad (3.11)$$

Eq. (3.8) becomes

$$G_l(r_1, r_2) = 2\sqrt{2\pi} \int_{-\infty}^\infty k^2 j_l(kr_1) j_l(kr_2) \hat{f}(k) \, dk. \qquad (3.12)$$

If we want to use contour integration to evaluate Eq. (3.12), there are convergence problems associated with the exponential behavior of the j_l function as $|k| \to \infty$. This difficulty can be removed by a simple transformation. We note the symmetry[24]

$$n_l(-x) = (-)^{l+1} n_l(x). \qquad (3.13)$$

From Eqs. (3.7), (3.10), (3.11), and (3.13), Eq. (3.12) can be reexpressed as

$$G_l(r_1, r_2) = 2\sqrt{2\pi} \int_{-\infty}^\infty k^2 h_l^{(1)}(kr_>) j_l(kr_<) \hat{f}(k) \, dk, \qquad (3.14)$$

which we want to evaluate by contour integration in the upper half of the complex k plane.

For Eq. (3.14) we now show that there are no problems associated with the behavior of the Bessel functions either at $k = 0$ or as $|k| \to \infty$. First, we have[24]

$$k^2 h_l^{(1)}(kr_1) j_l(kr_2) \underset{k\to 0}{\sim} k. \qquad (3.15a)$$

Then, if Eq. (2.13b) is satisfied and if we further allow only *integer* powers of k for the leading terms of $\hat{f}(k)$ as $k \to 0$, the worst allowed singularity is for the case

$$\hat{f}(k) \underset{k\to 0}{\sim} k^{-2}. \qquad (3.15b)$$

From Eqs. (3.15), it is clear that the most singular behavior leads to a simple pole at $k = 0$ for the integral in Eq. (3.14). However, the contribution from such a singularity is the (principal-value) residue for a pole situated *on* the contour.[25] If we write k as a complex variable

$$k = k_R + i k_I \quad (k_I \geqslant 0)$$
$$= |k| e^{i\phi}, \quad 0 < \phi \leqslant \pi, \qquad (3.16)$$

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then it can be shown that for large k_l [24]

$$h_l^{(1)}(kr_>)j_l(kr_<) \sim e^{-k_l(r_> - r_<)} \underset{k_l \to \infty}{\longrightarrow} 0. \qquad (3.17a)$$

Assume that Eq. (2.13a) generalizes to

$$\lim_{|k| \to \infty} \hat{f}(k) = 0, \qquad 0 \leqslant \phi \leqslant \pi. \qquad (3.17b)$$

From Eqs. (3.17) it is clear that the integrand in Eq. (3.14) approaches zero as $|k| \to \infty$. Thus, this integral may be evaluated by contour integration in the upper half of the complex k plane. We emphasize that not all functions of physical interest obeying Eq. (3.10) satisfy the restriction (3.17b). For example, the Gaussian function

$$f(\rho) = e^{-\rho^2/a^2}$$

can be Fourier transformed to obtain

$$\hat{f}(k) = (a^3/2\sqrt{2})e^{-a^2k^2/4},$$

which does *not* satisfy Eq. (3.17b). Thus, for the Gaussian function one cannot close the contour with an infinite semicircle, but one can use Eq. (3.8) or (3.9) to find $G_l(r_1, r_2)$.

We have used Eqs. (3.8), (3.9), and (3.14) to evaluate the multipole radial functions $G_l(r_1, r_2)$ for various standard two-body functions. For the Coulomb and Gaussian functions we used Eq. (3.9), with the definite integrals evaluated from Refs. 26 and 27; the results agree with the multipole expansions found in standard texts. [16] From Eq. (3.14) we then used complex contour integration to find $G_l(r_1, r_2)$ for the two-body functions: $e^{-\rho/a}$ and $\rho e^{-\rho/a}$; these results do not appear in the literature. However, the multipole expansions for $e^{-\rho/a}$ and $\rho e^{-\rho/a}$ can also be obtained from the Yukawa expression [16,17] by differentiating with respect to a^{-1}.

Finally, we obtain the multipole expansion for $f(\rho) = \rho^{-2}$. From Eq. (3.9) and Ref. 28, we find that

$$G_l(r_1, r_2) = \frac{4^{(l+1)}\pi(l!)^2}{(2l+1)!} \frac{r_<^l}{r_>^{(l+2)}}$$

$$\times F\left[\tfrac{1}{2}, l+1; l+3/2; \left(\frac{r_<}{r_>}\right)^2\right], \qquad (3.18)$$

where $F[a,b;c;d]$ is the Gaussian hypergeometric function. [29] This expansion for $f(\rho) = \rho^{-2}$ appears to be new, and the $G_l(r_1, r_2)$ functions are considerably more complicated than the functions obtained for $f(\rho) = \rho^{-1}$. [16] For small values of l, the G_l in Eq. (3.18) simplify to logarithmic functions, [29] e.g.,

$$G_0(r_1, r_2) = \frac{2\pi}{r_< r_>} \ln\left[\frac{1 + (r_</r_>)}{1 - (r_</r_>)}\right], \qquad (3.19a)$$

$$G_1(r_1, r_2)$$

$$= \frac{\pi}{r_<^2}\left\{\left[1 + \left(\frac{r_<}{r_>}\right)^2\right] \ln\left[\frac{1 + (r_</r_>)}{1 - (r_</r_>)}\right] - 2(r_</r_>)\right\}. \qquad (3.19b)$$

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$$a\left(\frac{\rho}{a}\right)^{-3}\left[-6 + e^{-\rho/a}\left(6 + 6\frac{\rho}{a} + \frac{3\rho^2}{a^2} + \frac{\rho^3}{a^3}\right)\right].$$

The other "surface" functions in this table are correct as listed.
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Isometries compatible with asymptotic flatness at null infinity: A complete description[a]

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The following results concerning isometries of space–times which are asymptotically empty and flat at null infinity are established: (i) The isometry group is necessarily a subgroup of the Poincaré group; (ii) if the asymptotic Weyl curvature is nonzero—more precisely, in the standard notation, if $K_{abcd}n^d$ does not vanish identically on $\mathscr{J}$—the space–time cannot admit more than two Killing fields unless the metric is Schwarzschildean in a neighborhood of $\mathscr{J}$; if it does admit two Killing fields, they necessarily commute; one (and only one) of them is a translation; the radiation field as well as the Bondi news vanishes everywhere on $\mathscr{J}$; and, finally, if the translational Killing field is timelike in a neighborhood of $\mathscr{J}$, the other Killing field is necessarily rotational. Several implications of these results are pointed out.

1. INTRODUCTION

The purpose of this paper is to present an essentially exhaustive analysis of isometries admitted by space–times which are asymptotically flat at null infinity.

Because asymptotically flat space–times resemble, in an appropriate sense, Minkowski space and because they represent isolated bodies in general relativity, one expects, from one's intuition derived from the geometrical properties of Minkowski space and from permissible symmetries of isolated bodies in Newtonian theory, that asymptotic flatness would severely constrain the possible isometries in such space–times. For example, one expects that the isometry group of any asymptotically flat space–time must be a subgroup of the Poincaré group; that an asymptotically flat space–time cannot admit more than one translational Killing field unless it is flat; that, in the framework of general relativity, an isolated system cannot be axisymmetric about two different axes unless it is spherically symmetric; that an (asymptotically flat) radiating solution cannot admit more than one Killing field; that if an asymptotically flat stationary space–time admits an additional Killing field, it is necessarily a rotational Killing field; etc. The expectation that such statements are true is so strong that their validity is often assumed in the analysis of isolated systems in general relativity.[1] Indeed, if it should turn out that a notion of asymptotic flatness fails to guarantee the validity of such statements, one would seriously consider the possibility of modifying the definition.

Why, then, have such statements remained unproven? Perhaps the main reason is that one does not *a priorily* know the precise degree of asymptotic flatness required for their validity. Does the space–time have to be asymptotically flat both in null and in spacelike directions? Which of these statements are likely to be true if it is asymptotically flat in one regime but not the other? To what extend does the validity of these statements depend on the details of the definition of asymptotic flatness? Is the null boundary, $\mathscr{J}$, of the

space–time required to be topologically $S^2 \times \mathbb{R}$, or, can one have a few generators (of $\mathscr{J}$) "missing?" Are the statements true if one has "only a piece of $\mathscr{J}$,"or, must the generators of $\mathscr{J}$ be complete?

In this paper, we shall analyze these and related issues.

The basic ideas underlying the present investigation are the following. Killing fields on the physical space–time admit natural extensions as conformal Killing fields on the conformally completed space–time. Moreover, the rigidity associated with conformal Killing fields permits one to investigate their structure in terms of their properties in the neighborhood of a single point. Hence, we will be able to analyze the structure of Killing fields in the physical space–time by examining the behavior of the corresponding conformal Killing fields on the null boundary $\mathscr{J}$. This examination will lead us to the result that the isometry group of an asymptotically flat space–time is necessarily a subgroup of the Poincaré group. If, furthermore, the asymptotic Weyl curvature does not fall off "too rapidly"—i.e., if, for example, the Bondi mass associated with *some* cross section of $\mathscr{J}$ is nonzero—one obtains additional constraints on the permissible isometries. (One might intuitively think of these constraints as follows: If a space–time is asymptotically flat but nonflat, it cannot admit Killing fields whose integral curves originate in the "interior" of the space–time and then go out "to infinity." A requirement such as the nonvanishing of Bondi mass ensures that the space–time is nonflat.)

Section 2 is devoted to preliminaries: The definition of asymptotic flatness, used in the rest of the paper, is given and the notions of conformal Killing transport and conformal Killing data are recalled. In Sec. 3 it is shown that if an asymptotically flat space–time admits a Killing field whose restriction to $\mathscr{J}$ is (the generator of) a BMS supertranslation, the Bondi-news function—and hence also the radiation field—must vanish everywhere on $\mathscr{J}$, and the restriction of the Killing field to $\mathscr{J}$ is necessarily a BMS *translation*. The second of these results implies, in turn, that the isometry group of any asymptotically flat space–time is a subgroup of the Poincaré group. In Sec. 4, a mild restriction is made on the class of space–times being considered: It is assumed that the asymptotic Weyl curvature does not fall off "too rapidly." (More precisely, it is assumed that the field $K_{abcd}n^d$ —or, equivalently, at least one of Ψ_4^0, Ψ_3^0, Ψ_2^0, and Ψ_1^0

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—does not vanish identically on $\mathscr{I}$.) It is then shown that a space–time satisfying this condition cannot admit more than two Killing fields unless the space–time metric is that of Schwarzschild's at least in a neighborhood of $\mathscr{I}$. Furthermore, if such a space–time does admit two Killing fields, it is shown that they commute and one (and only one) of them is a translation. Some implications of these results are discussed in Sec. 5.

2. PRELIMINARIES
A. Asymptotic flatness

Consider a smooth (that is C^∞) manifold $\hat{\mathbf{M}}$ (without boundary) equipped with a smooth metric $\hat{g}_{ab}$ of signature $(-+++)$. $(\hat{\mathbf{M}},\hat{g}_{ab})$ represents a possible space–time in general relativity.

Definition 1: $(\hat{\mathbf{M}},\hat{g}_{ab})$ will be said to be *asymptotically flat at null infinity* if there exists a smooth manifold $\mathbf{M}$ with boundary $\mathscr{I}$ ($\equiv\partial\mathbf{M}$) equipped with a smooth (nondegenerate) metric g_{ab} together with a diffeomorphism Ψ from $\hat{\mathbf{M}}$ onto $(\mathbf{M}-\mathscr{I})$ satisfying the following conditions:

(i) there exists a smooth function Ω on $\mathbf{M}$ such that $\Psi_*(g_{ab}) = \Psi_*(\Omega^2)\,\hat{g}_{ab}$ on $\hat{\mathbf{M}}$, $\Omega=0$, and $\nabla_a\Omega\neq0$ on $\mathscr{I}$;

(ii) there exists a neighborhood $\mathscr{N}$ of $\mathscr{I}$ in $\mathbf{M}$ such that in $\mathscr{N}\cap\Psi(\hat{\mathbf{M}})$, $\hat{g}_{ab}$ satisfies the vacuum Einstein's equation; and

(iii) If Ω is so chosen that $\nabla^a\nabla_a\Omega|_{\mathscr{I}}=0$ [where ∇ is the derivative operator on $(\mathbf{M},g_{ab})$], the vector field n^a $:=g^{ab}\nabla_b\Omega$ is complete on $\mathscr{I}$ and the space $\mathscr{S}$ of orbits of $n^a|_{\mathscr{I}}$ is diffeomorphic to $\mathbb{S}^2$ (where the differential structure on $\mathscr{S}$ is the one induced by that on $\mathscr{I}$).

Throughout this paper we shall restrict ourselves to space–times satisfying the conditions of this definition. Conditions (i) and (ii) are the familiar ones while (iii) ensures that $\mathscr{I}$ is topologically $\mathbb{S}^2\times\mathbb{R}$ and that its generators are complete. An asymptotically weakly simple space–time (in the sense of Penrose[2]) satisfies this definition provided the generators of $\mathscr{I}$ are complete while an asymptotically Minkowskian space–time (in the sense of Geroch[3]) satisfies the definition provided it is asymptotically empty. [It was recently realized[4] that even in an asymptotically simple space–time, the generators need not be complete. Hence the condition (iii) has been introduced explicitly in the above definition. The restriction to asymptotically empty space–times, on the other hand, has been made only for simplicity; we do expect that the present analysis will not be substantially altered if condition (ii) were weakened to allow, for example, zero rest mass fields near $\mathscr{I}$.] Note that, in the above definition, no restriction is made on null geodesics in $(\hat{\mathbf{M}},\hat{g}_{ab})$; in this sense, the notion of asymptotic flatness used in this paper is substantially weaker than that expressed in the definition of asymptotic simplicity. Finally, we do not require the existence of both $\mathscr{I}^+$ and $\mathscr{I}^-$; it suffices that at least one of them exists.

Since most of the standard results about null infinity do not refer to the condition on null geodesics in the definition of asymptotic simplicity, they continue to be valid for space–times satisfying our definition. [Note that, although

in the asymptotically simple case the result that $\mathscr{I}$ is topologically $\mathbb{S}^2\times\mathbb{R}$ depends on this condition, it now follows directly from condition (iii).] In particular, we have the following. The vector field n^a is null on $\mathscr{I}$; $\mathscr{I}$ is a null 3-surface. The Weyl tensor C_{abcd} of $(\mathbf{M},g_{ab})$ vanishes on $\mathscr{I}$ and furthermore the tensor field $K_{abcd}=\Omega^{-1}C_{abcd}$ is smooth on $\mathscr{I}$. Finally, given a space–time which satisfies the first two conditions of the definition, one can always choose a conformal factor Ω satisfying $\nabla^a\nabla_a\Omega|_{\mathscr{I}}=0$ and the two conditions. *Throughout this paper, we shall assume that such a choice has been made for Ω.* Using condition (ii), it is easy to check that, with this choice, the following equations hold:

$$\nabla_a\nabla_b\Omega|_{\mathscr{I}}=\nabla_a n_b|_{\mathscr{I}}=0\ ,$$
$$\lim_{\to\mathscr{I}}\Omega^{-1}n^a n_a=0\ , \tag{1}$$
$$S_{ab}n^b|_{\mathscr{I}}=kn_a|_{\mathscr{I}},$$

where $S_{ab}=R_{ab}-\frac{1}{6}Rg_{ab}$, R_{ab} being the Ricci tensor,[5] and where k is a scalar field. We shall use Eq. (1) several times in Secs. 3 and 4. Next, given a conformal factor Ω which satisfies $\nabla^a\nabla_a\Omega|_{\mathscr{I}}=0$, $\Omega'=\omega\Omega$ also satisfies (the conditions in Definition 1 and) $\nabla'^a\nabla'_a\Omega'|_{\mathscr{I}}=0$ if and only if ω is a smooth, nowhere vanishing function on $\mathbf{M}$ with $\mathscr{L}_n\omega=0$ on $\mathscr{I}$. This rescaling represents the restricted conformal freedom available. If $\Omega'=\omega\Omega$, then $n'^a=\omega^{-1}n^a$ on $\mathscr{I}$. Since $\mathscr{L}_n\omega=0$, n^a is complete if and only if n'^a is complete. Thus, condition (iii) is satisfied by a conformal factor Ω with $\nabla^a\nabla_a\Omega|_{\mathscr{I}}=0$ if and only if it is satisfied by *all* such conformal factors. Finally, the tensor field $S_{ab}{}^{cd}|_{\mathscr{I}}:=h_{ab}n^c n^d|_{\mathscr{I}}$ on $\mathscr{I}$, where h_{ab} is the pullback to $\mathscr{I}$ of the metric g_{ab} on $\mathbf{M}$, is conformally invariant; it is a part of the universal structure of $\mathscr{I}$. Using this fact and the Schmidt–Walker–Sommers[6] characterization of the BMS group, it follows that the group of asymptotic symmetries of the class of space–times being considered is the BMS group. This fact will be used crucially in Sec. 2.

B. Conformal Killing fields on $(\mathbf{M},g_{ab})$

Let ξ^a denote a conformal Killing field on $(\mathbf{M},g_{ab})$, i.e., a vector field satisfying $\mathscr{L}_\xi g_{ab}=2\phi g_{ab}$. Set $F_{ab}=\nabla_{[a}\xi_{b]}$ and $k_a=\nabla_a\phi$. Then, it is straightforward to show that given any curve in $\mathbf{M}$, with tangent vector ζ^a, one has

$$\zeta^a\nabla_a\xi_b=\zeta^a F_{ab}+\phi\zeta_b\ ,$$
$$\zeta^a\nabla_a F_{bc}=\zeta^a R_{mabc}\xi^m+2k_{[b}\zeta_{c]}\ ,$$
$$\zeta^a\nabla_a\phi=k_a\zeta^a\ , \tag{2}$$
$$\zeta^a\nabla_a k_b=-\tfrac{1}{2}\zeta^a\xi^m\nabla_m S_{ab}$$
$$\qquad\qquad -\phi S_{ab}\zeta^a-S^m{}_{(a}F_{b)m}\zeta^a$$

where, as before, $S_{ab}=R_{ab}-\frac{1}{6}Rg_{ab}$. These equations enable one to "transport" the quadruplet (ξ^a, F_{ab}, ϕ, k_a) from a given point of the curve to any other point. Hence, it follows that *the quadruplet, (ξ^a, F_{ab}, ϕ, k_a)$_p$, evaluated at any point p of $\mathbf{M}$ suffices to characterize the conformal Killing field ξ^a completely*: If the quadruplet vanishes at p, ξ^a must vanish *everywhere* on $\mathbf{M}$.[7] The quadruplet will therefore be referred to as the conformal Killing data of ξ^a at p, relative to g_{ab}.

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3. GENERAL ASYMPTOTICALLY FLAT SPACE-TIMES

Fix a space–time $(\widehat{M}, \hat{g}_{ab})$ which is asymptotically flat at null infinity in the sense of definition 1. Denote by (M, g_{ab}) a conformal completion and by $\mathscr{I}$ the boundary of M. (As remarked in Sec. 2, we assume that the conformal factor Ω satisfies $\nabla^a \nabla_a \Omega|_{\mathscr{I}} = 0$.) The purpose of this section is to show that the isometry group of $(\widehat{M}, \hat{g}_{ab})$ is a subgroup of the Poincaré group. However, to obtain this result, first we need to prove several facts about Killing fields on $(\widehat{M}, \hat{g}_{ab})$.

Let $\hat{\xi}^a$ denote a Killing field on $(\widehat{M}, \hat{g}_{ab})$. Then, we have the following:

Lemma 1.1: $\hat{\xi}^a$ admits a smooth extension ξ^a to M (on $\widehat{M}, \hat{\xi}^a = \xi^a$). ξ^a is a conformal Killing field on M whose restriction to $\mathscr{I}$ is a generator of the BMS group.[8]

Our purpose now is to show that the mapping defined by Lemma 1.1 from the isometry group of $(\widehat{M}, \hat{g}_{ab})$ into the BMS group is one to one and that the image of the isometry group under this mapping is always contained in a Poincaré subgroup of the BMS group.

Recall, first, that since ξ^a is a conformal Killing field on (M, g_{ab}), it is completely characterized by its data at any point on M. In particular, one can examine the data at a point of $\mathscr{I}$. It turns out that since ξ^a is not only a conformal Killing field on (M, g_{ab}) but also a Killing field on $(\widehat{M}, \hat{g}_{ab})$, this data is constrained:

Lemma 1.2: The conformal Killing data of ξ^a at any point of $\mathscr{I}$ is tangential to $\mathscr{I}$. That is, $\xi_a n^a|_{\mathscr{I}} = 0$, $F_{ab} n^b|_{\mathscr{I}} = 0$, and $k_a n^a|_{\mathscr{I}} = 0$ where, as before, $F_{ab} = \nabla_{[a} \xi_{b]}$, $k_a = \frac{1}{4}(\nabla_a \nabla_b \xi^b) = 0$, and $n^a = \nabla^a \Omega$.

Proof: Since $\mathscr{L}_\xi \hat{g}_{ab} = 0$,

$\mathscr{L}_\xi g_{ab} = 2(\Omega^{-1} \mathscr{L}_\xi \Omega) g_{ab} = 2\phi g_{ab}$ everywhere on $\widehat{M}$,

where $\phi = \frac{1}{4}\nabla_a \xi^a$. Since both g_{ab} and ξ^a are smooth on M, it follows that ϕ is smooth on M. Hence, on $\mathscr{I}$,

$\mathscr{L}_\xi \Omega = \xi_m n^m$ must vanish. Next consider $k_a = \nabla_a \phi$. On M, $\Omega k_a = (\nabla_a n_b) \xi^b + n^b F_{ab}$. Since k_a is smooth on $\mathscr{I}$, and since Ω is so chosen that $\nabla_a n_b|_{\mathscr{I}} = 0$ [Eq. (1)], we have $F_{ab} n^b|_{\mathscr{I}} = 0$. Finally, consider $n^a k_a$. On $\widehat{M}$,

$n^a k_a = \Omega^{-1} (n^a \nabla_a n_b) \xi^b = \frac{1}{2}[\xi^a \nabla_a \gamma + \Omega^{-1} \gamma(\xi^a n_a)]$,

where $\gamma = \Omega^{-1} n^a n_a$. Recall, however, from Eq. (1) that on $\mathscr{I}$, $\gamma = 0$, and hence $\nabla_a \gamma$ is proportional to n_a. Consequently, $n^a k_a|_{\mathscr{I}} = 0$. □

Remarks: (i) Note that the notion of conformal Killing data is itself not conformally invariant: In defining F_{ab}, ϕ and k_a we have used a derivative operator and hence a preferred metric in the conformal class of metrics available. Therefore, the statement that the conformal Killing data of ξ^a (at a point of $\mathscr{I}$) are tangential to $\mathscr{I}$ is true only in certain conformal frames. In particular, if the conformal factor were chosen such that $\nabla^a \nabla_a \Omega|_{\mathscr{I}} \neq 0$, this statement need not have been true.

(ii) Consider the vector space V_p of data $(\xi^a, F_{ab}, \phi, k_a)_p$, at any point p on $\mathscr{I}$, satisfying $\xi^a n_a|_p = 0$, $F_{ab} n^b|_p = 0$, and $k_a n^a|_p = 0$. V_p is precisely ten dimensional. This reflects the fact that the maximum

number of (independent) Killing fields that $(\widehat{M}, \hat{g}_{ab})$ can admit is ten.

Using the result of Lemma (1.2) and the conformal Killing transport equations (Sec. 2 B), it is easy to show the following:

Lemma 1.3: Consider the restriction $\xi^a|_{\mathscr{I}}$ of ξ^a to $\mathscr{I}$. If $\xi^a|_{\mathscr{I}}$ vanishes in an open set *within* $\mathscr{I}$, ξ^a must vanish everywhere. Hence, in particular, the mapping of Lemma 1.1 from the space of Killing fields into the space of generators of the BMS group is one to one.

Consider, next, the case when $\xi^a|_{\mathscr{I}}$ is the generator of a supertranslation. Then, $\xi^a|_{\mathscr{I}} = \alpha n^a$, where $\mathscr{L}_n \alpha = 0$. (That is, α is a pullback to $\mathscr{I}$ of a smooth scalar field on the space $\mathscr{S}$ of generators.) Hence,

$F_{ab}|_{\mathscr{I}} = \nabla_{[a} \xi_{b]}|_{\mathscr{I}} = n_{[b} D_{a]} \alpha|_{\mathscr{I}}$ where $D_a \alpha|_{\mathscr{I}}$ is the pullback to $\mathscr{I}$ of the gradient of α on $\mathscr{S}$. Next, since

$\mathscr{L}_n \xi^a|_{\mathscr{I}} = (n^b \nabla_b \xi^a - \xi^b \nabla_b n^a)|_{\mathscr{I}}$

$= n^b \nabla_b \xi^a|_{\mathscr{I}} = \phi n^a|_{\mathscr{I}}$, and $\mathscr{L}_n(\alpha n^a)|_{\mathscr{I}} = 0$, it follows that $\phi|_{\mathscr{I}} = 0$. Consequently, $k_a|_{\mathscr{I}} = \beta n_a|_{\mathscr{I}}$ for some scalar field β. Using these facts, we can now establish the following result.

Lemma 1.4[9]: Let $\xi^a|_{\mathscr{I}}$ be the generator of a BMS supertranslation. Then it is necessarily the generator of a BMS *translation*. Furthermore, n^a is a repeated principal null direction of the asymptotic conformal curvature tensor K_{abcd}, $K_{abcd} n^a n^c \propto n_b n_d$, and the Bondi-news function vanishes everywhere on $\mathscr{I}$.

Proof: First we shall show that if $\xi^a|_{\mathscr{I}}$ is the generator of a BMS supertranslation, the radiation field must vanish on $\mathscr{I}$, i.e., that $K_{abcd} n^a n^c h^b{}_m h^d{}_n = 0$ on $\mathscr{I}$, where, as before, ΩK_{abcd} is the Weyl tensor on (M, g_{ab}) and h_{ab} the pullback to $\mathscr{I}$ of the metric g_{ab} on M. (That is, h_{ab} is the intrinsic, degenerate metric on $\mathscr{I}$. Indices are raised and lowered using g_{ab}.) From Eq. (2) of the conformal Killing transport and the facts that the third piece ϕ of the conformal Killing data of ξ^a vanishes on $\mathscr{I}$ and the fourth piece equals βn_a, it follows that, on $\mathscr{I}$, $h_a{}^m h_b{}^n \xi^r \nabla_r S_{mn} = 0$. Since, on M, $\nabla_{[r} S_{m]n} = -\nabla_p C_{rmn}{}^p$, we have, on $\mathscr{I}$,

$\nabla_{[r} S_{m]n} = -K_{rmnp} n^p$. Hence, on $\mathscr{I}$,

$h_a{}^m h_b{}^n \xi^r (K_{rmnp} n^p - \frac{1}{2}\nabla_m S_{rn}) = 0$. Finally, using $\nabla_a n_b = 0$ and $S_{ab} n^b = kn_a$ on $\mathscr{I}$ [Eq. (1)] and $\xi^a = \alpha n^a$ on $\mathscr{I}$, we have $\alpha K_{abcd} n^a n^c h^b{}_m h^d{}_n = 0$ on $\mathscr{I}$. Since by Lemma 1.3, α cannot vanish in any open set within $\mathscr{I}$ and since all fields under consideration are smooth, we have $K_{abcd} n^a n^c h^b{}_m h^d{}_n = 0$ on $\mathscr{I}$ (or, equivalently, in the Newman–Penrose[10] notation, $\Psi_4^0 = 0$ everywhere on $\mathscr{I}$). This implies that n^a is a principal null direction of K_{abcd}; $K_{abcd} n^a n^c = v_{(b} n_{d)}$ for some vector field v_a on $\mathscr{I}$ with $v_a n^a|_{\mathscr{I}} = 0$. From now on we set $K_{ab} = K_{ambp} n^m n^p$.

Next, we show that v_a must be proportional to n_a on $\mathscr{I}$ (or, equivalently, Ψ_3^0 also vanishes on $\mathscr{I}$). Note, first, that it follows from Bianchi identities on (M, g_{ab}) that $\nabla_{[a} K_{bc]de} = 0$ on M. Next, since ξ^a is a conformal Killing field on (M, g_{ab}), $\mathscr{L}_\xi C_{abcd} = \mathscr{L}_\xi \Omega^{-1} K_{abcd} = 0$ on M. Hence, $\mathscr{L}_\xi K_{abcd} = \phi K_{abcd}$. However, since $\xi^a|_{\mathscr{I}}$ is a supertranslation, $\phi|_{\mathscr{I}} = 0$. Thus, $\mathscr{L}_\xi K_{abcd} = 0$ on $\mathscr{I}$ and con-

sequently $n^{m}\nabla_{m}K_{abcd}$ also vanishes on $\mathscr{I}$. Using this fact and the equation $\nabla_{[m}K_{ab]cd}=0$ on $\mathbf{M}$, it follows that $K_{ab}=gn_{a}n_{b}+n_{(a}D_{b)}\mu$, where g is a function on $\mathscr{I}$, μ on $\mathscr{I}$ and $D_{b}\mu$, the pullback of the gradient of μ on $\mathscr{I}$. Next, using $\nabla_{[m}K_{ab]cd}=0$ on $\mathbf{M}$, and $n^{P}\nabla_{p}K_{abcd}=0$ on $\mathscr{I}$, it follows that $h^{ab}\nabla_{a}K_{bc}=0$ on $\mathscr{I}$. Substituting $gn_{a}n_{b}+n_{(a}D_{b)}\mu$ for K_{ab} it follows that μ is constant on $\mathscr{I}$. Hence $D_{a}\mu=0$ and $K_{ab}=gn_{a}n_{b}$ on $\mathscr{I}$.

Next, we shall show that the Bondi-news function vanishes on $\mathscr{I}$. Set $P_{ab}=h_{a}{}^{m}h_{b}{}^{n}S_{mn}$ .Using the relation between the Riemann tensors of $(\mathbf{M},g_{ab})$ on $\mathscr{I}$ and $(\mathscr{I},h_{ab})$, it follows that $h^{ab}P_{ab}={}^{2}\mathscr{R}$, where ${}^{2}\mathscr{R}$ is the scalar curvatuve of $(\mathscr{I},h_{ab})$. Next, using the fact that $h_{a}{}^{m}h_{c}{}^{n}n^{r}\nabla_{r}S_{mn}=0$ on $\mathscr{I}$, it follows that the covariant tensor field P_{ab} on $\mathscr{I}$ is the pullback to $\mathscr{I}$ of a tensor field on $\mathscr{I}$ which we also denote by P_{ab} . Since $\nabla_{[a}S_{b]c}=-K_{abcp}n^{P}$ on $\mathscr{I}$ and since $K_{abcd}n^{a}n^{c}=gn_{b}n_{d}$, one can show that $D_{[a}P_{b]c}=0$ on $\mathscr{I}$, where D is now the derivative operator on $(\mathscr{I},h_{ab})$. Let us now assume that we have a Bondi conformal frame. That is, let h_{ab} on $\mathscr{I}$ be of constant scalar curvature. Then, since $D_{[a}P_{b]c}=0$ and since $h^{ab}P_{ab}={}^{2}\mathscr{R}$ on $\mathscr{I}$, it follows that $P_{ab}=\frac{1}{2}{}^{2}\mathscr{R}h_{ab}$. Hence the Bondi-news function vanishes everywhere on $\mathscr{I}$. (In Newman–Penrose notation, we have the following. Since $\Psi_{4}^{0}=0$, Bondi news is constant. $P_{ab}\propto h_{ab}$ implies that the constant is zero: $R_{ab}m^{a}m^{b}=0$ on $\mathscr{I}$.)

Finally, we show that $\xi^{a}|_{\mathscr{I}}$ is the generator of translations. Fix a Bondi conformal frame. (So, $P_{ab}=\frac{1}{2}{}^{2}\mathscr{R}h_{ab}$.) Consider the conformal Killing transport equation for the second piece, F_{ab}, of the conformal Killing data of ξ^{a} [Eq. (2)]. Using the fact that $F_{ab}=n_{[b}D_{a]}\alpha$ and $k_{a}=\beta n_{a}$ on $\mathscr{I}$, one obtains, on $\mathscr{I}$, $D_{a}D_{b}\alpha=Ah_{ab}$, for some scalar field A. Thus, α is a scalar field whose gradient is a conformal Killing field on $(\mathscr{I},h_{ab})$ where h_{ab} is of constant curvature. (Thus α is a linear combination of $l=0$ and $l=1$ spherical harmonics.) Hence $\xi^{a}|_{\mathscr{I}}=\alpha n^{a}|_{\mathscr{I}}$ is the generator of a BMS translation. $\square$

We are now ready to prove the main result of this section:

Theorem 1[9]: The isometry group of any asymptotically flat space–time is a subgroup of the Poincaré group.

Proof: Let $\mathbf{G}$ denote the isometry group of $(\widehat{\mathbf{M}},\hat{g}_{ab})$. Then, by Lemma 1.1, there exists a canonical mapping i from $\mathbf{G}$ into the BMS group $\mathbf{B}$. By Lemma 1.3, i is an imbedding. Denote by $\mathbf{N}$, the (maximal) subgroup of $\mathbf{G}$ which is sent by i into the supertranslation subgroup $\mathbf{S}$ of $\mathbf{B}$; $i(\mathbf{N})\subset\mathbf{S}$. Since i is an imbedding and since $\mathbf{S}$ is a normal subgroup of $\mathbf{B}$, $\mathbf{N}$ is a normal subgroup of $\mathbf{G}$. Consider $\mathbf{G}/\mathbf{N}$. Since $\mathbf{B}/\mathbf{S}$ is isomorphic to the Lorentz group $\mathbf{L}$, it is clear that $\mathbf{G}/\mathbf{N}$ is a subgroup of $\mathbf{L}$. Also, since $\mathbf{B}$ is the semidirect product of $\mathbf{S}$ with $\mathbf{L}$, $\mathbf{G}$ is a semidirect product of $\mathbf{N}$ and $\mathbf{G}/\mathbf{N}$. Finally, since $\mathbf{G}/\mathbf{N}$ is a subgroup of $\mathbf{L}$ and since, by Lemma 1.4, $i(\mathbf{N})$ is a subgroup of the four-dimensional group $\mathbf{T}$ of translations, $\mathbf{G}$ is a subgroup of the semidirect product of $\mathbf{T}$ with $\mathbf{L}$; i.e., of the Poincaré group. $\square$

4. CONSTRAINTS ON ISOMETRIES IMPOSED BY THE ASYMPTOTIC CURVATURE

We now restrict ourselves to asymptotically flat space-

–times for which $K_{abcd}n^{d}$ does not vanish identically on $\mathscr{I}$. (The restriction, in the Newman–Penrose[10] notation is that at least one of Ψ_{4}^{0}, Ψ_{3}^{0}, Ψ_{2}^{0}, and Ψ_{1}^{0} must fail to be identically zero.) The motivation behind this condition is that we now wish to focus on space–times which are asymptotically flat but whose curvature does not fall off so rapidly as to make them physically uninteresting as models of isolated systems. The condition $K_{abcd}n^{d}|_{\mathscr{I}}\neq0$ is very mild; indeed, if it fails to hold, the Bondi mass associated with every cross section of $\mathscr{I}$ must, in particular, vanish.

We shall first show that the nonvanishing of asymptotic curvature imposes severe restrictions on possible isometries in the physical space–time. Fix a point p of $\mathscr{I}$ at which $K_{abcd}n^{d}$ fails to vanish. Since $K_{abcd}n^{d}$ is smooth on $\mathscr{I}$, there exists a neighborhood N_{p} (in $\mathscr{I}$) of p such that $K_{abcd}n^{d}$ is nonzero in N_{p}. We can now show the following result.

Lemma 2.1: There exists a curvature scalar f on $\mathscr{I}$ and a point p' in N_{p} such that, given any Killing field ξ^{a} on $(\widehat{\mathbf{M}},\hat{g}_{ab})$, in N_{p}, $\xi^{m}\nabla_{m}f=-6\phi f$, with $f|_{p'}\neq0$, where $\phi=\frac{1}{4}(\nabla_{m}\xi^{m})$ is the third piece of the conformal Killing data of ξ^{a} (with respect to g_{ab}). (Thus, at p', the conformal Killing data of ξ^{a} are constrained.)

Proof: We shall repeatedly use the following facts: $\mathscr{L}_{\xi}K_{abcd}=\phi K_{abcd}$ on $\mathscr{I}$, $\mathscr{L}_{\xi}n^{a}=-\phi n^{a}$ on $\mathscr{I}$ and $\mathscr{L}_{\xi}g_{ab}=2\phi g_{ab}$ on $\mathbf{M}$. Consider the scalar $K_{ab}K^{ab}$. (Recall that $K_{ab}=K_{ambp}n^{m}n^{p}$.) It is easy to check that $\mathscr{L}_{\xi}(K_{ab}K^{ab})=-6\phi K_{ab}K^{ab}$ on $\mathscr{I}$. Hence, if $K_{ab}K^{ab}$ is nonzero at some point p' in N_{p}, we can set $f=K_{ab}K^{ab}$. Let us now consider the case when $K_{ab}K^{ab}$ vanishes everywhere in N_{p}. Then it follows that $h_{m}{}^{a}h_{n}{}^{c}K_{abcd}n^{b}n^{d}$ must also vanish everywhere in N_{p}. (In the Newman–Penrose[10] notation, $K_{ab}K^{ab}=2|\Psi_{4}^{0}|^{2}$.) Hence, in N_{p}, $K_{ab}=n_{(a}v_{b)}$ for some vector field v_{b} with $v_{a}n^{a}=0$. Again, it is easy to check that $\mathscr{L}_{\xi}(v^{a}v_{a})=-6\phi v^{a}v_{a}$. Hence if the function $v^{a}v_{a}$ is nonzero at any point p' in N_{p}, we can set $f=v^{a}v_{a}$. (In the Newman–Penrose notation, $v^{a}v_{a}$ is proportional to $|\Psi_{3}^{0}|^{2}$.) We next consider the case when $v^{a}v_{a}=0$ in N_{p}. Then there exists a scalar field g in N_{p} such that $v_{a}=gn_{a}$ and $K_{ab}=gn_{a}n_{b}$. Again $\mathscr{L}_{\xi}g^{2}=-6\phi g^{2}$. Hence, if g does not vanish identically in N_{p}, we can choose $f=g^{2}$. If $g=0$ in N_{p}, then $K_{ab}=0$ in N_{p}. We can then consider the scalar $K_{abcd}K^{abcp}n^{d}n_{p}$ which satisfies the desired equation. If this scalar also vanishes everywhere in N_{p}, $K_{abcd}n^{d}K^{abmp}n_{p}=jn_{c}n^{m}$ for some scalar field j which can server as the required f unless it vanishes everywhere in N_{p}. Proceeding in this manner, it is easy to check that a curvature scalar satisfying the conditions of the lemma must exist if $K_{abcd}n^{d}$ is not identically zero on $\mathscr{I}$. (In the Newman–Penrose notation, we can set $f=|\Psi^{0}{}_{4}|^{2}$ unless it vanishes everywhere in N_{p}. If it does, Ψ_{3}^{0} is gauge invariant and we can set $f=|\Psi_{3}^{0}|^{2}$ and so on.)

Since the curvature scalar of Lemma 2.1 is nonzero at p', it is nonzero in some neighborhood $N_{p'}$ of p'. In this neighborhood, $\phi=-\frac{1}{6}(\xi^{m}\nabla_{m}\log f)$. (Note that f can always be chosen to be positive.) Hence, given any vector v^{a} with $v^{a}n_{a}$

$$6v^a k_a|_{p'} = 6v^a \nabla_a \phi|_{p'}$$
$$= [-v^a \xi^m \nabla_a \nabla_m \log f$$
$$-v^a F_a{}^m \nabla_m \log f$$
$$+\tfrac{1}{6}(\xi^m \nabla_m \log f)v^n \nabla_n \log f]_{p'}.$$

Thus, given *any* Killing field $\hat{\xi}^a$ in $(\widehat{M},\hat{g}_{ab})$, the first two pieces, $\xi^a|_{p'}$, $F_{ab}|_{p'}$ of the conformal Killing data (w.r.t. g_{ab}) at p', of its extension ξ^a to M already determine the third piece $\phi|_{p'}$ completely, and the fourth piece $K_a|_{p'}$ up to a multiple of $n_a|_{p'}$: for Killing fields on $(\widehat{M},\hat{g}_{ab})$, the conformal Killing data at p' is further constrained by the presence of f. Hence, we now have a stronger version of Lemma 1.2: There is a natural imbedding of the vector space underlying the Lie algebra $\mathscr{L}_G$ of the isometry group G of $(\widehat{M},\hat{g}_{ab})$ into the *seven*-dimensional vector space $V_{p'}$ of quadruplets $(\xi^a,F_{ab},\phi,k_a)_{p'}$ at p' satisfying $\xi^a n_a|_{p'}=0$, $n^a F_{ab}|_{p'}=0$, $k_a n^a|_{p'}=0$, $(F_{ab}+F_{ba})|_{p'}=0$, and $\phi|_{p'}=-\tfrac{1}{6}\xi^m \nabla_m \log f|_{p'}$ and

$$6v^a k_a|_{p'} = -v^a \xi^m \nabla_m \nabla_a \log f|_{p'}$$

$$-v^a F_a{}^m \nabla_m \log f|_{p'}$$
$$+\tfrac{1}{6}(\xi^m \nabla_m \log f)v^r \nabla_r \log f|_{p'}.$$

This fact, together with the result that if $\xi^a|_{\mathscr{I}}$ is the generator of a BMS supertranslation, its data is necessarily of the form $(\xi^a=\alpha n^a, F_{ab}=n_{[b}D_{a]}\alpha, \phi=0, k_a=\beta n_a)$, leads one to the following consequence of the assumption $K_{abcd}n^d \neq 0$ on $\mathscr{I}$:

Lemma 2.2: In the terminology of Theorem 1, the dimension of the group G/N cannot exceed three.

Next, we establish another direct consequence of the assumption that $K_{abcd}n^d$ does not vanish everywhere on $\mathscr{I}$:

Lemma 2.3: The dimension of the group N cannot exceed one (i.e., there exists at most one Killing field whose restriction to $\mathscr{I}$ is a generator of a BMS supertranslation).

Proof: It suffices to show that $(\widehat{M},\hat{g}_{ab})$ does not admit two independent Killing fields $\hat{\xi}^a$ and $\hat{\xi}'^a$ such that restrictions of ξ^a and ξ'^a to $\mathscr{I}$ are generators of BMS translations. Let us assume it does. Let $\xi^a|_{\mathscr{I}}=\alpha n^a$ and $\xi'^a|_{\mathscr{I}}=\alpha'n^a$. Then, by Lemma 1.3, α and α' as well as $\alpha-\alpha'$ cannot vanish in an open neighborhood, within $\mathscr{I}$, of a point on $\mathscr{I}$. Also, $\mathscr{L}_\xi K_{abcd}$ and $\mathscr{L}_{\xi'} K_{abcd}$ must both vanish on $\mathscr{I}$. Using the fact that $\nabla_a n_b$ vanishes on $\mathscr{I}$, one can now easily show that $K_{abcd}n^d$ must vanish everywhere on $\mathscr{I}$. This contradicts the assumption. $\square$

As a direct consequence of Lemmas 2.2 and 2.3, we have

Theorem 2: If $K_{abcd}n^d$ does not vanish identically on $\mathscr{I}$, the dimension of the isometry group G of the given (physical) space–time $(\widehat{M},\hat{g}_{ab})$ cannot exceed four. Furthermore, the dimension of the (normal) subgroup N of G (whose action on $\mathscr{I}$ yields BMS supertranslations), cannot exceed one and that of G/N cannot exceed three.

The rest of this section is devoted to the discussion of various possibilities.

Consider, first, the case when G is four-dimensional. Then, G contains one translation. Hence, the curvature scalar f (of Lemma 2.1) now satisfies $n^a \nabla_a f=0$ on $\mathscr{I}$. That is, f is now a pullback to $\mathscr{I}$ of a function on $\mathscr{S}$ which we also denote by f. Consider, now, the Lie algebra $\mathscr{L}_{G/N}$. This is of dimension three. Next, there exists a natural imbedding of $\mathscr{L}_{G/N}$ into the six-dimensional Lie algebra of conformal Killing fields on $(\mathscr{S},h_{ab})$ (i.e., into the Lie algebra $\mathscr{L}_L$ of the Lorentz group L). Furthermore, every conformal Killing field ξ^a on $(\mathscr{S},h_{ab})$, induced by this imbedding satisfies $\mathscr{L}_\xi h_{ab}=2\phi h_{ab}$ where $\phi=-\tfrac{1}{6}(\xi^m D_m \log f)$. It is easy to check from the definition of f (Lemma 2.1) that, under a conformal rescaling $h_{ab}\to h'_{ab}=\omega^2 h_{ab}$, $f\to f'=\omega^{-6}f$ and hence $D_m \log f\to D'_m \log f'= D_m \log f-6 D_m \log \omega$. Hence by a suitable conformal transformation, we can always make ϕ vanish.[11] (Recall, from Sec. 2.1 that the restricted conformal freedom is $g_{ab}\to g'_{ab}=\omega^2 g_{ab}$ with ω nonzero, smooth on M and satisfying $\mathscr{L}_n \omega|_{\mathscr{I}}=0$; $\omega|_{\mathscr{I}}$ can be the pullback of an arbitrary, nonzero, smooth function on $\mathscr{S}$.) Let us now assume that we have chosen such a conformal frame. Then, since $\mathscr{L}_\xi h_{ab}=-\tfrac{1}{3}(\xi^m D_m \log f)h_{ab}$ on $\mathscr{S}$, each of the three conformal Killing fields on $(\mathscr{S},h_{ab})$ is, in fact, a *Killing* field on $(\mathscr{S},h_{ab})$. Since three independent Killing fields cannot all be proportional everywhere on $\mathscr{S}$, their integral manifold is all of $\mathscr{S}$. Hence, the scalar curvature $^2\mathscr{R}$ of h_{ab} is constant on $\mathscr{S}$. (That is, our chosen conformal frame has turned out to be a Bondi frame.) Consequently, $\mathscr{L}_{G/N}$ is isomorphic with the Lie algebra of the rotation group and $\mathscr{L}_G$ is isomorphic with the Lie algebra of Killing fields in the Schwarzschild space–time.[12] Recall, however, that there exists a neighborhood $\mathscr{N}$ of M such that in $\mathscr{N}\cap\widehat{M}$, the metric $\hat{g}_{ab}$ satisfies Einstein's equation. Therefore, by Birkhoff's[13] theorem, we have, under our assumption on $K_{abcd}n^d$,

Lemma 3.1: If G is four dimensional, $\hat{g}_{ab}$ is isometric to the Schwarzschild metric in $\mathscr{N}\cap\widehat{M}$.

Next, consider the case when G is three dimensional. Using arguments similar to those which led us to Lemma 3.1, it is easy to show that N is necessarily zero dimensional. Thus G/N is of dimension three ($G=G/N$). Recall, however, that $\mathscr{L}_{G/N}$ is necessarily a sub-Lie algebra of the Lorentz Lie algebra $\mathscr{L}_L$. Furthermore, every three-dimensional sub-Lie algebra of $\mathscr{L}_L$ except $\mathscr{L}_{SO(3)}$, itself admits a two-dimensional sub-Lie algebra. Using this fact one can show that $\mathscr{L}_G$ must be isomorphic to $\mathscr{L}_{SO(3)}$, and hence, by Birkhoff's theorem, $\hat{g}_{ab}$ is isometric with the Schwarzschild metric in a neighborhood of $\mathscr{I}$. (Let us suppose that $\mathscr{L}_G$ is not isomorphic to $\mathscr{L}_{SO(3)}$. Consider a two-dimensional sub-Lie algebra of $\mathscr{L}_G$. Fix, on $\mathscr{I}$, an orbit of the action of this sub-Lie algebra, passing through a point at which the curvature scalar f fails to vanish. This orbit is a 2-sphere[14] cross section of $\mathscr{I}$. By repeating the argument which led us to Lemma 3.1, we can choose a Bondi conformal frame in which the generators of the two-dimensional subalgebra turn out to be Killing fields on the 2-sphere cross section. This contradicts the assumption that the two fields are closed under commutation since the metric on the cross section is of constant curvature.)

Thus one has[15]:

Lemma 3.2: If $K_{abcd}n^d$ does not vanish identically on $\mathscr{I}$, and if G is of dimension three, the space–time metric $\hat{g}_{ab}$ is Schwarzschildean in the intersection of $\widehat{M}$ with a neighborhood of $\mathscr{I}$ in M.

Next, consider the case when the dimension of G is two. Then by an argument similar to the one used for Lemma 3.1, it follows that G/N cannot be two dimensional. Since by Theorem 2, N cannot be two dimensional, it follows that each of N and G/N must be of dimension one. Denote the two Killing fields on ($\widehat{M},\hat{g}_{ab}$) by ξ^a and ξ'^a (ξ^a being the generator of N). Since N is necessarily a normal subgroup of G (Theorem 1), we must have $[\xi,\xi']^a = \lambda\xi^a$ on M, for some constant λ. Since $\xi^a|_{\mathscr{I}}$ is the generator of a BMS translation, we have, $\xi^a|_{\mathscr{I}} = \alpha n^a|_{\mathscr{I}}$, and $\nabla_a\xi_b|_{\mathscr{I}} = n_{[b}D_{a]}\alpha|_{\mathscr{I}}$. Hence, on $\mathscr{I}$, $[\xi,\xi']^a = \phi'\alpha n^a - \frac{1}{2}(\xi'^m D_m\alpha)n^a = \lambda\alpha n^a$. Since G contains one translation, the curvature scalar f is a pullback to $\mathscr{I}$ of a function f on $\mathscr{S}$, which satisfies $\xi'^m D_m f = -6\phi'f$ on $\mathscr{I}$. The last two equations give $\xi'^m D_m(\alpha^3 f) = -6\lambda(\alpha^3 f)$. If $\lambda\neq 0$, since $\alpha^3 f$ is smooth on $\mathscr{S}$ and since $\mathscr{S}$ is compact, this would imply $\alpha^3 f = 0$ on $\mathscr{S}$. Consider a Bondi frame. Since α has zeros only in a set of measure zero (as a linear combination of $l=0$ and $l=1$ spherical harmonics) and since f does not vanish anywhere in some open set of $\mathscr{S}$ (the projection of N_p into $\mathscr{S}$), we have a contradiction. Therefore, the constant λ should vanish. Thus, one has,

Lemma 3.3[16]: If ($\widehat{M},\hat{g}_{ab}$) admits exactly two Killing fields, and if $K_{abcd}n^d$ does not vanish everywhere on $\mathscr{I}$, one (and only one) of the Killing fields is a translation and the two Killing fields must commute everywhere.

Finally, let us suppose that ($\widehat{M},\hat{g}_{ab}$) does admit exactly two Killing fields and that the translation Killing field ξ^a is timelike in the intersection of $\widehat{M}$ with a neighborhood of $\mathscr{I}$ in M. Then, on $\mathscr{I}$, $\xi^a = \alpha n^a$ with α nowhere vanishing. Since, under a conformal rescaling $g_{ab}\to g'_{ab} = \omega^2 g_{ab}$, $\alpha\to\alpha' = \omega\alpha$, there exists a *unique* conformal frame such that $\alpha = 1$, i.e., $\xi^a = n^a$ on $\mathscr{I}$. Furthermore, as is well known, this is a Bondi frame. Since the two fields ξ^a and ξ'^a commute, we have $[\xi,\xi']^a = \phi' n^a = 0$; ϕ' must vanish in this Bondi frame. Thus, the projection of ξ'^a on $\mathscr{S}$ is a rotational Killing field; G/N is isomorphic with SO(2). Hence one has:

Lemma 3.4: If a stationary asymptotically flat space–time ($\widehat{M},\hat{g}_{ab}$) with the property that $K_{abcd}n^d$ does not vanish identically on $\mathscr{I}$ admits another Killing field, it is either Schwarzschildean in a neighborhood of $\mathscr{I}$, or G/N is isomorphic with SO(2).

Collecting the results of 3.1, 3.2, 3.3, and 3.4, one obtains:

Theorem 3: Let $K_{abcd}n^d$ not vanish on $\mathscr{I}$ identically. Then the dimension of the isometry group G of ($\widehat{M},\hat{g}_{ab}$) cannot exceed two unless it is Schwarzschildean in the intersection of $\widehat{M}$ with a neighborhood of $\mathscr{I}$ in M. If G is of dimension two, it is Abelian, N is one-dimensional and the

Bondi news vanishes identically on $\mathscr{I}$. Furthermore,[17] if the generator of N is timelike in the intersection of $\widehat{M}$ with a neighborhood of $\mathscr{I}$, G/N is isomorphic with SO(2).

5. DISCUSSION
A. Assumptions

Throughout this analysis, we have assumed that the given space–time satisfies the following three asymptotic conditions: (i) generators of $\mathscr{I}$ are complete; (ii) $\mathscr{S}$ is diffeomorphic to S²; and, (iii) the physical metric $\hat{g}_{ab}$ satisfies Einstein's vacuum equation in the intersection of $\widehat{M}$ with a neighborhood of $\mathscr{I}$ in M. How sensitive is the dependence of our results on these assumptions? The first two assumptions were introduced to ensure that the group of asymptotic symmetries is the BMS group. Thus, for example, without the first assumption, one cannot even formulate an universally applicable and unambiguous notion of supertranslations. Thus, essentially every result of this paper depends on the validity of these two assumptions. (A notable exception is the first part analysis of the constraints imposed by the presence of asymptotic Weyl curvature on isometries. Thus, using the notion of conformal Killing data, even in the absence of these two assumptions, one can show that, in the conformal frame in which $\mathscr{I}$ is divergence-free, the data must be tangential to $\mathscr{I}$ (Lemma 1.2), and that the presence of asymptotic curvature leads to the determination of the third piece, and the fourth piece up to a multiple of n^a. Thus, we can still show that the space–time cannot admit more than seven Killing fields. The subsequent analysis, however, does not go through since it refers to the BMS supertranslations.) The assumption about the vacuum Einstein equation, on the other hand, has not played such a crucial role in our analysis: In essence, it is used only to guarantee the existence of limits to $\mathscr{I}$ of fields such as K_{abcd}. Hence, our analysis will presumably continue to be valid without any major modification if the vacuum Einstein equation is replaced by the Einstein–Maxwell equation or if the matter sources are assumed to fall of faster than a suitable power of Ω.

B. Corollaries

Several of the "intuitively obvious" results about symmetries of isolated systems in general relativity follow as immediate corollaries of Theorems 2 and 3 of Sec. 4. Let us consider some examples. Fix a space–time ($\widehat{M},\hat{g}_{ab}$) which is asymptotically flat and whose asymptotic curvature $K_{abcd}n^d$ does not vanish identically on $\mathscr{I}$. Then ($\widehat{M},\hat{g}_{ab}$) cannot be stationary in two distinct ways, i.e., with respect to two distinct Killing fields. (This result is of relevance to quantum field theory in curved space–times, in the analysis of the uniqueness of the vacuum state in asymptotically flat, nonflat space–times.) Next, if ($\widehat{M},\hat{g}_{ab}$) is to represent an isolated system emitting gravitational radiation, then it can admit at most one Killing field. Finally, suppose that the matter sources are confined to a world tube. Then ($\widehat{M},\hat{g}_{ab}$) cannot be axisymmetric about two distinct axes unless $\hat{g}_{ab}$ is isometric to the Schwarzschild metric, outside the world tube.

C. Applications to the analysis of exact solutions

There exist several methods of generating new solutions of Einstein's vacuum equation with one or more Killing fields. It is often the case that, although using these techniques one can obtain new solutions with relative ease, one cannot associate a simple physical interpretation with these solutions. A major obstacle in this respect is that it is usually rather difficult to determine whether or not a given solution is asymptotically flat. Results obtained in this paper might prove to be especially useful in this regard. For example, if a vacuum solution admits more than two Killing fields, and if it is not isometric with the Schwarzschild solution, one might regard it as unsuitable for representing isolated systems in general relativity: Either its asymptotic curvature falls off too rapidly making its mass vanish identically on any cross section of $\mathcal{I}$, or it is not asymptotically flat. Similarly, if a (stationary) space–time admits exactly two Killing fields which fail to commute, it is again unsuitable for representing isolated bodies. Finally, a space–time admitting more than one Killing field cannot represent an isolated system emitting gravitational radiation.[18]

Note finally that the notion of asymptotic flatness at null infinity was initially introduced in order to describe gravitational radiation in general relativity and not for the express purpose of making an analysis of the isometries that such space–times can admit. The fact that this definition enables an essentially complete analysis of the isometries and that it furthermore leads us to the results about symmetries of isolated systems which are intuitively expected to hold, provides a strong independent support in favor of this definition.

Note added in proof: (1) In the proof of Lemma 1.4, we have often used the tensor field $h^a{}_b$ with this index structure. An implicit choice of a Bondi frame has been made in "raising the index of the pullback h_{ab} to $\mathcal{I}$ of g_{ab}."

(2) If one is interested only in the Lie algebra of Killing fields—rather than the structure of isometry groups—one can drop the assumption on completeness of the generators of $\mathcal{I}$. In this case, all results continue to be valid in their "infinitesimal form."

(3) For a nonvacuum, asymptotically flat space–time, the situation may be summarized as follows. If the stress energy T_{ab} falls off to order four in the sense of Geroch,[3] most of the analysis goes through. (This falloff ensures that K_{abcd} is C^2 on $\mathcal{I}$ and that one can impose $\nabla_a n_b|_{\mathcal{I}} = 0$.) In this case, Theorems 1 and 2 are still valid. It also follows that if a space–time admits four Killing fields, the radiation field must vanish on $\mathcal{I}$. However, we do not have access to Birkhoff's theorem. As a consequence, one cannot rule out the possibility that, in a neighborhood of infinity, the space–time admits exactly three Killing fields. Indeed, an explicit example of such a space–time is provided by the Vaidya solution: In this case the stress energy does fall off to order four, the space–time admits precisely three Killing fields, and, although there is no gravitational radiation, the Bondi news fails to vanish on $\mathcal{I}$! For space–times with precisely two

Killing fields, on the other hand, previous results (Lemma 3.3) continue to be valid.

[1]See, e.g., S.W. Hawking, Commun. Math. Phys. **25**, 152 (1972) and L. Lindblom, Ap. J. **208**, 873 (1976). In both cases, stationary space–times are considered and the existence of an additional Killing field is established. It is then concluded that the additional Killing field must be a (spatial) rotation by making an appeal to asymptotic flatness.

[2]R. Penrose, in *Battelle Rencontres, Lectures in Mathematical Physics*, edited by C. DeWitt and J.A. Wheeler (Benjamin, New York, 1968).

[3] R. Geroch, in *Asymptotic Structure of Space–time*, edited by F.P. Esposito and L. Witten (Plenum, New York, 1977).

[4]R. Geroch and G.T. Horowitz, Phys. Rev. Lett. **40**, 203 (1978).

[5]Our convention is as follows: $\nabla_{[a}\nabla_{b]}k_c = \frac{1}{2}R_{abc}{}^d k_d$, $R_{ab} = R_{amb}{}^m$, and $R = R_a{}^a$.

[6]B.G. Schmidt, M. Walker, and P. Sommers, Gen. Rel. Grav. **6**, 489 (1975).

[7]One can also introduce a bracket on the 15-dimensional vector space of data such that the resulting algebra has the property that the conformal isometry Lie algebra of (M,g_{ab}) is necessarily its sub-Lie algebra. The existence of this 15-dimensional algebra turns out to be useful in the analysis of conformal Killing fields on (M,g_{ab}). For details, see A. Ashtekar and A. Magnon-Ashtekar, "A technique for analyzing the structure of isometries," J. Math. Phys. **19**, 1567 (1978).

[8]The proof is straightforward. See, e.g., p. 30, Ref. 3.

[9]A result analogous to Lemma 1.4 is proved in Ref. 3, under the assumption that the function α defined on $\mathcal{I}$ by $\xi^a|_{\mathcal{I}} = \alpha n^a|_{\mathcal{I}}$ is nowhere vanishing. (In terms of the physical space–time, the assumption is that ξ^a is timelike at infinity.) Only when this assumption is removed can one obtain Theorem 1.

[10]E.T. Newman and R. Penrose, Proc. Roy. Soc. (London) A **305**, 175 (1968).

[11]To obtain a suitable conformal rescaling, we need f to be nonzero *everywhere* on $\mathcal{I}$. Recall, however, that f is nonzero in a neighborhood $N_{p'}$ of p' within $\mathcal{I}$. Hence f is nonzero in an open region within $\mathcal{I}$. Consider a Bondi frame. Then each of the three conformal Killing fields on ($\mathcal{I}$,h_{ab}) which arise from $\mathcal{L}_{G/N}$ satisfies $\mathcal{L}_\xi f = -6\phi f$, where ϕ is a linear combination of $l = 1$ spherical harmonics (ϕ, of course, depends on the choice of ξ^a). Thus, we have three smooth vector fields on $\mathcal{I}$, whose integral manifold is $\mathcal{I}$, which satisfy $\mathcal{L}_\xi f = -6\phi f$ where ϕ can vanish only on a set of measure zero. From this, and from the fact that f is nonzero in an open set in $\mathcal{I}$, it follows that f is everywhere nonzero.

[12]Recall that G is necessarily a semidirect product of N with G/N. (Theorem 1.)

[13]G.D. Birkhoff, *Relativity and Modern Physics* (Harvard U.P., Cambridge, Mass., 1923).

[14]This follows from the properties that each integral manifold is two dimensional and its image, under the projection mapping, is all of $\mathcal{I}$.

[15]Thus, although by assumption, the isometry group ($\hat{M}$,$\hat{g}_{ab}$) is only three dimensional, in a neighborhood of $\mathcal{I}$, $\hat{g}_{ab}$ must acquire a fourth Killing field. For example, ($\hat{M}$,$\hat{g}_{ab}$) may represent the gravitational field of a radially oscillating spherical star.

[16]Compare, B.G. Carter, Commun. Math. Phys. **17**, 233 (1970). Note however that Carter's theorem does *not* require that Einstein's equation be satisfied to any order.

[17]Thus, the small gap in Hawking's and Lindblom's analyses, referred to in Ref. 1, is now filled.

[18]This of course is at least a partial explanation of the fact that no asymptotically flat radiating solutions are explicitly known.

Geometry of the roots of matrices with Onsager–Casimir symmetry

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The condition under which a matrix with Onsager–Casimir symmetry describing the coupling between m variables that are even (odd) under time reversal and 1 variable that is odd (even) has a pair of complex eigenvalues are analyzed, both graphically as well as algebraically.

I. INTRODUCTION

The problem of the maximum number of complex eigenvalues of matrices with Onsager–Casimir symmetry has been studied by Lekkerkerker and Laidlaw,[1,2] McLennan,[3] and Grmela and Iscoe.[4] Using the Onsager–Casimir symmetry relations it has been shown that the maximum number of complex roots of a hydrodynamic matrix describing the coupling between n variables that are even under time reversal and m variables that are odd is $2n$ or $2m$, whichever is smaller. In addition Lekkerkerker and Van Oost[5] have shown that in the dissipation-free limit there are $2n$ or $2m$ (whichever is smaller) purely imaginary roots and $|n-m|$ roots that are zero.

Further in the purely dissipative limit the hydrodynamic matrix is Hermitian and has only real roots. Knowing the limit situations (dissipation-free limit and purely dissipative limit) there remains the problem to determine the threshold conditions at which a pair of complex roots changes into real roots. In this paper we treat this problem for the simplest case, i.e., that of m even (odd) variables coupled to 1 odd (even) variable. In Sec. II we present a graphical analysis of this problem and in Sec. III we give an algebraic treatment.

II. GRAPHICAL ANALYSIS

The analysis presented in this section is closely related to the graphical determination of the changes of the eigenvalues caused by adding a single state to the Hamiltonian matrix in quantum mechanics.[6] It can be shown[1] that the hydrodynamic matrix describing the coupling of m variables that are even (odd) under time reversal and 1 variable that is odd (even) without loss of generality can be written in the form

$$
M = \begin{bmatrix}
a_1 & 0 \text{---} \cdots \text{---} 0 & b_1 \\
0 & a_2 \text{---} \cdots \text{---} 0 & b_2 \\
\vdots & \ddots & \vdots \\
0 & 0 \text{---} \cdots \text{---} a_m & b_m \\
-b_1^* & -b_2^* \text{---} \cdots \text{---} b_m^* & c
\end{bmatrix} ,
$$

(1)

where $a_1, a_2 \cdots a_m$ and c are real and if we assume the system to be stable they are in addition positive. The eigenvalue equation

$$ MV = \lambda V \qquad (2) $$

can be written as

$$ (a_j - \lambda) V_j + b_j V_{m+1} = 0 \quad (j = 1,2,\cdots,m), \qquad (3) $$

$$ -\sum_{j=1}^{m} b_j^* V_j + (c - \lambda) V_{m+1} = 0. \qquad (4) $$

Substituting the values of V_j that follow from (3) in (4) gives the equation

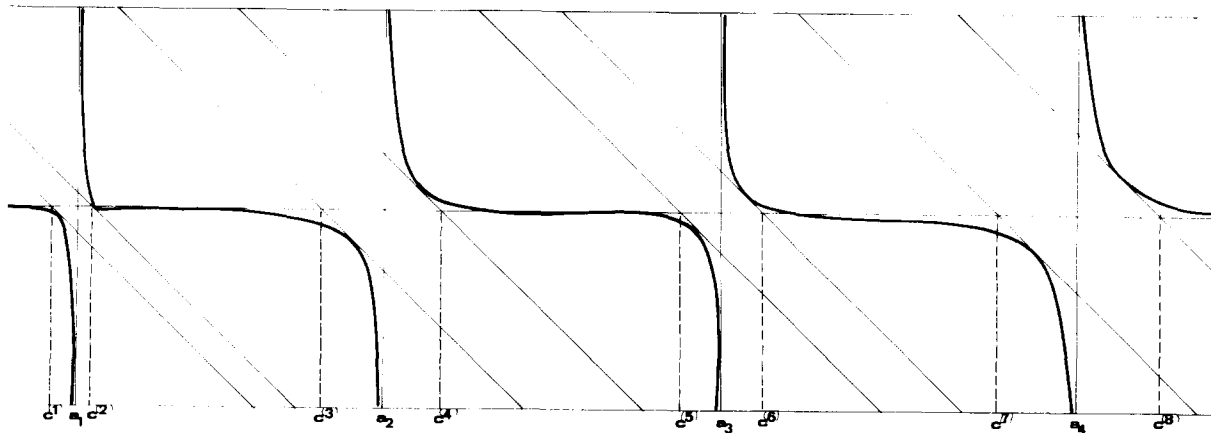

FIG. 1 Graphical determination of values of c for which M has complex eigenvalues. In this example $a_1 = 2$, $a_2 = 32$, $a_3 = 65$, $a_4 = 100$, $|b_1| = 1$, $|b_2| = 3$, $|b_3| = 2$, $|b_4| = 4$. One obtains $c^{(1)} = -.51$, $c^{(2)} = 3.46$, $'c^{(3)} = 25.70$, $c^{(4)} = 37.65$, $c^{(5)} = 60.87$, $c^{(6)} = 68.79$ $c^{(7)} = 92.28$, $c^{(8)} = 108.24$.
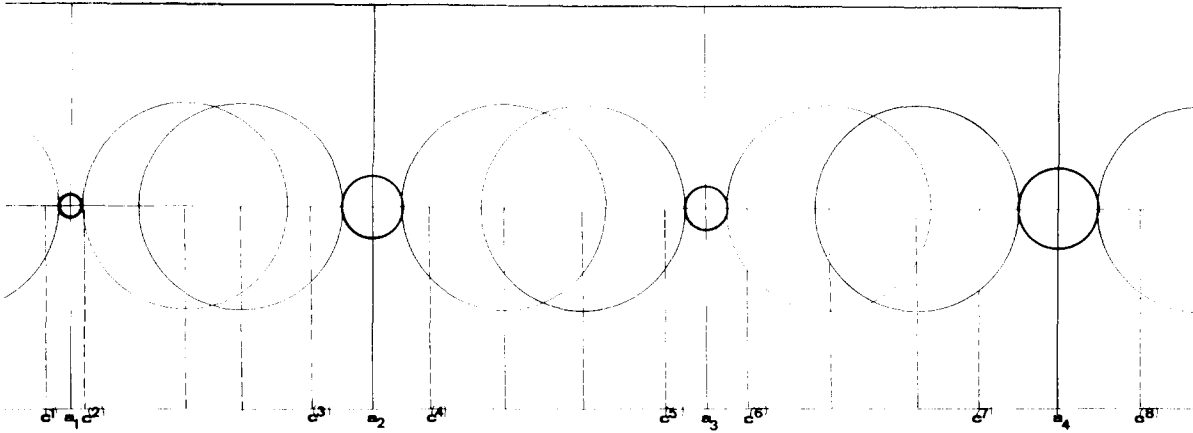
FIG. 2. Geršgorin's disks for the example treated in Sec. II (See caption Fig. 1).

$$\sum_{j=1}^{m} \frac{|b_j|^2}{\lambda - a_j} = (c - \lambda) \tag{5}$$

which is equivalent to the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{M}| = 0.$$

In Fig. 1 the heavy curves represent the function

$$g(\lambda) = \sum_{j=1}^{m} \frac{|b_j|^2}{\lambda - a_j}$$

for the example noted and the light lines represent the lines $f(\lambda) = c - \lambda$ for values of c such that $f(\lambda)$ is tangent to $g(\lambda)$. It is clear that for values of c situated in the union of open intervals $]c^{(2k-1)}, c^{(2k)}[$ $(k = 1, \cdots, 4)$, $f(\lambda)$ intersects $g(\lambda)$ only three times [or in the general case $(m - 1)$ times]. Thus for values of c situated in these intervals there are $(m - 1)$ real roots and two complex roots whereas for all other c values there are $(m + 1)$ real roots.

III. ALGEBRAIC ANALYSIS

In addition to a graphical analysis of the problem it is worthwhile to have an algebraic method at one's disposal to determine the values of the parameters for which the matrix $\mathbf{M}$ given by (1) has two complex eigenvalues. Using Geršgorin's theorem[7] it is possible to write down sufficient (but not necessary) conditions for which $\mathbf{M}$ has no complex eigenvalues. From the graphical analysis presented in the previous section it follows that the union of the m

Geršgorin's disk

$$|z - a_j| \leqslant |b_j| \quad (j = 1, 2, \cdots, m) \text{ contains at least } m - $$

1 real eigenvalues if the Geršgorin disk

$$|z - c| \leqslant \sum_{j=1}^{m} |b_j|$$

has no points in common with this union. Of the remaining two roots one is located in the isolated Geršgorin disk

$$|z - c| \leqslant \sum_{j=1}^{m} |b_j|$$

and the other in the union of the remaining disks. This means that these roots are real since complex roots would appear in the same Geršgorin disk. Thus a sufficient condition for the absence of complex eigenvalues is

$$|c - a_j| > |b_j| + \sum_{j=1}^{m} |b_j|, \quad j = 1, \cdots, m. \tag{6}$$

This condition is illustrated in Fig. 2 for the same example as was treated in Sec. II.

Since the characteristic polynomial of $\mathbf{M}$ is real and has at most two complex roots it appears logical to try to establish an extension of the relation $\Delta = A'_1{}^2 - 4A'_2 < 0$ which indicates that the real second degree polynomial $p_2(\lambda) = \lambda^2 + A'_1 \lambda + A'_2$ has complex eigenvalues. Indeed it is possible to obtain such a generalization using determinant sequences.[8]
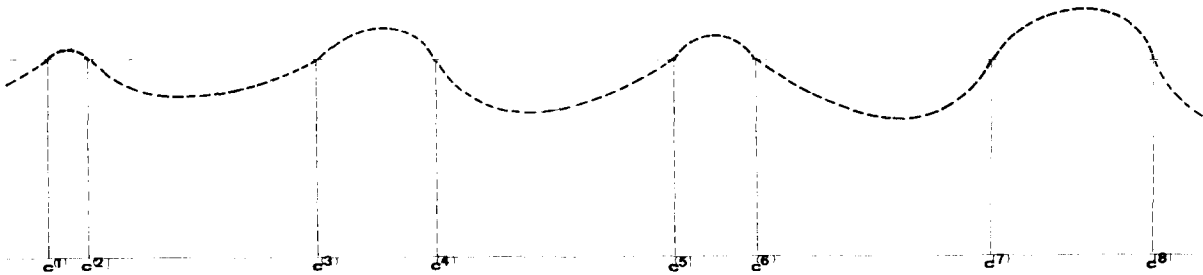

FIG. 3. Schematic illustration of the behavior of $\Delta(5,5)$ for the example of Sec. II. The values of $c^{(k)}$ for $k = 1, \ldots 8$ are the same as in Fig. 1.
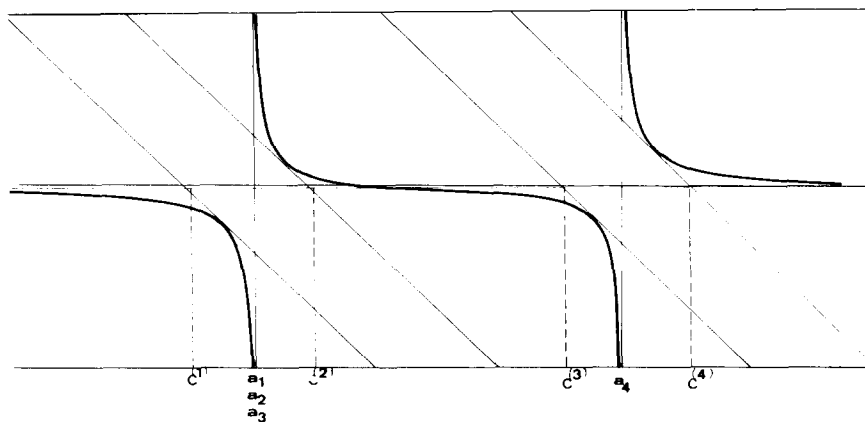
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FIG. 4. Graphical determination of the values of c for which a hydrodynamic matrix with a twofold degenerate root has complex eigenvalues. In this example $a_1 = a_2 = a_3 = 4$, $a_4 = 40$, $|b_1| = 1$, $|b_2| = |b_3| = 2$, $|b_4| = 3$. One obtains $c^{(1)} = -2.23$, $c^{(2)} = 9.73$, $c^{(3)} = 34.27$, and $c^{(4)} = 46.23$.

Consider the nth degree polynomial

$$f(\lambda) = p_n(\lambda) - i \frac{dp_n(\lambda)}{d\lambda}$$

$$= \lambda^n + A_1 \lambda^{n-1} + \cdots + A_n, \qquad (7)$$

where $p_n(\lambda)$ is the characteristic polynomial of **M**. Let the coefficients of $f(\lambda)$ be written in the form $A_k = A'_k + iA''_k$ where A'_k and A''_k are real. Let $A(n,k)$ ($k = 1, \cdots, n$) denote the maxtrix formed from the first $(2k - 1)$ elements in the first $(2k - 1)$ rows of the matrix.

$$\begin{bmatrix} A''_1 & A''_2 & --- & A''_n & 0 & --- & 0 \\ 1 & A'_1 & --- & A'_{n-1} & A'_n & --- & 0 \\ 0 & A''_1 & --- & & A''_n & --- & 0 \\ | & | & & | & | & & | \\ | & | & & | & | & & | \\ | & | & 1 & A'_1 & ----------- & A'_n \\ 0 & ----- & 0 & A''_1 & A''_2 & --- & A''_n \end{bmatrix}. \qquad (8)$$

Further let $\Delta(n,k)$ denote the determinant of the matrix $A(n,k)$. It is possible to prove the following theorem.

Theorem I: Let $p_n(\lambda)$ be a real polynomial with at most two complex roots. Then $p_n(\lambda)$ has $(n-l)$ distinct roots iff $\Delta (n,n) = \cdots = \Delta (n,n-l+1) = 0$ and $\Delta (n,n-l) \neq 0$. Further two of these are complex iff sign $\Delta (n,n-l) =$ sign $(-1)^{n-l+1}$. (The proof of this theorem is given in the Appendix.)

Let us apply Theorem I to the example analyzed in Sec. II. The behavior of $\Delta(5,5)$ is schematically illustrated in Fig. 3. As a further illustration of the application of Theorem I, we consider a hydrodynamic matrix with a twofold degener-

ate root. In Fig. 4 we first give a graphical analysis of such a case. For c values situated in the open intervals $]c^{(1)}, c^{(2)}[$ and $]c^{(3)}, c^{(4)}[$ there are complex eigenvalues.

The determinant $\Delta(5,5)$ for the example considered is zero for all real c and the behavior of $\Delta(5,4)$ is schematically illustrated in Fig. 5.

We hope that the simple examples treated here sufficiently demonstrate the usefulness of theorem I. An extension of theorem I to deal with the general case of n variables that are even under time reversal coupled to m variables that are odd would be desirable but so far we have not been able to find such an extension.
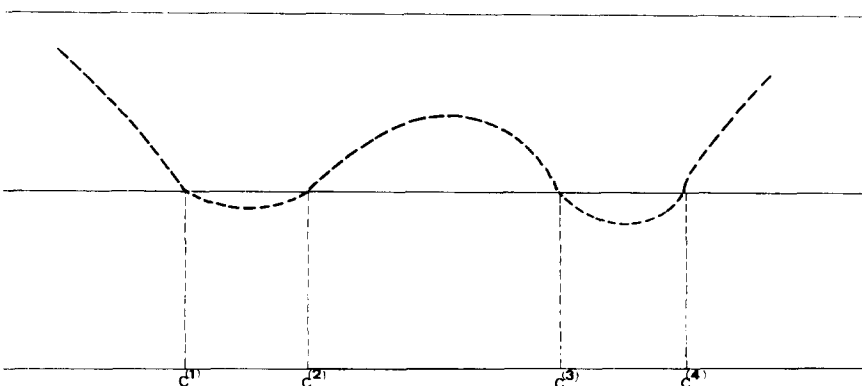

FIG. 5. Schematic illustration of the behavior of $\Delta(5,4)$ for the example given in Fig. 4. The values of $c^{(1)}$, $c^{(2)}$, $c^{(4)}$ are the same as in Fig. 4.

APPENDIX

Let the real polynomial
$p_n(\lambda) = \lambda^n + A'_1 \lambda^{n-1} \cdots + A'_n$ be written in the form

$$p_n(\lambda) = \prod_{i=1}^{r}(\lambda - \alpha_i)^{m_i} \qquad (A1)$$

where the roots are ordered such that the first j roots have multiplicity higher than one.

Lemma 1: If $p_n(\lambda)$ has $n-1$ different roots then
$\Delta(n,n) = \Delta(n,n-1) = \cdots = \Delta(n,n-s+1) = 0$.

Proof: Consider the real functions γ and κ of the integer i_j for $j \geqslant 1$

$$\gamma(i_j) = \sum_{i_{j-1}=0}^{i_j} \cdots \sum_{i_0=0}^{i_1}\left[\prod_{k=1}^{j}\alpha_k^{i_k - i_{k-1}}\binom{m_{k-2}+i_k-i_{k-1}}{i_k-i_{k-1}}\right]A'_{i_0}$$

$$\kappa(i_j) = \sum_{i_{j-1}=0}^{i_j} \cdots \sum_{i_0=0}^{i_1}\left[\prod_{k=1}^{j}\alpha_k^{i_k-i_{k-1}}\binom{m_{k-2}+i_k-i_{k-1}}{i_k-i_{k-1}}\right]$$

$$(n-i_0)A'_{i_0}$$

where $A'_{i_0} = 0$ for $i_0 > n$ and $A'_0 = 1$.

After some algebra we find that

$$\frac{p_n(\lambda)}{\prod_{i=1}^{r}(\lambda-\alpha_i)^{m_i-1}} = \sum_{i_j=0}^{r}\gamma(i_j)\lambda^{r-i_j}$$

and

$$\frac{dp_n(\lambda)/d\lambda}{\prod_{i=1}^{r}(\lambda-\alpha_i)^{m_i-1}} = \sum_{i_j=0}^{r-1}\kappa(i_j)\lambda^{r-1-i_j}.$$

Further one can show that for any $k \geqslant 0$

$$\gamma(r+k+1)$$

$$= \frac{1}{(m_j-2)!}\frac{d^{(m_j-2)}}{d\lambda^{(m_j-2)}}$$

$$\times\left(\lambda^k \cdot \frac{p_n(\lambda)}{\prod_{i=1}^{j-1}(\lambda-\alpha_i)^{m_i-1}}\right)\bigg|_{\lambda=\alpha_j}$$

and

$$\kappa(r+k)$$

$$= \frac{1}{(m_j-2)!}\frac{d^{(m_j-2)}}{d\lambda^{(m_j-2)}}$$

$$\times\left(\lambda^k \cdot \frac{dp_n(\lambda)/d\lambda}{\prod_{i=1}^{j-1}(\lambda-\alpha_i)^{m_i-1}}\right)\bigg|_{\lambda=\alpha_j}.$$

From the last two relations it is clear that

$$\gamma(r+k+1) = 0, \quad \text{for } k \geqslant 0, \qquad (A2)$$

$$\kappa(r+k) = 0, \quad \text{for } k \geqslant 0. \qquad (A.3)$$

If $p_n(\lambda)$ has $n-s$ different roots, we use relation (A.2) and (A.3) to construct a linear combination of the columns of the matrices $A(n,n), \cdots, A(n,n-s+1)$, which is zero.

Lemma 2: Let α_i be a root of $p_n(\lambda)$ with multiplicity m_i and let $\Delta(n-g,n-g)$ denote the determinant associated

with the polynomial

$$p_n(\lambda)/(\lambda-\alpha_i)^g, \text{ for } g \leqslant m_i$$
then

$$\Delta(n,n-g+1)$$

$$= -g\left(\frac{1}{g!}\frac{d^{(g)}}{d\lambda^{(g)}}(p_n(\lambda))\right)\bigg|_{\lambda=\alpha_i}^{2}\cdot\Delta(n-g,n-g). (A.4)$$

Proof: Let S and T be $2(n-g+1)-1$ dimensional upper triangular square matrices

$$S = \begin{bmatrix} 1 & g & -\alpha_i & 0 & - & - & - & - & - & - & - & 0 \\ 0 & 1 & 0 & -\alpha_i & 0 & - & - & - & - & - & - & 0 \\ 0 & 0 & 1 & g & -\alpha_i & 0 & - & - & - & - & - & 0 \\ & & & & & & & & & & & | \\ & & & & & & & & & & & | \\ & & & & & & & & & -1 & g & -\alpha_i \\ & & & & & & & & & & 1 & 0 \\ & & & & & & & & & & & 1 \end{bmatrix}$$

$$(A5)$$

and $T = (t_{h,k})$ with $t_{h,k} = 0$ if $h > k$.

For $h \leqslant k$

$$t_{n,k} = \binom{g-h+k}{k-h}\cdot\alpha_i^{k-h}, \quad 1 \leqslant k \leqslant 2n-2g$$

and

$$t_{n,k} = \binom{2n-g-h}{2n-2g+1-h}\cdot\alpha_i^{2n-2g+1-h}, \text{ if } k = 2n-2g+1. \qquad (A6)$$

The matrix $S \cdot A(n,n-g+1) \cdot T$ has the form

$$S \cdot A(n,n-g+1)\cdot T = \begin{bmatrix} A(n-g,n-g) & | & 0 \\ - & - & - & - & - & - & + & - \\ X & | & B \end{bmatrix},$$

where 0 is a $(2n-2g-1) \times 2$ dimensional zeroblock

$$B = \begin{bmatrix} \frac{1}{g!}\frac{d^{(g)}p_n(\lambda)}{d\lambda^{(g)}}\bigg|_{\lambda=\alpha_i} & 0 \\ x & -\frac{g}{g!}\frac{d^{(g)}p_n(\lambda)}{d\lambda^{(g)}}\bigg|_{\lambda=\alpha_i} \end{bmatrix}.$$

Taking into account that $\det(S) = \det(T) = 1$ it follows that

$\det(S \cdot A(n,n-g+1) \cdot T)$

$$= \Delta(n,n-g+1) = \Delta(n-g,n-g)\cdot\det B$$

$$= -g\left(\frac{1}{g!}\frac{d^{(g)}p_n(\lambda)}{d\lambda^{(g)}}\bigg|_{\lambda=\alpha_i}\right)^{2}\cdot\Delta(n-g,n-g)$$

which was to be proved. $\qquad\qquad\square$

Lemma 3: If $p_n(\lambda)$ has $n-s$ different roots and we denote by $A(n-m_k,n-m_k)$ the matrix of the polynomial
$[p_n(\lambda)/(\lambda-\alpha_k)]^{m_k}$,

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where α_k is a m_k—fold degenerate root of $p_n(\lambda)$ then,

$$\Delta(n,n-s)$$
$$= -m_k \Delta(n-m_k,n-s-1)$$
$$\times \left(\frac{d}{d\lambda} \left(\frac{p_n(\lambda)}{\Pi_{i=1}^r (\lambda-\alpha_i)^{m_i-1}} \right)\Big|_{\lambda=\alpha_k} \right)^2.$$

Proof: The proof follows the same lines as that of Lemma 2. We introduce the matrices $\mathbf{S}_{n-s}$ and $\mathbf{T}_{n-s}$ formed from the last $2(n-s)-1$ elements from the last $2(n-s)-1$ rows of $\mathbf{S}$ and $\mathbf{T}$ as defined by (A5) and (A6).

Then

$$\mathbf{S}_{n-s}\cdot\mathbf{A}(n,n-s)\cdot\mathbf{T}_{n-s} = \begin{pmatrix} \mathbf{A}(n-m_k,n-s-1) & \mathbf{C} \\ \mathbf{X} & \mathbf{B} \end{pmatrix},$$

where $\mathbf{C}$ consists of the last two elements of the first $2(n-s)-3$ rows of $\mathbf{A}(n-m_k,n-s)$.

From Lemma 1 it follows that $\Delta(n-m_k,n-s)=0$. Using the structure of the matrices $\mathbf{A}(n,k)$ for $1 \leqslant k \leqslant n$, this relation implies that there exists linear combinations of the columns of $\mathbf{S}_{n-s}\mathbf{A}(n,n-s)\,\mathbf{T}_{n-s}$ which transform $\mathbf{C}$ into a zeroblock. Let $\mathbf{B}'$ be the result of this transformation applied to $\mathbf{B}$, then

$$\det(B') = -m_k \left(\frac{d}{d\lambda} \left(\frac{p_n(\lambda)}{\Pi_{i=1}^r(\lambda-\alpha_i)^{m_i-1}} \right)\Big|_{\lambda=\alpha_k} \right)^2.$$

which leads to the relation that we had to prove. □

Lemma 4: If $p_n(\lambda)$ has $n-s$ different roots and $\mathbf{A}(n-s,n-s)$ denotes the matrices associated to the polynomial

$$p_n(\lambda) \Big/ \prod_{i=1}^r (\lambda-\alpha_i)^{m_i-1}$$

then

$$\Delta(n,n-s) = \prod_{i=1}^r m_i \cdot \Delta(n-s,n-s). \qquad (A7)$$

Proof: Consider first the case that $p_n(\lambda)$ has one degenerate root α_1 with multiplicity m_1 and thus $s=m_1-1$. Using Lemma 2 and the relation

$$\frac{p_n(\lambda)}{(\lambda-\alpha_i)^k}\Big|_{\lambda=\alpha_i} = \frac{1}{k!} \frac{d^{(k)}p_n(\lambda)}{d\lambda^k}\Big|_{\lambda=\alpha_i},$$

for $0 \leqslant k \leqslant m_i$, where m_i is the multiplicity of the root α_i of $p_n(\lambda)$, we obtain (A7) for $s=m_1-1$. Let us now assume that (A7) is valid for any polynomial with at most $j-1$ degenerate roots

This implies that

$$\Delta(n-m_1,n-s-1) = m_2 \cdots m_r \Delta(n-s-1,n-s-1), \qquad (A8)$$

where $\Delta(n-m_1,n-s-1)$ is the determinant associated with the polynomial

$$p_n(\lambda)/(\lambda-\alpha_i)^{m_1}$$

and $\Delta(n-s-1,n-s-1)$ is associated with

$$p_n(\lambda)/(\lambda-\alpha_1)^{m_1}\,\Pi_{i=2}^r(\lambda-\alpha_i)^{m_i-1}.$$

Then

$$\Delta(n,n-s)$$
$$= -m_1 \left(\frac{d}{d\lambda} \left(\frac{p_n(\lambda)}{\Pi_{i=1}^r(\lambda-\alpha_i)^{m_i-1}} \right)\Big|_{\lambda=\alpha_i} \right)^2$$
$$\cdot \Delta(n-m_1,n-s-1) \quad (\text{Lemma 3})$$
$$= -m_1 \left(\frac{d}{d\lambda} \left(\frac{p_n(\lambda)}{\Pi_{i=1}^r(\lambda-\alpha_i)^{m_i-1}} \right)\Big|_{\lambda=\alpha_i} \right)^2$$
$$\cdot m_2 \cdots m_r \Delta(n-s-1,n-s-1) \qquad (A9)$$
$$= \prod_{i=1}^r m_i \Delta(n-s,n-s) \quad (\text{Lemma 2}). \qquad \square$$

Proof of Theorem I: We first prove the theorem for the case that $l=0$ by induction on n, the degree of the polynomial $p_n(\lambda)$. For $n=2$, the proof is trivial since $\Delta(2,2)=A_1'^2-4A_2'$. Let us assume that the theorem is valid for any real $(n-1)$th degree polynomial with a maximum of two complex roots. From Lemma 1 it follows that

$$\Delta(n,n) = -\left(\frac{dp_n(\lambda)}{d\lambda}\Big|_{\lambda=x} \right)^2 \Delta(n-1,n-1), \qquad (A10)$$

where x is any root of $p_n(\lambda)$ and $\Delta(n-1,n-1)$ the determinant associated to $p_n(\lambda)/(\lambda-x)$. From (A7) it follows that $\Delta(n,n) \neq 0$ iff

$$\frac{dp_n(\lambda)}{d\lambda}\Big|_{\lambda=x} \neq 0$$

and $\Delta(n-1,n-1) \neq 0$, which is equivalent to say that $p_n(\lambda)$ has n different roots.

If moreover we assumed that x is real (which is always possible for $n \geqslant 3$) it follows from (A7) that $\Delta(n,n)$ and $\Delta(n-1,n-1)$ have opposite signs, and since p_n has complex roots iff $p_n(\lambda)/(\lambda-x)$ has complex roots, the theorem is proved for $l=0$.

We now consider the case that $l \neq 0$.

Let us first assume that $p_n(\lambda)$ has $(n-l)$ distinct roots. From Lemma 1 it follows then that

$$\Delta(n,n) = \cdots \Delta(n,n-l+1)=0.$$ Further the $(n-l)$th degree polynomial

$$p_n(\lambda) \Big/ \prod_{i=1}^r (\lambda-\alpha_i)^{m_i-1}$$

has $(n-l)$ distinct roots and thus statisfies the conditions of the first part of the proof. This means that $\Delta(n-l,n-l) \neq 0$ and sign $\Delta(n-l,n-l) = \text{sign }(-1)^{n-l+1}$ iff

$$p_n(\lambda) \Big/ \prod_{i=1}^r (\lambda-\alpha_i)^{m_i-1}$$

has complex roots. Using Lemma 4 the above means that $\Delta(n,n-l) \neq 0$ and sign$\Delta(n,n-l)=\text{sign}(-1)^{n-l+1}iffp_n(\lambda)$

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has complex roots. Finally we still have to show that if $\Delta (n,n) = \cdots \Delta (n,n-l+1) = 0$ and $\Delta (n,n-l) \neq 0$ that $p_n(\lambda)$ has $(n-l)$ distinct roots. Indeed if the number of distinct roots of $p_n(\lambda)$ is different from $(n-l)$, this leads immediately to contradictions. $\square$

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Fibre bundle analysis of topological charges in spontaneously broken gauge theories

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We study the topological properties of spontaneously broken gauge theories in the context of fibre bundle theory. In particular, we discuss the conditions under which the topological charges of gauge and Higgs fields are the same.

I. INTRODUCTION

The aim of the present paper is to study topological properties of spontaneously broken gauge theories, giving an explicit geometrical description in terms of connections and cross sections in a principal and associated fibre bundles.

As it was noted by Popov,[1] this approach not only enables one to "calculate" but renders calculations more transparent.

In Sec. II we study the simple case of an Abelian Higgs theory in $d = 2$ (Euclidean) dimensions and show how the geometrical interpretation leads naturally to the identification of the topological charges of both gauge fields and Higgs field.

The discussion of the general case is done in Sec. III: for a compact Lie group G, conditions under which a single K-uple of integers labels the topological charges of gauge and Higgs fields are obtained. These conditions, already obtained by Woo[2] for the case $d = 4$ (Euclidean) dimensions, using rather different techniques, arises in a transparent manner in the context of the fibre bundle theory. Several examples are discussed at the end of this section.

II. THE ABELIAN CASE

We will consider in this section the Abelian Higgs model: an Abelian gauge field A_μ coupled to a complex scalar field ϕ, with dynamics determined by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu + iA_\mu)\phi^*(\partial_\mu - iA_\mu)\phi - \mathcal{U}(\phi), \quad (2.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$
$$\phi = \phi_1 + i\phi_2, \quad (2.2)$$

and

$$\mathcal{U}(\phi) = \tfrac{1}{2}\lambda(\phi^*\phi - \mu^2/2\lambda)^2. \quad (2.3)$$

This Lagrangian density is invariant under the local

gauge transformations generated by the group $G = \mathcal{U}(1)$; we are interested in the case $\mu^2 > 0$, that is, when the symmetry is spontaneously broken. It was noted by Nielsen and Olesen[3] that this model allows for vortex solutions-static, cylindrically symmetric field configurations of finite energy per unit length. The Nielsen—Olesen solution depends only on two variables. Then, the vortex field can be considered as a pseudoparticle configuration (with finite action) in the Euclidean version of the two-dimensional theory. If the action is to be finite, then, at Euclidean infinity (that is, on the sphere S^1_∞)

$$\lim_{r \to \infty} F_{\mu\nu} = 0, \quad (2.4)$$

$$\lim_{r \to \infty} (\partial_\mu - iA_\mu)\phi = 0, \quad (2.5)$$

$$\lim_{r \to \infty} \mathcal{U}(\phi) = 0, \quad (2.6)$$

where $r^2 = x_1^2 + x_2^2$.

These conditions can be interpreted geometrically in terms of the theory of fibre bundles. To this end, we consider S^1_∞ as the base space of a principal fibre bundle $P(S^1_\infty, \mathcal{U}(1))$ with structure group $\mathcal{U}(1)$. If we call $\mathcal{J}$ the associated vector bundle with fibre $\mathbb{C}$—the group $\mathcal{U}(1)$ acts on the left on $\mathbb{C}$ by usual multiplication—then the scalar field ϕ can be considered as a global cross section on $\mathcal{J}$[4].

Because $\phi \neq 0$ on S^1_∞ [from Eq. (2.3) and condition (2.6)] $\mathcal{J}$ is trivial and so is P.

The one form $A_\mu dx^\mu$ defines a connection on P. We can then consider on $\mathcal{J}$ the covariant derivative D induced by the connection [its explicit expression is $(\partial_\mu - iA_\mu)$].

Let γ be a given curve in S^1_∞ from x_0 to x. If $(x_0, g) \in P_{x_0}$ and $(x_0, v) \in \mathcal{J}_{x_0}$ [$\mathcal{J}_{x_0}(P_{x_0})$ is the fibre of $\mathcal{J}(P)$ over x_0], we will call $(x, T_\gamma g) \in P_x$ the parallel displacement of (x_0, g) along γ, given by the connection $A_\mu dx^\mu$, and $(x, T_\gamma v) \in \mathcal{J}_x$ the parallel displacement of (x, v) along γ given by D. Since D is induced by the connection,

$$\widetilde{T}_\gamma v = (T_\gamma g)v/g \quad (2.7)$$

and

Lemma: for any curve γ' in S^1_∞ from x_0 to x, $T_{\gamma'}g = T_\gamma g$ $\forall$ $g \in \mathcal{U}(1)$ and $\widetilde{T}_{\gamma'}v = \widetilde{T}_\gamma v$ $\forall$ $v \in \mathbb{C}$.

Proof: From Eq. (2.5), $\phi(x) = \tilde{T}_\gamma \phi(x_0)$; then for any $g \in U(1)$

$$\phi(x) = (T_\gamma g)\phi(x_0)/g. \tag{2.8}$$

But Eq. (2.5) also implies that $\phi(x) = \tilde{T}'_\gamma \phi(x_0)$. Hence, $\phi(x) = (T'_\gamma g) \phi(x_0)/g$. Since $\phi(x_0) \neq 0$, it follows $T_\gamma g = T'_\gamma g$. Now, from (2.7) it is evident that $\tilde{T}_\gamma v = \tilde{T}'_\gamma v$. That is, the parallel displacement from x_0 to x is independent of the particular choice of γ in S^1_∞.

Hence, if we fix a point $x_0 \in S^1_\infty$, the given connection $A_\mu dx^\mu$ can be identified with a well-defined mapping

$$\tilde{A} : S^1_\infty \to U(1),$$
$$\tilde{A}(x) = T_\gamma(I), \tag{2.9}$$

where I is the identity of $U(1)$ and γ is any curve from x_0 to x in S^1_∞. We note that the explicit form of $\tilde{A}(x)$ reads

$$\tilde{A}(x) = \exp(i \int_{x_0}^x A_\mu dx^\mu).$$

Taking $g = I$ in Eq. (2.7), we have

$$\phi(x) = \tilde{A}(x)\phi(x_0). \tag{2.10}$$

Now, condition (2.6) shows that ϕ defines a mapping $\phi : S^1_\infty \to S^1$. Because $U(1) \equiv S^1$, $\tilde{A}$ can be also considered as a mapping $\tilde{A} : S^1_\infty \to S^1$. Then, relation (2.10) implies that ϕ and $\tilde{A}$ *belong to the same class in* $\Pi(S^1) \cong Z$. Hence, *the same integer characterizes topologically the gauge field A_μ and the scalar field ϕ.*

This integer, the topological charge "n" is related to the quantization of the magnetic flux of the vortex. In terms of the gauge field A_μ it is given by the expression

$$n = \frac{1}{2\pi} \oint_{S^1_\infty} A_\mu dx^\mu$$

that, according to Stokes theorem, can be written as

$$n = \frac{1}{4\pi} \int_{\mathbb{R}^2} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

Our derivation of the topological equivalence between Higgs and gauge fields can be understood intuitively as follows: The principal fibre bundle P can be visualized as a torus (A) (see Fig. 1). Then the topological charge, associated with the gauge field counts the num-
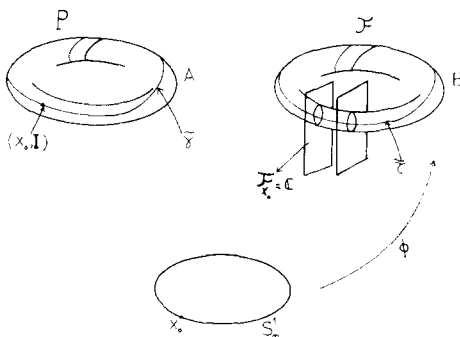
ber of times that the curve $\tilde{\gamma}$ joining the points $(x, \tilde{A}(x))$ in P winds around the torus. On the other hand, we can take on each fibre $\mathbb{C}$ of the vector bundle $\mathcal{J}$ the circle $|z| = a_0 (a_0 = (\mu^2/2\lambda)^{1/2})$, that is, all the possible values of ϕ on S^1_∞. We thus obtain another torus (B) and we can take on it the curve $\tilde{\tau}$ joining the points $(x, \phi(x))$. The topological charge associated with the Higgs field corresponds to the number of times the curve $\tilde{\tau}$ winds around this torus. Relation (2.10) states the equality between both numbers.

III. HIGHER-DIMENSIONAL CASES

In this section, we consider the general case of a spontaneously broken gauge theory in a Euclidean d-dimensional space ($d \geq 2$). Let G be a compact Lie group and

$$L = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} D_\mu \phi D^\mu \phi - U(\phi) \tag{3.1}$$

a Yang–Mills Lagrangian density for G. The field ϕ transforms in accordance with an N-dimensional unitary irreducible representation of G, and the fields $A_\mu = (A^1_\mu, \ldots, A^n_\mu)$ assume values in the algebra of the Lie Group G. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, D_μ is the covariant derivative, and $U(\phi)$ is a G-invariant function which has a minimum (this minimum is assumed to be equal to zero). As in the Abelian case, finiteness of the action imposes the following conditions on S^{d-1}_∞:

$$\lim_{r \to \infty} F_{\mu\nu} = 0, \tag{3.2}$$
$$\lim_{r \to \infty} D_\mu \phi = 0, \tag{3.3}$$
$$\lim_{r \to \infty} U(\phi) = 0, \tag{3.4}$$

where $r^2 = x_1^2 + \ldots + x_d^2$. This situation corresponds to the following geometrical description: a flat connection $A_\mu dx^\mu$ [condition (3.2)] on a principal fibre bundle $P(S^{d-1}_\infty, G)$ with base manifold S^{d-1}_∞ and structure group G.

Since S^{d-1}_∞ is simply connected ($d > 2$) and the connection is flat, then it follows that P is trivial and also that the parallel displacement along a curve γ in S^{d-1}_∞ from x_0 to x, induced by the connection is independent of the particular choice of the curve γ. (These results follow from the general theory of flat connections, see, for example, Ref. 4).

As we did in Sec. II, let us fix a point $x_0 \in S^{d-1}_\infty$. The connection $A_\mu dx^\mu$ determines a well defined mapping

$$\tilde{A} : S^{d-1}_\infty \to G, \tag{3.5}$$
$$\tilde{A}(x) = T_\gamma I, \tag{3.6}$$

where I is the identity of G and $(x, T_\gamma I) \in P_x$ is the parallel displacement of $(x_0, I) \in P_{x_0}$ along any curve γ in S^{d-1}_∞ from x_0 to x.

Let $\mathcal{J}$ be the associated bundle with fibre K, where K is a vector space-tensor space—according to the choice of ϕ. (The group G acts on the left on K in the usual way).

Let us call D the covariant derivative induced on $\mathcal{J}$ by the connection given on P. From Eq. (3.3) the Higgs field ϕ can be considered as a global cross section of the bundle $\mathcal{J}$ with

FIG. 1. We represent at the bottom the base space S^1_∞ of the principal fiber bundle P (torus A). Taking on each fibre C of the vector bundle $\mathcal{J}$ the circles $|Z| = a_0$, we obtain another torus (B).

$$\phi(x) = \tilde{T}_\gamma \phi(x_0), \tag{3.7}$$

where $(x, \tilde{T}_\gamma \phi(x_0)) \in \mathcal{F}_x$ is the parallel displacement of $(x_0, \phi(x_0)) \in \mathcal{F}_{x_0}$ along any curve γ from x_0 to x, induced by D.

Now Eq. (3.7) implies that

$$\phi(x) = \tilde{A}(x)\phi(x_0). \tag{3.8}$$

Let us call $a_0 \in K$ a fixed element of K satisfying $U(a_0) = 0$ on S_∞^{d-1}. Because $U(\phi)$ is G-invariant, ga_0 also satisfies condition (3.4) for every $g \in G$. We will assume that all the zeros of U on S_∞^{d-1} are of this form.[5] Hence, the set of zeros of U can be identified with the left coset space G/H where $H = \{h \in G/ha_0 = a_0\}$, the isotropy group of a_0, is called the unbroken group. Then ϕ can be regarded as a mapping $\phi = S_\infty^{d-1} \to G/H$.

From this and relation (3.8) the mapping ϕ can be identified with the mapping $\mathrm{Pr} \cdot \tilde{A}$ where Pr is the projection

$$\mathrm{Pr}: G \to G/H.$$

The element of $\Pi_{d-1}(G)$ represented by ϕ is then $\mathrm{Pr}*[\tilde{A}]$ where

$$\mathrm{Pr}*: \Pi_{d-1}(G) \to \Pi_{d-1}(G/H) \tag{3.9}$$

is the mapping induced by the projection Pr and $[\tilde{A}]$ is the class of $\tilde{A}$ in $\Pi_{d-1}(G)$.

Now, $\Pi_{d-1}(G)$ is an Abelian group, then $[\tilde{A}]$ is represented by a k-uple of integers $(n_1, \ldots, n_k)$ (the signs and ordering depending on the choice of the generators; note that if $\Pi_{d-1}(G)$ is not free, some of these integers n_i are elements of Z_{p_i}). Then, the equivalence between topological charges of Higgs and gauge fields can be characterized in the following way: *One can choose the generators of $\Pi_{d-1}(G/H)$ in order to have the class of ϕ represented by the same k-uple $(n_1, \ldots, n_k)$ if and only if $\mathrm{Pr}*$ is an isomorphism.*

We wish to stress at this point that the interpretation of Yang—Mills fields in terms of connection in the principal fibre bundle and the Higgs field as a global cross section in the associated fibre bundle leads naturally to a condition for the equivalence between Higgs fields and gauge fields topological charges. [in the Abelian case (with $d = 2$) studied in Sec. II, $H = \{I\}$, and the precedent discussion was not necessary.] From the exact sequence of homotopy

$$\cdots \Pi_{d-1}(H) \xrightarrow{i*} \Pi_{d-1}(G) \xrightarrow{\mathrm{Pr}*} \Pi_{d-1}(G/H) \to \Pi_{d-2}(H) \to \cdots$$

it can be deduced that $\mathrm{Pr}*$ is an isomorphism if

$$\Pi_{d-1}(H) = 0, \tag{3.10a}$$

$$\Pi_{d-2}(H) = 0. \tag{3.10b}$$

For the case $d = 4$, Eq. (3.10b) is always verified since H is compact (H is closed in G), and Eq. (3.10a) reduces to the condition derived by Woo[2]: $\Pi_3(H) = 0$. This is the case when the gauge group is broken (i) entirely, (ii) down to a subgroup of Abelian factors. Then, the topological charges of the gauge and Higgs fields are the same.

We consider lastly the example of the group $G = SO(3)$ and a scalar field forming an isovector ϕ, with the usual potential $U(\phi) = \frac{1}{2}\lambda(|\phi|^2 - 1)^2$, that is, the Gerogi—Glashow model without fermions. The group G can be identified in this case with P^3 (the three-dimensional projective space) and $H = U(1)$.

If $d = 3$, then $\Pi_2(G) = 0$. On the other hand, the Higgs field topological charge is characterized by $\Pi_2(G/H) = Z$. What happens in this example is that, although condition (3.10a) is satisfied, (3.10b) is not, since $\Pi_1(H) = Z$. Then, our analysis does not apply. If $d = 4$, then $\Pi_3(G) = Z$, and $\Pi_3(G/H) = Z$, that is, a single integer labels the Higgs field topological charge. Note that $A: S_\infty^3 \to P^3$ while $\phi: S_\infty^3 \to G/H \equiv S^2$.

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Mechanical response and the initial value problem

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Solutions to canonical mechanical equations are considered as functionals of a source function σ, which parametrizes the source. The coefficients of the functional Taylor series expansions in σ for the solutions are constructed from an infinite sequence of time ordered Poisson brackets, which in turn can be expressed in terms of the derivatives of the solution to the homogeneous equation with respect to its initial canonical momenta. Thus the functional Taylor series expansion is formally defined once the initial value problem for the possibly nonlinear homogeneous equation has been solved. A time ordered exponential operator acting on the homogeneous solution, which may be viewed as a "response" extension of the Lie series operation, simply represents the series expansion.

I. INTRODUCTION

For linear mechanical systems described by second order equations of motion a system's retarded Green's function is intimately connected with the solution to the initial value problem.[1] The Green's function and its derivative specify the coordinates of the system in terms of the coordinates and velocities of the system at some previous time. In fact, in the linear case the retarded Green's function may be viewed as the functional derivative defining the variation produced in a generalized coordinate due to a change in the initial generalized momentum. Here the generalization of this point of view to integrable[2] nonlinear systems is considered.

The generalization is rather straight forward if based upon the notion of Poisson brackets. The "PB" approach also has the advantage of a more obvious quantum analog, and in fact one can recognize a correspondence to the functional techniques employed in quantum field theory. Thus the dynamical arguments of the Poisson brackets considered here belong to different times. The causal requirement that a variation in the canonical variables at one instant must only affect the system variables at a future time leads to the interpretation of the fundamental Poisson bracket as a tensor Green's function defining the first order response of the system to a change in its environment. The definition of the PB then leads to the connection with the initial value problem.

The next section presents the Poisson bracket properties needed to formally construct the functional power series expansion for the inhomogeneous solution to a nonlinear equation of motion. The series is developed for the case of one degree of freedom and shown to be equivalent to a time ordered exponential operator acting on the homogeneous solution, where the operation is defined if the "initial value problem" for the possibly nonlinear homogeneous solution has been solved. Section IV is concerned with the relationship between the derived functional power series expansion and the series obtained by iterating nonlinear "Volterra" or "Born" type integral equations. Section V considers the situation where more than one source term is present

and relates the formalism to the Lie series method.[3] The latter sheds light upon the existence of the functional power series expansion. In the following section the one-dimensional analysis is generalized to a formalism applicable to the case where an arbitrary number of degrees of freedom, possibly an infinite number, are present. Finally two examples are briefly discussed. The Mathieu equation is used to illustrate the "renormalization" technique allowed by the response theory, and the solution to the nonlinear Kepler orbit equation is used to construct an example of a first Green's function approximate resulting from the initial value approach.

II. THE TIME ORDERED POISSON BRACKET

Let f and g denote two dynamical variables, functions of the canonical variables q_κ and p_κ ($\kappa = 1, \ldots, n$). Time is treated only as a parameter. The Poisson bracket is defined in the usual way by

$$[f,g] = \sum_\kappa \left(\frac{\partial f}{\partial q_\kappa} \frac{\partial g}{\partial p_\kappa} - \frac{\partial g}{\partial q_\kappa} \frac{\partial f}{\partial p_\kappa} \right). \tag{1}$$

Since the time development of a system can be viewed as a canonical transformation under which (1) is invariant,[4] the differentiation in (1) can be viewed as defined in terms of some initial canonical variables $q_\kappa(t')$ and $p_\kappa(t')$, where $t' < t$. The functions of f and g can be considered to be defined by canonical variables at later, possibly different, times and the PB can be evaluated in principle if the solution to the equations of motion are known. The canonical variables at any time are assumed to be known functions of the canonical variables at any previous time. This notion will be emphasized with the notation

$$[f_i, g_j]_k = \begin{cases} \sum_\kappa \left(\dfrac{\partial f_i}{\partial q_{\kappa_k}} \dfrac{\partial g_j}{\partial p_{\kappa_k}} - \dfrac{\partial g_j}{\partial q_{\kappa_k}} \dfrac{\partial f_i}{\partial p_{\kappa_k}} \right) \\ 0, \quad t_k > t_i, t_j, \end{cases} \tag{2}$$

where

$$f_i = f_i(t_i, q_{\kappa_i}, p_{B_i}), \quad g_j = g_j(t_j, q_{\kappa_j}, p_{\kappa_j})$$

$$q_{\kappa_i} = q_\kappa(t_i), \quad p_{\kappa_i} = p_\kappa(t_i).$$

The PB vanishes for $t_k > t_i, t_j$ as a consequence of the causality requirement, a variation in the canonical variables at a particular instant can only affect the canonical variables in the future.

The two PB's playing a fundamental role in the discussion below are

$$[q_{\kappa_i}, q_{\beta_j}]_j = \begin{cases} -\dfrac{\partial q_{\kappa_i}}{\partial p_{\beta_j}}, & t_i \geq t_j, \\[2ex] 0, & t_i < t_j, \end{cases} \tag{3}$$

and

$$[p_{\kappa_i}, q_{\beta_j}]_j = \begin{cases} -\dfrac{\partial p_{\kappa_i}}{\partial p_{\beta_j}}, & t_i \geq t_j, \\[2ex] 0, & t_i < t_j. \end{cases} \tag{4}$$

Here we will assume that we have a general Hamiltonian system, so that p and q satisfy the equations

$$\frac{dq_\kappa}{dt} = -[H, p_\kappa], \quad \frac{dq_\kappa}{dt} = -[H, q_\kappa],$$

and will assume that these equations can be placed in the form

$$\frac{dq_\kappa}{dt} = \sum_\beta a_{\kappa\beta}(q)p_\beta + b_\kappa(q), \tag{5}$$

$$\frac{dp_\kappa}{dt} + f_\kappa(q,p) = \sigma(t)S_\kappa(q,p), \tag{6}$$

where $\sigma(t)$ is some function which vanishes if t is less than the time t_0. In the following it will be assumed that $a_{\kappa\beta}$, b_κ, f_κ, and S_κ are holomorphic functions of the canonical variables in the region of phase space in which the functional response series is developed.

In the following it will be useful to note some PB identities needed below. First,

$$\frac{d}{dt}[q_\kappa, q_{\beta_j}]_j = \left[\frac{dq_\kappa}{dt}, q_{\beta_j}\right]_j,$$

$$\frac{d}{dt}[p_\kappa, q_{\beta_j}]_j = \left[\frac{dp_\kappa}{dt}, q_{\beta_j}\right]_j. \tag{7}$$

Secondly one has the Jacobi identity

$$[[f,g],h] + [[g,h],f] + [[h,f],g] = 0, \tag{8}$$

and finally

$$[f, gh] = g[f,h] + [f,g]h,$$
$$[fg, h] = f[g,h] + [f,h]g. \tag{9}$$

It is not difficult to see that these identities apply equally well to PB's of the type (2).

It is helpful to note that (3) and (4) together define a type of retarded tensor Green's function. It is evident from (4) that the fundamental PB (3) has a discontinuity in its first derivative as $t_i \to t_j$ which holds for all canonical systems.

III. NONLINEAR RESPONSE IN ONE DIMENSION

With the equations of motion (5) and (6) the canonical variables may be viewed as functionals of the source function $\sigma(t)$. The inhomogeneous solutions to (5) and (6) will be formally constructed from the homogeneous solutions for $q(t)$ and $p(t)$ (assumed known) by expanding $q(t, \sigma)$ and $p(t, \sigma)$ in functional Taylor series about $q(t)$ and $p(t)$. Thus we look for the expansion coefficients in

$$q(t, \sigma) = q(t) + \int \frac{\delta q(t, \sigma)}{\delta\sigma(t_1)}\sigma(t_1)dt_1 + \frac{1}{2!}\int \frac{\delta^2 q(t, \sigma)}{\delta\sigma(t_1)\delta\sigma(t_2)}$$

$$\times \sigma(t_1)\sigma(t_2)dt_1 dt_2 \tag{10}$$

$$+ \cdots + \frac{1}{n!}\int \frac{\delta^n q}{\delta\sigma_1 \cdots \delta\sigma_n}\sigma_1 \cdots \sigma_n dt_1 \cdots dt_n + \cdots$$

and

$$p(t, \sigma) = p(t) + \int \frac{\delta p(t, \sigma)}{\delta\sigma(t_1)}\sigma(t_1)dt_1 + \frac{1}{2!}\int \frac{\delta^2 p(t, \sigma)}{\delta\sigma(t_1)\delta\sigma(t_2)}$$

$$\times \sigma(t_1)\sigma(t_2)dt_1 dt_2 \tag{11}$$

$$+ \cdots + \frac{1}{n!}\int \frac{\delta^n p}{\delta\sigma_1 \cdots \delta\sigma_n}\sigma_1 \cdots \sigma_n dt_1 \cdots dt_n + \cdots,$$

where

$$\frac{\delta^n q}{\delta\sigma_1 \cdots \delta\sigma_n} = \frac{\delta^n q(t, \sigma)}{\delta\sigma_1 \cdots \delta\sigma_n}\bigg|_{\sigma=0},$$

$$\frac{\delta^n p}{\delta\sigma_1 \cdots \delta\sigma_n} = \frac{\delta^n p(t, \sigma)}{\delta\sigma_1 \cdots \delta\sigma_n}\bigg|_{\sigma=0}.$$

and where for the sake of simplicity the indices on q and p have been suppressed, that is, we first analyze the one-dimensional system.

Direct functional differentiation of (5) leads to

$$\frac{d}{dt}\frac{\delta q(t)}{\delta\sigma(t')} = \frac{\partial a(q)}{\partial q}\frac{\delta q(t)}{\delta\sigma(t')} + \frac{a(q)\delta p(t)}{\delta\sigma(t')} + \frac{\partial b}{\partial q}\frac{\delta q(t)}{\delta\sigma(t')} \tag{12}$$

while differentiation of (6) leads to

$$\frac{d}{dt}\frac{\delta p(t)}{\delta\sigma(t')} + \frac{\partial f}{\partial q}\frac{\delta q(t)}{\delta\sigma(t')} + \frac{\partial f}{\partial p}\frac{\delta p(t)}{\delta\sigma(t')}$$

$$= \delta(t - t')S(q,p) + \sigma(t)\left\{\frac{\partial S}{\partial q}\frac{\delta q(t)}{\delta\sigma(t')} + \frac{\partial S}{\partial p}\frac{\delta p(t)}{\delta\sigma(t')}\right\}. \tag{13}$$

Now we demonstrate formally that

$$\frac{\delta q(t)}{\delta\sigma(t_1)} = -[q(t), q(t_1)]_1 S_1, \tag{14}$$

$$\frac{\delta p(t)}{\delta\sigma(t_1)} = -[p(t), q(t_1)]_1 S_1. \tag{15}$$

From (7) and (5) and (6) one finds that

$$\frac{d}{dt}[q, q_1]_1 = [a(q)p, q_1]_1 + [b(q), q_1]_1,$$

$$\frac{d}{dt}[p, q_1]_1 = -[f(q,p), q_1]_1 + \sigma(t)[S(q,p), q_1]_1.$$

Now from the definition of the PB (2), for any function $h(q, p)$

$$[h(q,p), q_1]_1 = -\frac{\partial h(q,p)}{\partial p_1} = -\left(\frac{\partial h}{\partial q}\frac{\partial q}{\partial p_1} + \frac{\partial h}{\partial p}\frac{\partial p}{\partial p_1}\right)$$

or

$$[h(q,p), q_1]_1 = \frac{\partial h}{\partial q}[q, q_1]_1 + \frac{\partial h}{\partial p}[p, q_1]_1.$$

With this one finds that $[q, q_1]_1$, and $[p, q_1]_1$ satisfy

$$\frac{d}{dt}[q,q_1]_1 = \frac{\partial a}{\partial q}[q,q_1]_1 p + a(q)[p,q_1]_1 + \frac{\partial b}{\partial q}[q,q_1]_1,$$

$$\frac{d}{dt}[p,q_1]_1 = -\frac{\partial f}{\partial q}[q,q_1]_1 - \frac{\partial f}{\partial p}[p,q_1]_1$$
$$+ \sigma(t)\left\{\frac{\partial S}{\partial q}[q,q_1]_1 + \frac{\partial S}{\partial p}[p,q_1]_1\right\}.$$

Thus (14) and (15) satisfy (12) and (13) when $t > t'$, and therefore the functional derivatives are defined up to a numerical factor. This factor is fixed by noting that the first order variation in p must be given by

$$\delta p(t) = -\int_{t_0}^t [p,q_1]_1 \sigma_1 S_1 dt_1,$$

a consequence of (4) and Leibnitz' rule for differentiation of an integral. Thus $[q,q_1]_1$, and $[p,q_1]_1$, must both be multiplied by $-S(q,p)_1$, leading to (14) and (15).

The higher order functional derivatives in (10) and (11) can be expressed in terms of more general Poisson brackets. Thus, for example, (14) leads to

$$\frac{\delta^2 q}{\delta\sigma_2 \delta\sigma_1} = -\left[\frac{\delta q}{\delta\sigma_2},q_1\right]_1 S_1 - \left[q,\frac{\delta q_1}{\delta\sigma_2}\right]_1 S_1 - [q,q_1]_1 \frac{\delta S_1}{\delta\sigma_2} \qquad (16)$$

for $t_2 < t$. Now the causality requirement implies that

$$\frac{\delta^2 q}{\delta\sigma_2 \delta\sigma_1} = (-1)^2 [[q,q_2]_2 S_2, q_1]_1 S_1, \quad t > t_2 > t_1, \qquad (17)$$

which in turn suggests

$$\frac{\delta^2 q}{\delta\sigma_2 \delta\sigma_1} = (-1)^2 [[q,q_1]_1 S_1, q_2]_2 S_2, \quad t > t_1 > t_2. \qquad (18)$$

This proposition is consistent with (16). If $t > t_1 > t_2$, the subscript 1 on the PB in (16) can be replaced by 2; then, using (14) and the fact that

$$\frac{\delta S_1}{\delta\sigma_2} = -\frac{\partial S_1}{\partial q_1}[q_1,q_2]_2 S_2 - \frac{\partial S_1}{\partial p_1}[p_1,q_2]_2 S_2, \qquad (19)$$

or

$$\frac{\delta S_1}{\delta\sigma_2} = -[S_1,q_2]_2 S_2,$$

which follows from the chain rule, one finds that

$$\frac{\delta^2 q}{\delta\sigma_2 \delta\sigma_1} = [[q,q_2]_2 S_2, q_1]_2 S_1 + [q,[q_1,q_2]_2 S_2]_2 S_1$$
$$+ [q,q_1]_1 [S_1, q_2]_2 S_2.$$

The first two terms can be simplified using (9). One obtains

$$\frac{\delta^2 q}{\delta\sigma_2 \delta\sigma_1} = \{[[q,q_2]_2,q_1]_2 + [q,[q_1,q_2]_2]_2\}S_1 S_2$$
$$+ \{[q,q_2]_2[S_2,q_1]_2 + [q_1,q_2]_2[q,S_2]_2\}S_1$$
$$+ [q,q_1]_1[S_1,q_2]_2 S_2.$$

Now adding and subtracting (18) with the subscript 1 on the inner PB replaced by 2, which is possible since $t_2 < t_1$, one finally obtains

$$\frac{\delta^2 q}{\delta\sigma_2 \delta\sigma_1} = \{[[q,q_2]_2,q_1]_2 + [q,[q_1,q_2]_2]_2 - [[q,q_1]_2,q_2]_2\}S_1 S_2$$
$$+ \{[q,q_2]_2[S_2 q,q_1]_2 + [q_1,q_2]_2[q,S_2]_2\}S_1$$

$$+ \{[q,q_1]_2[S_1,q_2]_2 - [q,q_1]_2[S_1,q_2]_2\}S_2$$
$$+ [[q,q_1]_2 S_1, q_2]_2 S_2.$$

The third term in curly brackets obviously vanishes while the first vanishes as a consequence of the Jacobi identity. That the second term in curly brackets vanishes follows from the definition (2), use of the chain rule, and (3) and (4). Thus the proposition (18) is correct, since the subscript 2 on the inner PB in the last term can be replaced by 1.

Note that (17) and (18) can be summarized by

$$\frac{\delta^2 q}{\delta\sigma_2 \delta\sigma_1} = T[[q,q_1]_1 S_1, q_2]_2 S_2, \qquad (20)$$

where T is a time ordering operator. In the same way one can show

$$\frac{\delta^2 p}{\delta\sigma_2 \delta\sigma_1} = T[[p,q_1]_1 S_1, q_2]_2 S_2. \qquad (21)$$

One, of course, can also directly verify that (20) and (21) satisfy the same equations as those for the second functional derivatives of (12) and (13).

In general for equations of the form (5) and (6) one finds[5]

$$\frac{\delta^n(q)}{\delta\sigma_n \cdots \delta\sigma_1} = (-1)^n T[[\cdots[(q),q_n]_n S_n, q_{n-1}]_{n-1} S_{n-1}, \cdots, q_1]_1 S_1. \qquad (22)$$

This expression defines the coefficients for the functional Taylor series expansion relating $q(t,\sigma)$ and $p(t,\sigma)$ to the homhgeneous solution q and p. One obtains the results

$$q(t,\sigma) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} T \int_{t_0}^t dt_1 \cdots dt_n \sigma_1 \cdots \sigma_n$$

$$\times \{[[\cdots[q,q_n]_n S_n, q_{n-1}]_{n-1} S_{n-1}, \cdots, q_1]_1 S_1\}, \qquad (23)$$

$$p(t,\sigma) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} T \int_{t_0}^t dt_1 \cdots dt_n \sigma_1 \cdots \sigma_n$$

$$\times \{[[\cdots[q,q_n]_n S_n, q_{n-1}]_{n-1} S_{n-1}, \cdots, q_1]_1 S_1\}.$$

One can also easily show along the same lines worked out above that for an arbitrary dynamical quantity

$$f(q,p,t,\sigma) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} T \int_{t_0}^t dt_1 \cdots dt_n \sigma_1 \cdots \sigma_n$$

$$\times \{[[\cdots[f,q_n]S_n, q_{n-1}]_{n-1} S_{n-1}, \cdots, q_1]_1 S_1\}. \qquad (24)$$

These expressions have obvious quantum analogs in quantum response theory.

From the classical point of view a much more simple and convenient form is obtained from (2), which implies

$$[q_i, q_{i-1}]_{i-1} S_{i-1} = -S_{i-1} \frac{\partial q_i}{\partial p_{i-1}}.$$

Thus (20) can be written

$$\frac{\delta^2 q}{\delta\sigma_2 \delta\sigma_1} = S_2 \frac{\partial}{\partial p_2}\left[S_1 \frac{\partial q}{\partial p_1}\right], \quad t_1 > t_2.$$

Note that here q must be first expressed in terms of "initial coordinates" q_1 and p_1 so that the first differentiation can be carried out, and then the resulting function of q_1 and p_1 must be expressed as a function of q_2 and p_2 so that the second differentiation can be carried out. Then one finds that (24) can be written as

$$q(t, \sigma) = T^{-1} \exp \int_{t_0}^{t} dz\, \sigma(z) S(q(z), p(z)) \frac{\partial}{\partial p(z)} q(t). \tag{25}$$

Thus the inhomogeneous solution is expressed as the result of a simple time ordered exponential operator acting on the homogeneous solution and has meaning if the original homogeneous solution can be expressed in terms of initial canonical variables. The existence of such solutions is discussed in Sec. V.

In a similar way one can show for an arbitrary dynamical variable $f(q, p, t)$ that

$$f(q, p, t, \sigma) = T^{-1} \exp \int_{t_0}^{t} dz\, \sigma(z) S(q(z), p(z)) \frac{\partial}{\partial p(z)} t(q, p, t). \tag{26}$$

Thus the initial value problem plays a fundamental role in classical response theory.

IV. COMPARISON WITH THE BORN SERIES

Let (5) and (6) be equivalent to the equation

$$m \frac{d^2 q}{dt^2} + d_0 \frac{dq}{dt} + f_0 q = \sigma(t) S(q). \tag{27}$$

This equation can be converted into the integral equation

$$q(t, \sigma) = \int_{t_0}^{t} G(t, t_1) \sigma(t_1) S(q(t_1, \sigma)) dt_1 + q(t) \tag{28}$$

where the Green's function satisfies

$$\frac{d^2}{dt^2} G(t, t_1) + d_0 \frac{d}{dt} G(t, t_1) + f_0 G(t, t_1) = \delta(t - t_1).$$

If this equation is iterated about the homogeneous solution, then one arrives at a nonlinear "Born" series. Note that iteration is equivalent to expanding the function $S(q(t, \sigma))$ for each t in a power series expansion in

$$\delta q(t, \sigma) = q(t, \sigma) - q(t)$$

and then replacing $q(t, \sigma)$ in $\delta q(t, \sigma)$ with the integral equation and expanding $S(q(t, \sigma))$ again, ad inf. To third order in $\sigma(t)$ one obtains

$$q(t, \sigma) = q(t) + \int_{t_0}^{t} dt_1\, \sigma_1 S_1 G(t, t_1) + \int_{t_0}^{t} \int_{t_0}^{t_1} dt_1 dt_2 \sigma_1 \sigma_2 \frac{\partial S_1}{\partial q_1}$$

$$\times S_2 G_{12} G(t, t_1) + \int_{t_0}^{t} \int_{t_0}^{t_1} \int_{t_0}^{t_2} dt_1 dt_2 dt_3 \sigma_1 \sigma_2 \sigma_3 \frac{\partial S_1}{\partial q_1} \frac{\partial S_2}{\partial q_2}$$

$$\times S_3 G_{12} G_{23} G(t, t_1) + \frac{1}{2} \int_{t_0}^{t} dt_1 \sigma_1 \frac{\partial^2 S_1}{\partial q_1^2} G(t, t_1)$$

$$\times [\int_{t_0}^{t_1} dt_2 \sigma_2 S_2 G_{12}]^2 + \cdots. \tag{29}$$

On the other hand (25) can be expressed in the form

$$q(t, \sigma) = q(t) + \int_{t_0}^{t} dt_1\, \sigma_1 S_1 \frac{\partial q}{\partial p_1}$$

$$+ \int_{t_0}^{t} \int_{t_0}^{t_1} dt_1 dt_2 \sigma_1 \sigma_2 S_2 \left\{ \frac{\partial S_1}{\partial q_1} \frac{\partial q_1}{\partial p_2} \frac{\partial q}{\partial p_1} \right.$$

$$\left. + S_1 \frac{\partial^2 q}{\partial p_2 \partial p_1} \right\} + \cdots.$$

Now the solution of a linear differential equation must be a linear function of its initial conditions. This means that all partial derivatives of $q(t)$ with respect to initial momenta vanish if of higher order than first. As a result one finds that

$$q(t, \sigma) = q(t) + \int_{t_1}^{t} dt_1\, \sigma_1 S_1 \frac{\partial q}{\partial p_1} + \int_{t_0}^{t} \int_{t_0}^{t} dt_1 dt_2 \sigma_1 \sigma_2 S_2$$

$$\times \frac{\partial S_1}{\partial q_1} \frac{\partial q_1}{\partial p_2} \frac{\partial q}{\partial p_1} + \int_{t_0}^{t} \int_{t_0}^{t_1} \int_{t_0}^{t_2} dt_1 dt_2 dt_3 \sigma_1 \sigma_2 \sigma_3 S_3$$

$$\times \frac{\partial S_2}{\partial q_2} \frac{\partial q_2}{\partial p_3} \frac{\partial S_1}{\partial q_1} \frac{\partial q_1}{\partial p_2} \frac{\partial q}{\partial p_1} + \int_{t_0}^{t} \int_{t_0}^{t_1} \int_{t_0}^{t_2} dt_1 dt_2 dt_3 \sigma_1 \sigma_2 \sigma_3$$

$$\times S_3 S_2 \frac{\partial^2 S_1}{\partial q_1^2} \frac{\partial q_1}{\partial p_3} \frac{\partial q_1}{\partial p_2} \frac{\partial q}{\partial p_1} + \cdots,$$

where in deriving the above use has been made of the chain rule

$$\frac{\partial h(q_1)}{\partial p_2} = \frac{\partial h_1}{\partial q_1} \frac{\partial q_1}{\partial p_2}.$$

Identifying now

$$G_{ij} = \frac{\partial q_i}{\partial p_j},$$

one finds

$$q(t, \sigma) = q(t) + \int_{t_0}^{t} dt_1 \sigma_1 S_1 G(t, t_1) + \int_{t_0}^{t} \int_{t_0}^{t_1} dt_1 dt_2 \sigma_1 \sigma_2 \frac{\partial S_1}{\partial q_1}$$

$$\times G_{12} G(t, t_1) + \int_{t_0}^{t} \int_{t_0}^{t_1} \int_{t_0}^{t_2} dt_1 dt_2 dt_3 \sigma_1 \sigma_2 \sigma_3$$

$$\times \frac{\partial S_2}{\partial q_2} G_{23} \frac{\partial S_1}{\partial q_1} G_{12} S_3 G(t, t_1) + \int_{t_0}^{t} \int_{t_0}^{t_1} \int_{t_0}^{t_2}$$

$$\times dt_1 dt_2 dt_3\, \sigma_1 \sigma_2 \sigma_3 S_2 S_3 \frac{\partial^2 S}{\partial q_1^2} G_{13} G_{12} G(t, t_1)$$

$$+ \cdots.$$

Since in the last term on the rhs above one can write

$$\int_{t_0}^{t_1} \int_{t_0}^{t_2} dt_2 dt_3 \sigma_2 \sigma_3 S_3 S_2 G_{12} G_{13}$$

$$= \frac{1}{2} \{ \int_{t_0}^{t_1} dt_2\, G_{12} S_2 \sigma_2 \int_{t_0}^{t_2} dt_3 G_{13} S_3 \sigma_3$$

$$+ \int_{t_0}^{t_1} dt_3 G_{13} S_3 \sigma_3 \int_{t_0}^{t_3} dt_2 \sigma_2 S_2 G_{12} \}$$

$$= \frac{1}{2} [\int_{t_0}^{t} dt_2 \sigma_2 S_2 G_{12}]^2,$$

it should be evident that to third order (25) agrees with (29). Since the general proof that (25) gives the "non-

linear source" Born series for (27) is straightforward but tedious, we omit the complete proof for the sake of brevity. In the case of a linear homogeneous differential equation, one expects that the Born series obtained by iterating (28) is equivalent to the functional power series expression (25), or conversely (25) is equivalent to the Born series. For more general situations, where the homogeneous solution to (27) is that to a nonlinear differential equation, the function power series expansion goes beyond the Born series, since it is generally then not possible to write (28), that is, the Green's function G is unknown.

V. THE MULTIPLE SOURCE EXPANSION AND THE LIE SERIES

It is convenient also to be able to consider equations of motion having the form

$$m\frac{d^2}{dt^2}q + d(q)\frac{dq}{dt} + f(q) = \sigma(t)S(q) + \tau(t)T(q). \tag{30}$$

Using the techniques of Sec. II one determines readily that

$$\frac{\delta q}{\delta \sigma_i} = -[q,q_i]_i S_i, \quad \frac{\delta q}{\delta \tau_i} = -[q,q_i]_i T_i,$$

while

$$\frac{\delta^2 q}{\delta \sigma_i \delta \sigma_j} = T[[q,q_j]_j S_j, q_i]_i S_i, \quad \frac{\delta^2 q}{\delta \tau_i \delta \tau_j}$$

$$= T[[q,q_j]_j T_j, q_i]_i T_i,$$

and

$$\frac{\delta^2 q}{\delta \sigma_i \delta \tau_j} = T[[q,q_j]_j T_j, q_i]_i S_i.$$

In the latter case one should note that a change in the time ordering implies a change in the location of the functions as well as their arguments. For example,

$$\frac{\delta^2 q}{\delta \sigma_i \delta \tau_j} = \begin{cases} [[q,q_i]_i S_i, q_j]_j T_j, & t > t_i > t_j, \\ [[q,q_j]_j T_j, q_i]_i S_i, & t > t_j > t_i. \end{cases}$$

One obtains

$$q(t,\sigma,\tau) = T^{-1}\exp\left\{\int_{t_0}^t dz\, (\sigma(z)S(q(z)) + \tau(z)T(q(z))) \times \frac{\partial}{\partial p(z)}\right\} q(t). \tag{31}$$

This form is useful when it proves necessary to renormalize one of the parameters of the homogeneous equation, an application which will be illustrated below.

A more general case arises when the canonical equations of motion have the form

$$\frac{dq_\kappa}{dt} - \sum_\beta a_{\kappa\beta} p_\beta + b_\kappa(q) = \sigma(t)Q_\kappa(q,p), \tag{32}$$

$$\frac{dp_\kappa}{dt} + f_\kappa(q,p) = \sigma(t)S_\kappa(q,p).$$

It is not difficult to demonstrate that the response is given by

$$\frac{\delta q}{\delta \sigma_1} = [q,q_1]_1 S_1 + [q,p_1]_1 Q_1,$$

$$\frac{\delta p}{\delta \sigma_1} = [p,q_1]_1 S_1 + [p,p_1]_1 Q_1,$$

and

$$\frac{\delta f}{\delta \sigma_1} = [f,q_1]_1 S_1 + [f,p_1]_1 Q_1.$$

The time ordered exponential response operation (26) now becomes

$$f(q,p,t,\sigma) = T^{-1}\exp\left\{\int_{t_0}^t dt_1 \sigma_1(t_1)\left(S_1\frac{\partial}{\partial p_1} + Q_1\frac{\partial}{\partial q_1}\right)\right\} \times f(q,p,t), \tag{33}$$

where it is again assumed that the IVP for the homogeneous solution has been solved, that is, $f(q,p,t)$ is expressed in terms of the "initial values" q and p of the homogeneous solution.

The present form of the response is closely related to the Lie-series transformation.[3] Given a system of differential equations of the form

$$\frac{dZ_i}{dt} = \theta_i(Z), \quad i = 1,\ldots,n,$$

with the initial conditions $Z_i(0) = z_i$, the solutions $Z_i(t)$ are constructed from the initial values z_i via the Lie-series transformation

$$Z_i = e^{tD} z_i, \tag{34}$$

where

$$D = \sum_i \theta_i \frac{\partial}{\partial z_i}.$$

Thus we see that the response (33) is constructed from the Lie-series operators $\theta_i \partial/\partial z_i$ belonging to the source alone. The inhomogeneous Lie-series operators, acting on the homogeneous solution, generate the Green's function approximates for the functional power series expansion of the inhomogeneous solution.

The Lie theory insures that if the $\theta_i(z)$ in (34) are holomorphic functions over a closed bounded region of the space of the initial values z_i, then the Lie series (34) is absolutely and uniformly convergent and represents a holomorphic function in the $n+1$ variables z_i, t for $|t| \leq T$.[3] Thus the requirement that the nonderivative terms in (5) and (6) [or more generally (32)] are holomorphic functions insures both the existence of a solution to the IVP for the homogeneous solution and the existence of the terms in the functional power series, represented by (23) or (26), up to the time T. The convergence of the functional response series is, however, a question which at present must be resolved separately for each situation.

VI. MORE THAN ONE DEGREE OF FREEDOM

The generalization of (25) and (26) [or (23) and (24)] to systems having more than one degree of freedom is quite straightforward. Starting with the fundamental PB's (3) and (4) one finds with (5) and (6) that (14) and (15) become

$$\frac{\delta q_{\kappa}}{\delta \sigma_1} = -\sum_{\beta_1}[q_{\kappa}, q_{\beta_1}]_1 S_{\beta_1},$$

$$\frac{\delta p_{\kappa}}{\delta \sigma_1} = -\sum_{\beta_1}[p_{\kappa}, q_{\beta_1}]_1 S_{\beta_1}, \tag{35}$$

where again we use the notion introduced in (2). The functional derivatives of arbitrary order are given by

$$\frac{\delta^n q_{\kappa}}{\delta \sigma_n \circ \circ \delta \sigma_1} = (-1)^n \sum_{\beta_1 \cdots \beta_n} T[[\circ \circ \circ [p_{\kappa}, q_{\beta_n}]_n S_{\beta_n}, \circ \circ \circ]_2$$

$$\times S_{\beta_2}, q_{\beta_1}]_1 S_{\beta_1}, \tag{36}$$

$$\frac{\delta^n p_{\kappa}}{\delta \sigma_n \circ \circ \delta \sigma_1} = (-1)^n \sum_{\beta_1 \circ \circ \circ \beta_n} T[[\circ \circ \bullet [p_{\kappa}, q_{\beta_n}]_n S_{\beta_n}, \circ \circ \circ]_2$$

$$\times S_{\beta_2}, q_{\beta_1}]_1 S_{\beta_1}.$$

Then with the use of (3) and (4) it is not difficult to show that

$$q_{\kappa}(t, \sigma) = T^{-1} \exp\left\{\int_{t_0}^t dz\, \sigma(z) S(q(z), p(z))_\beta \frac{\partial}{\partial p_\beta(z)}\right\} q_{\kappa}(t),$$

$$p_{\kappa}(t, \sigma) = T^{-1} \exp\left\{\int_{t_0}^t dz\, \sigma(z) S(q(z), p(z))_\beta \frac{\partial}{\partial p_\beta(z)}\right\} p_{\kappa}(t). \tag{37}$$

Similarly one finds that for any dynamical variable

$$f(q, p, t, \sigma) = T^{-1} \exp\left\{\int_{t_0}^t dz\, \sigma(z) S(q(z), p(z))_\beta \frac{\partial}{\partial p_\beta(z)}\right\}$$

$$\times f(q, p, t). \tag{38}$$

It should be pointed out that the case where the canonical variables are not real is also provided for by (37) and (38) and (35) and (36). However, one must consider then that the indices κ and β include in their range the indices for the complex conjugate variables. That is, if there are n canonical coordinates q and q^*, then κ, $\beta = 1, \ldots, 2n$ and then (37) can be applied directly.

The transformations (37) and (38) lead easily to those for systems having an infinite number of degrees of freedom. The sum over the discrete index κ becomes an integral over spatial coordinates. For the sake of simplicity we consider a real scalar field, whose Hamiltonian is given by

$$H = \int d^3x \{\tfrac{1}{2}\pi^2(\mathbf{x}, x^0) + (\nabla \varphi(\mathbf{x}, x^0))^2$$

$$+ m^2\varphi^2 - \sigma(x)\, V(x, \varphi)\},$$

which leads to the equation of motion

$$g^{\mu\nu}\partial^2_{\mu\nu}\varphi + m^2\varphi = \sigma S(x, \varphi), \quad S(x, \varphi) = \frac{\partial V}{\partial \varphi}.$$

Here the generalizations of (3) and (4) become

$$[\varphi(\mathbf{x}, x^0), \varphi(\mathbf{y}, y^0)]_{y0} = -\frac{\delta \varphi(\mathbf{x}, x^0)}{\delta \pi(\mathbf{y}, y^0)},$$

$$[\partial_0 \varphi(\mathbf{x}, x^0), \varphi(\mathbf{y}, y^0)]_{y0} = -\frac{\delta \pi(\mathbf{x}, x^0)}{\delta \pi(\mathbf{y}, y^0)}.$$

Finally (37) becomes

$$\varphi(x, \sigma) = T^{-1} \exp\left\{\int dy\, \sigma(y) S(y, \varphi(y)) \frac{\delta}{\delta \pi(y)}\right\} \varphi(x),$$

$$\pi(x, \sigma) = T^{-1} \exp\left\{\left(dy\, \sigma(y) S(y, \varphi(y))\right) \frac{\delta}{\delta \pi(y)}\right\} \pi(x), \tag{39}$$

where the integration here is over the four-dimensional volume contained between the two constant time surfaces at x^0 and y^0.

VII. TWO EXAMPLES

It was stated in Sec. V that the multiple source expansion can be useful in situations where it is necessary to "renormalize" coefficients appearing in the homogeneous equation. The Mathieu equation

$$\frac{d^2 q}{dt^2} + aq = 2b \cos 2tq \tag{40}$$

is a useful example (albeit a linear one) for illustrating this process.

Let the desired solutions to (40) be those of period $n\pi$. The homogeneous solution to (40), defined in terms of the initial canonical variables, is given by

$$q(t) = q(z)\cos\sqrt{a}\,(t - z) + (p(z)/\sqrt{a})\sin\sqrt{a}\,(t - z),$$

$$\sqrt{a} = n\pi.$$

Direct application of (25) leads to the well-known secular difficulties. Terms appear, for example, in the resultant series proportional to $t^m \cos n\pi t$ and $t^m \sin n\pi t$. However, this difficulty can be removed by choosing

$$\left.\begin{array}{c} \sigma(t) = \theta(t - t_0)2b\cos 2t, \quad S(q) \\ \tau(t) = -\theta(t - t_0)\delta_a, \quad T(q) \end{array}\right\} = q \tag{41}$$

carrying out the expansion (14), and then choosing the expansion coefficients in

$$\delta_a = \sum_{n=1} 2nb^n \tag{42}$$

in such a way that the secular terms are removed. That is, δa is chosen so that the solution has a certain periodic property. One is essentially taking into account the fact that the eigenvalue a is a functional of the source. This procedure has been found to lead to the standard solutions to the Mathieu equation of period $n\pi$, as well as to the correct eigenvalues

$$e = a + \delta a. \text{[6]}$$

Thus besides providing for the situation where the source terms involve factors which are distinct functions of q and time, the double source expansion (31) permits in this case a solution of the "scalar" problem.

Note that according to the present example the frequencies of the fourier series expansions of the inhomogeneous solutions to (40) are predetermined by the frequencies of the homogeneous solution and the potential. This is in contrast to more historical nonlinear perturbative approaches, a detailed review of which is given in Giacaglia.[7] One might also contrast this technique with a recently developed anharmonic approximation method.[8] It is hoped that the present renormalization procedure can be extended to anharmonic systems.

The complex periodic Mathieu function solutions to (40) can be constructed in a similar way using the two-dimensional formalism. The complex solutions are associated with the Hamiltonian

$$H = \frac{pp^*}{m} + \omega^2 qq^* - 2bqq^*\cos 2t.$$

The appropriate solution to the initial value problem for the homogeneous equation is given by

$$q(t) = \tfrac{1}{2}\{q(t') + p^*(t')/im\omega\}e^{i\omega(t-t')} + \tfrac{1}{2}\{q(t') - p^*(t')/im\omega\}$$
$$\times e^{-i\omega(t-t')},$$

and this is to be used in (37), where here one sums over the momenta $p(z)$ and $p^*(z)$. It is interesting to note that the Floquet solutions to (40),

$$F(t) = e^{i\omega t}G(z), \quad G(z \pm \pi) = G(z),$$

for $\omega \neq n\pi$ are derived from (41) using (37) suitably modified as a double source expansion and choosing the initial $q(t_0)$ and $p(t_0)$ and the coefficients in (42) such that ω remains unchanged by the modified form of (37).

The construction of the first order Greens function approximate [the PB on the right-hand side of (14)] for a nonlinear system is conveniently illustrated with the well-known Kepler problem. The Lagrangian for the Kepler problem has the form

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \alpha/r.$$

The conserved angular momentum

$$l = mr^2\dot{\theta}$$

and the Euler—Lagrange equations of motion combine to lead to the nonlinear orbit equation

$$\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r - \frac{\alpha m}{l^2}r^2 = 0. \tag{43}$$

The latter is easily transformed to the canonical form of (5) and (6) through the use of the "Hamiltonian"

$$H = \frac{mr^4}{2l^2}p^2 + \frac{l^2}{2mr^2} + \frac{\alpha}{r}. \tag{44}$$

"Hamiltonians" equations of motion become

$$\frac{dp}{d\theta} = -\frac{\partial mr^3}{l^2}p^2 + \frac{l^2}{mr^3} + \frac{\alpha}{r^2},$$
$$\frac{dr}{d\theta} = \frac{mr^4}{l^2}p,$$

which are equivalent to (43). The initial value form of the solution is easily discerned from the orbit equation given in standard textbooks on mechanics[9]

$$r(\theta - \theta_0, r_0, p_0) = l\Big/\Big[-\frac{m\alpha}{l} + \Big(\frac{l}{r_0} + \frac{m\alpha}{l}\Big)\cos(\theta - \theta_0)$$
$$- \frac{mr_0^2}{l}p_0\sin(\theta - \theta_0)\Big]. \tag{45}$$

The Green's function approximate defining the first order response of (43),

$$G(\theta - \theta_0; r_0, p_0) = \frac{\partial r}{\partial p_0},$$

is readily calculated. One finds

$$G(\theta - \theta_0), r_0, p_0) = \frac{mr_0^2}{l^2}r^2(\theta - \theta_0; r_0, p_0)\sin(\theta - \theta_0).$$

From this result and (45) it is a straightforward matter to construct higher order terms in the functional power series expansion. Application of these results will be discussed elsewhere.

VIII. DISCUSSION

The response of mechanical systems has been studied by viewing the solution to an inhomogeneous equation of motion as a functional of a source function σ. The expansion coefficients in the functional Taylor series in σ were then identified with certain time ordered Poisson

brackets, which in turn could be evaluated if the initial value problem for the homogeneous system could be solved. Actually at this point one must be careful, since the PB's (20)—(22) are to be evaluated technically before the limit $\sigma \to 0$ is taken, while in writing down (37) and (38) one has implicitly taken the limit $\sigma \to 0$ before evaluating the PB. Thus the application of this initial value approach to mechanical response depends presently on the analytical behavior of the complete solution as $\sigma \to 0$, that is, on the ability to differentiate the solution with respect to its initial canonical momenta after the limit $\sigma \to 0$. However, one expects the class of integrable solutions to which this approach should be applicable to be quite large.

Since the coefficients in the functional Taylor series expansions (10) and (11) in σ can be viewed as Green's functions approximates,[10] it is clear that the solution to the initial value problem can lead to these Green's function approximates even when the homogeneous mechanical equations are non-linear in the dependent variables. The promise of obtaining Green's function approximates for integrable[2] nonlinear systems should lend considerable interest to this initial value approach. Also, while the development here has been purely classical, the connection between the initial value problem and the Poisson brackets outlined above should have applications to the quantum response of nonlinear systems, and in fact to the quantization of nonlinear systems in general. Up to the operator ordering problem one should be able to study how the correspondence between Poisson brackets and quantum commutators occurs. Applications of the above techniques to nonlinear oscillatory systems having exact closed form solutions,[11,12] and to systems involving collective coordinates,[13] are currently under investigation.

[1]J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), Chap. 6, p. 186.

[2]There are several distinct uses of the word integrable in the literature (for detailed discussion see Ref. 7 below). Here integrable means that the solution to the equation of motion should be a locally halomorphic function of the initial conditions.

[3]W. Gröbner, *Die Lie-Reihen und ihre Auwendungen* (Veb Deutscher Verlag der Wissenschaften, 1967), Zwerter Kapitel.

[4]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading Mass.) Chap. 8, p. 259.

[5]This result is more simply obtained via the causal requirements as used above and the requirement that the result should not be affected by the order of the functional differentiation.

[6]M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, AMS 55 (National Bureau of Standards, Washington, D.C., 1964), and references therein.

[7]G.E.O. Giacaglia, *Perturbative Methods in Non-linear Systems*, Appl. Math. Sci. 8 (Springer-Verlag, Berlin, 1972).

[8]C.R. Eminhizer, R.H.G. Helleman, and E.W. Montroll, J. Math. Phys. **17**, 212 (1976).

[9]L.D. Landau and E. Lifshitz, *Mechanics* (Addison-Wesley, Reading, Mass., 1960), Chap. III.

[10]The term Green's function approximate refers here to $\partial^n q/\partial p_1 \cdots \partial_n$.

[11]M. Lakshmanan, Lett. Nuovo Cimento **8**, 743 (1973).

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The energy eigenvalue equation for the linear potential

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Recently the solution of the linear potential problem for any value of the orbital quantum number l was given via a new class of combinatorics functions. This paper deals with the eigenvalue problem and presents a further reduction of the energy eigenvalue equation leading to a general method for obtaining numerical results. Originally the eigenvalue equation was obtained by requiring the vanishing of an infinite order polynomial, $H_l(t) = \sum_{m=0}^{\infty} 2(l+1)K_m(l)t^{m-1}$, that should reduce to the Airy function, $\mathrm{Ai}(-t)$, in the case $l = 0$. The expansion coefficients $K_m(l)$ were in turn given as the limit for infinitely large integer k values of some other coefficients $K_m^{(k)}(l)$. These latter coefficients were given explicitly in terms of combinatorics functions, $\sigma_l(k,p)$, that individually diverge in the limit as k goes to infinity. No real attempt was made to show that $H_0(t) \equiv \mathrm{Ai}(-t)$, to calculate the limit as $k \to \infty$, or even prove the cancellation of infinities. This work has three objectives: (i) Show the complete equivalence between the combinatorics function solution and the Airy function at $l = 0$; (ii) Obtain a closed form expression for the expansion coefficients $2(l+1)K_p(l)$ by relating them in a very simple manner to the finite part $f_p(l)$ of $\sigma_l(k,p)$ in the limit as $k \to \infty$; (iii) Calculate a few lower order coefficients $f_p(l)$ and obtain some numerical results.

I. INTRODUCTION

While the Coulomb and harmonic potential problems are exactly solvable, the linear potential problem has so far escaped a solution in terms of the tabulated special functions, since no transformation on the associated radial Schrödinger equation is possible to bring it into the hypergeometric form. The only other alternative was to start from the radial equation whose series solution leads to a three-term recursion relation with nonconstant coefficients. This motivated the development of a general formalism solving linear homogeneous recursion relations,[1] extended later by the author to include the solutions of the inhomogeneous equations.[2]

The major breakthrough in obtaining the solutions of linear finite-difference equations was to relate this problem to the partitions of an interval into parts of restricted lengths.[1] New classes of combinatorics functions were then introduced and called combinatorics functions of the first and second kind that are representations of partitions of an interval into parts. Combinatorics are "constrained" when the corresponding partitions have their first part constrained to be greater than some given length, and "unconstrained" or "special" when no constraint at all is present.

A solution of the linear potential was consequently obtained in terms of the above mentioned combinatorics functions.[3] In particular, the eigenvalue equation was derived by relating the boundary condition on the wavefunction at the origin to its boundary condition at infinity.

The binary system of reduced mass μ and total mass M is interacting through a central linear potential

$$V(r) = V_0 + Kr. \tag{1.1}$$

Upon introducing dimensionless energy and distance parameters, t and x,

$$t = (2\mu/K^2)^{1/3}(E - M - V_0), \quad x = (2\mu K)^{1/3}r, \tag{1.2}$$

the regular solution of the radial Schrödinger equation can then be expanded as

$$u_l(x,t) = x^{l+1} \sum_m b_m(l,t)x^m. \tag{1.3}$$

The expansion coefficients satisfy the recursion relation

$$m(m + 2l + 1)b_m = -tb_{m-2} + b_{m-3} \tag{1.4a}$$

with the initial conditions imposed by the regularity of the wavefunction at the origin being

$$b_{-m} = 0 \quad \text{for } m > 0. \tag{1.4b}$$

The immediate consequence of Eq. (1.4b) is

$$2(l + 1)b_1(l,t)/(-t) = b_{-1} = 0 \tag{1.5}$$

Since the solution of the radial equation for $l = 0$ was known to be given by the Airy function $\mathrm{Ai}(x - t)$ leading to the energy eigenvalue equation[4]

$$\mathrm{Ai}(-t) = 0, \quad \text{for } l = 0, \tag{1.6}$$

the author realized that Eq. (1.6) could be consistently derived directly from Eq. (1.5) where l is set to be zero.[5] This lead him to believe that the eigenvalue equation is essentially given by Eq. (1.5) for all values of the orbital quantum number l.

The crucial steps behind the derivation of an expression for b_1 as a function of l and t are the following[3]:

(i) Add to Eq. (1.5) the boundary condition at infinity expressed as

$$u_l(x,t) \underset{x \to \infty}{\sim} \mathrm{Ai}(x - t) \quad \text{for any } l. \tag{1.7a}$$

(ii) The comparison between the series expansion for $u_l(x,t)$, Eq. (1.3), and the series for the Airy function

$$\mathrm{Ai}(x - t) = \sum_{n=0}^{\infty} a_n(x - t)^n \tag{1.7b}$$

leads to the asymptotic result

$$b_N(l,t) \underset{N \to \infty}{\longrightarrow} \sum_{i=0}^{\infty} (-t)^i \frac{(N+l+i+1)!}{i!(N+l+1)!} a_{N+l+i+1}, \tag{1.7c}$$

where the a_n are the well known Airy expansion coefficients.[3,6]

(iii) Equation (1.7c) provides an explicit expression for $b_N(l,t)$ in the limit as N becomes infinitely large.

Thus the problem of obtaining b_1 as an explicit function of l and t is solved if one is able to relate b_1 to higher order coefficients $b_N(l, t)$ then take the limit as N goes to infinity. This is essentially what is meant by relating the boundary condition at the origin to the boundary condition at infinity.

(iv) In order to relate b_1 to higher order coefficients what was needed is the inversion of the recursion relation, Eq. (1.4a). A straightforward generalization of the work done in Ref. 1 enabled Antippa[7] to obtain the solution of inverted homogeneous recursion relations. The combinatorics of Ref. 1 are based on the ordering of the parts in a partition from the lower end to the upper end of the corresponding interval. The combinatorics that emerged in the solution of the inverted recursion relations were called "conjugate," because they are associated with partitions whose ordering have been reversed.[8,9]

For the details of the calculations leading to the actual eigenvalue equation we refer the reader to Ref. 3. This equation, (1.5), turns out to be equivalent to the vanishing of an infinite order polynomial in the energy parameter t, as postulated in Ref. 5. This is also in agreement with the limiting case $l = 0$, Eq. (1.6), since the Airy function $Ai(-t)$ has an infinite series representation. The major result of Ref. 3 is essentially the eigenvalue equation obtained as[3]

$$H_l(t) = \sum_{n=0}^{\infty} 2(l+1)K_n(l)t^{n-1} = 0, \tag{1.8}$$

where the expansion coefficients $K_n(l)$, as may be guessed from step (iii) mentioned above, are given as the limit for infinitely large integer k values of coefficients $K_n^{(k)}(l)$,[3,10]

$$K_n(l) = \lim_{k \to \infty} K_n^{(k)}(l), \tag{1.9}$$

expressed in terms of conjugate combinatorics functions. A major step was done in Ref. 3 by performing a reduction of these conjugate combinatorics functions. This was possible after introducing new quantities, $\sigma_l(n, p)$, defined as[11]

$$\sigma_l(n, p) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \sum_{i_p=1}^{i_{p-1}} \left[\prod_{k=1}^{p} \gamma_l(i_k, k, p) \right], \quad p = 1, 2, \cdots, \tag{1.10a}$$

with

$$\gamma_l(i, k, p) = \frac{\Gamma(i + (p+1-k)/3)\Gamma(i + (p+2l+2-k)/3)}{\Gamma(i + (p+2-k)/3)\Gamma(i + (p+2l+3-k)/3)} \tag{1.10b}$$

and the normalization condition

$$\sigma_l(n, 0) = 1. \tag{1.10c}$$

This lead to a rather simplified version of $K_n^{(k)}(l)$, namely

$$K_{3r+i}^{(k)}(3l' + j)$$
$$= (1 - \delta_{i0}\delta_{j0} - \delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1})(1 - 2\delta_{i2}\delta_{j2})$$

$$\times \frac{(-1)^j}{3^{2(l+j-1)/3}\Gamma(\frac{1}{3})} \frac{\Gamma(\frac{2}{3})}{3^{2(r+l')}} \frac{(-1)^r a_0}{\Gamma(2l' + (2j+5)/3)}$$

$$\times \sum_{q=0}^{r} \left\{ \frac{(-1)^q 3^{3r+3q+i}\Gamma(k+r-q+l'+1+(i+j)/3)}{(3r-3q+i)!\,\Gamma(k+l'+(j+5)/3)} \right.$$

$$\times \left[\frac{\Gamma(k+\frac{4}{3})\Gamma(k+2l'+(2j+5)/3)}{\Gamma(k+l'+(j+3)/3)\Gamma(k+l'+(j+4)/3)} \sigma_{3l'+j}(k-q+1, 3q) \right.$$

$$- (r - q + i/3)\frac{\Gamma(k+\frac{5}{3})\Gamma(k+2l'+(2j+6)/3)}{\Gamma(k+l'+(j+4)/3)\Gamma(k+l'+(j+6)/3)}$$

$$\times \sigma_{3l'+j}(k-q, \ 3q+1)$$

$$+ (r - q + i/3)[r - q + (i-1)/3]$$

$$\times \frac{\Gamma(k+2)\Gamma(k+2l'+(2j+7)/3)}{\Gamma(k+l'+(j+6)/3)\Gamma(k+l'+(j+7)/3)}$$

$$\left. \times \sigma_{3l'+j}(k-q, \ 3q+2)\right] \right\}$$

for $i, j = 0, 1, 2$, and $r, l' = 0, 1, \ldots, \infty$. (1.11)

Also in Ref. 3, we have been able to calculate a closed form expression for $K_0(l)$, and a few higher order coefficients for $l = 0$. As a matter of fact, beyond remarking the property[3]

$$\sigma_l(n, p) = \sum_{i=1}^{n} \gamma_l(i, 1, p)\sigma_l(i, p - 1), \tag{1.12}$$

the authors of Ref. 3 were only able to show how closed form expressions for $\sigma_0(n, p)$ could be obtained by combining Eq. (1.12) and the relation[12]

$$\sum_{j=1}^{n} \frac{\Gamma(j+a)}{\Gamma(j+b)} = \frac{1}{a-b+1}\left[\frac{\Gamma(n+a+1)}{\Gamma(n+b)} - \frac{\Gamma(a+1)}{\Gamma(b)}\right]. \tag{1.13}$$

Obviously, the limit as $k \to \infty$ of Eq. (1.11) can be calculated only after evaluating the combinatorics functions $\sigma_l(n, p)$. In Sec. II we give a complete study of the case $l = 0$, based on a direct evaluation of $\sigma_0(n, p)$ for any p. We then show in Sec. III the cancellation of infinities for arbitrary values of l. In Sec. IV we derive a closed form expression for the expansion coefficient $2(l+1)K_p(l)$. Finally, Secs. V and VI, and Appendixes A and B deal with the practical problem of obtaining numerical results.

II. THE EIGENVALUE PROBLEM FOR l=0

As mentioned earlier the heart of the problem is the explicit evaluation of the combinatorics functions $\sigma_l(n, p)$. In the case $l = 0$, it is possible to reduce the p summations of a product of p terms, as evidenced in Eq. (1.10a), to one single summation. This we are able to do due to a simplification occuring at $l = 0$, namely,

$$\gamma_0(i, k, p) = \frac{\Gamma(i + (p+1-k)/3)}{\Gamma(i + (p+3-k)/3)}. \tag{2.1}$$

A major step in this work is to provide an expression for $\sigma_0(n, p)$ making explicit its n-dependence so crucial for dealing effectively with the eigenvalue equation (1.11). The actual proof and details of the derivation are available elsewhere.[13] The result is[13,14]

$$\sigma_0(n,p) = \frac{3^p}{p!} \sum_{j=0}^{p} (-1)^j \binom{p}{j} \Gamma(1+j/3) \frac{\Gamma(n+1+p/3)}{\Gamma(n+1+j/3)} .$$
(2.2)

The large n behavior of $\sigma_0(n,p)$ can then be easily computed to be

$$\sigma_0(n,p) \underset{n\to\infty}{\sim} \frac{3^p}{p!} n^{p/3},$$
(2.3)

which obviously is a divergent quantity except for $p=0$. As will be shown in the next section, this behavior continues to hold for $l \neq 0$.

A. Closed from expression for K_n (0)

We are now in a position to evaluate the coefficients $K_n^{(k)}(0)$ and deduce closed form expressions for the expansion coefficients $K_n(0)$. In Eq. (1.11) we set $l'=j=0$, then multiply and divide the right hand side of this equation by the same quantity $(3r+i)!$. Using the Γ-function triplication formula for the factor $(3r+i)!$ appearing in the denominator and Eq. (2.2), one finds $2K_{3r+i}^{(k)}(0)$

$$= \frac{(-1)^r(1-\delta_{i0})\Gamma(\frac{2}{3})a_0}{3^{2r+2(i-1)/3}\Gamma(r+(i+1)/3)\Gamma(r+(i+2)/3)}$$

$$\times \sum_{q=0}^{r} \sum_{j=0}^{3q+2} \frac{(-1)^{q+j}\Gamma(k-q+r+(i+3)/3)\Gamma(1+j/3)}{\Gamma(r+(i+3)/3)\Gamma(k-q+1+j/3)}$$

$$\times \frac{k+1}{k+1-q+j/3} \left[\binom{3r+i}{3q}\binom{3q}{j} - \binom{3r+i}{3q+1}\binom{3q+1}{j} \right.$$

$$\left. + \binom{3r+i}{3q+2}\binom{3q+2}{j} \right] .$$
(2.4)

Using the property on the binomial coefficient

$$\binom{n}{j} = 0 \quad \text{for } j > n,$$
(2.5)

one can extend the summation over j in Eq. (2.4) up to $j=3r+i$, thus making it easy to exchange the order of the summations over q and j. The result is

$$2K_{3r+i}^{(k)}(0) = \frac{(-1)^r(1-\delta_{i0})\Gamma(\frac{2}{3})a_0}{3^{2r+2(i-1)/3}\Gamma(r+(i+1)/3)\Gamma(r+(i+2)/3)}$$

$$\times \sum_{j=0}^{3r+i} \frac{(-1)^j\Gamma(1+j/3)}{\Gamma(r+(i+3)/3)} \sum_{q=0}^{r} T_q(j)$$
(2.6a)

with $T_q(j)$ easily identified to be

$$T_q(j) = \frac{(-1)^q\Gamma(k-q+r+(i+3)/3)}{\Gamma(k-q+1+j/3)} \left[\frac{k+1}{k+1-q+j/3} \right.$$

$$\times \binom{3r+i}{3q}\binom{3q}{j} - \binom{3r+i}{3q+1}\binom{3q+1}{j}$$

$$\left. + \binom{3r+i}{3q+2}\binom{3q+2}{j} \right] .$$
(2.6b)

In the special case where $r=0$ and $i=0$, the double summation, over j and q, in Eq. (2.6a) contains one single term identically equal to one. Excluding this case for the time being, it is then possible to single out the contribution coming from $j=3r+i$ and write

$$\sum_{j=0}^{3r+i} \frac{(-1)^j\Gamma(1+j/3)}{\Gamma(r+(i+3)/3)} \sum_{q=0}^{r} T_q(j) = (-1)^{3r+i} \sum_{q=0}^{r} T_q(3r+i)$$

$$+ \sum_{j=0}^{3r+i-1} \frac{(-1)^j\Gamma(1+j/3)}{\Gamma(r+(i+3)/3)} \sum_{q=0}^{r} T_q(j).$$
(2.7)

Setting $j=3r+i$ in Eq. (2.6b), it is easy to show that

$$T_q(3r+i) = (-1)^{r+i}\delta_{qr}.$$
(2.8)

Combining Eqs. (2.7) and (2.8), one finds

$$\sum_{j=0}^{3r+i} \frac{(-1)^j\Gamma(1+j/3)}{\Gamma(r+(i+3)/3)} \sum_{q=0}^{r} T_q(j)$$

$$= 1 + \sum_{j=0}^{3r+i-1} \frac{(-1)^j\Gamma(1+j/3)}{\Gamma(r+(i+3)/3)} \sum_{q=0}^{r} T_q(j).$$
(2.9)

So far Eq. (2.9) is meaningful for both $r \neq 0$ and $i \neq 0$. After a long and tedious checking, it is possible to show that

$$\sum_{q=0}^{r} T_q(j) = 0 \quad \text{for } 0 \le j \le 3r+i-1.$$
(2.10)

Among other things, the proof is based on a binomial property that will be used in Sec. III, namely

$$\sum_{q=0}^{n} (-1)^q \binom{n}{q}\binom{q}{j} = 0 \quad \text{for } 0 \le j \le n-1.$$
(2.11)

A more appropriate version of the above property is

$$\sum_{q=0}^{r} (-1)^q \left[\binom{3r+i}{3q}\binom{3q}{j} \right.$$

$$- \binom{3r+i}{3q+1}\binom{3q+1}{j} + \binom{3r+i}{3q+2}\binom{3q+2}{j} \right] = 0$$

for $i=0,1,2$ and $0 \le j \le 3r+i-1$
(2.12)

Finally, combining Eqs. (2.6), (2.9), and (2.10), one obtains

$$2K_{3r+i}^{(k)}(0)$$

$$= \frac{(-1)^r(1-\delta_{i0})\Gamma(\frac{2}{3})a_0}{3^{2r+2(i-1)/3}\Gamma(r+(i+1)/3)\Gamma(r+(i+2)/3)}$$

for $r \ge 0$ and $i=0,1,2$.
(2.13)

This not only shows the explicit k independence of the coefficients $K_n^{(k)}(0)$ (as claimed in Ref. 3) but also gives a direct relationship to the Airy expansion coefficient[3,6], a_{n-1}, namely

$$2K_n^{(k)}(0) = 2K_n(0) = (-1)^{n-1}a_{n-1}, \quad n \ge 0,$$
(2.14a)

with the initial condition

$$a_{-1} = 0.$$
(2.14b)

B. The function $H_0(t)$

According to Eq. (2.31), the eigenvalue equation, Eq. (1.8), for $l=0$ becomes

$$H_0(t) = \sum_{n=0}^{\infty} 2K_n(0)t^{n-1} = \sum_{n=0}^{\infty} a_{n-1}(-t)^{n-1}$$

$$= \sum_{n=0}^{\infty} a_n(-t)^n = 0. \qquad (2.15)$$

Thus, according to Eqs. (1.6) and (1.7b), this completes the proof that

$$H_0(t) \equiv \mathrm{Ai}(-t). \qquad (2.16)$$

It also shows the complete equivalence for $l = 0$ between the solution of the linear potential problem in terms of the combinatorics functions in one hand, and the conventional solution in terms of the tabulated special functions on the other hand. While the known special functions provide a solution for $l = 0$, no solution in terms of them is available for $l \neq 0$. The new special function that has to be studied is $H_l(t)$, and we now proceed into a systematic study of this function for $l \neq 0$. As will become apparent, three separate cases arise naturally, $l = 3l'$, $l = 3l' + 1$, and $l = 3l' + 2$, where $l' = 0, 1, \ldots, \infty$.

III. CANCELLATION OF INFINITIES: THE GENERAL CASE

In Sec. II we were able to show the k independence of

the coefficients $K_n^{(k)}(l)$ for $l = 0$, thus trivially solving the problem of cancellation of infinities. In general, $K_n^{(k)}(l)$ are k-dependent, and a first step toward a practical evaluation of these coefficients in the limit as $k \to \infty$ is to show the cancellation of their leading parts. The discussion for $l = 0$ will be our guide.

Let $f_p(l)$ be the finite part of $\sigma_l(n, p)$ in the limit as $n \to \infty$. In particular, we have

$$f_0(l) = \sigma_l(n, 0) = 1 \qquad (3.1)$$

and, for $l = 0$,

$$f_p(0) = (-1)^p (3^p / p!) \Gamma(1 + p/3). \qquad (3.2)$$

The above result follows from Eq. (2.2). It can also be shown that the leading terms in the asymptotic expansion of $\sigma_l(n, p)$ are[13]

$$\sigma_l(n, p) \underset{n \to \infty}{\sim} \sum_{j=0}^{2} \frac{3^{p-j}}{(p-j)!} f_j(l) n^{(p-j)/3} + O(n^{p/3-1}). \qquad (3.3)$$

Equation (3.3) holds for $p \geqslant 0$.

We now go back to the general expression of $K_n^{(k)}(l)$ as given by Eq. (1.11), where k is assumed to be very large, so that the σ's can be replaced by their asymp-

totic expansion, Eq. (3.3). One finds

$$K_{3r+i}^{(k)}(3l' + j) = (1 - \delta_{i0}\delta_{j0} - \delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1})(1 - 2\delta_{i2}\delta_{j2}) \frac{(-1)^{j+r}\Gamma(\frac{2}{3})a_0}{3^{r+2i+2(i+j)/3} \cdot \Gamma(\frac{1}{3})\Gamma(2l' + (2j+5)/3)(3r+i)!}$$

$$\times \sum_{q=0}^{r} (-1)^q \left\{ \frac{\Gamma(k+\frac{4}{3})\Gamma(k+2l'+(2j+5)/3)\Gamma(k+r-q+l'+1+(i+j)/3)}{\Gamma(k+l'+(j+3)/3)\Gamma(k+l'+(j+4)/3)\Gamma(k+l'+(j+5)/3)} \left[\sum_{p=0}^{2} \binom{3r+i}{3q} \binom{3q}{p} p! f_p k^{q-p/3} + O(k^{q-1}) \right] \right.$$

$$- \frac{\Gamma(k+\frac{5}{3})\Gamma(k+2l'+(2j+6)/3)\Gamma(k+r-q+l'+1+(1+j)/3)}{\Gamma(k+l'+(j+4)/3)\Gamma(k+l'(j+5)/3)\Gamma(k+l'+(j+6)/3)} \left[\sum_{p=0}^{2} \binom{3r+i}{3q+1} \binom{3q+1}{p} p! f_p k^{q-(p-1)/3} + O(k^{q-2/3}) \right]$$

$$\left. + \frac{\Gamma(k+2)\Gamma(k+2l'+(2j+7)/3)\Gamma(k+r-q+l'+1+(1+j)/3)}{\Gamma(k+l'+(j+5)/3)\Gamma(k+l'+(j+6)/3)\Gamma(k+l'+(j+7)/3)} \left[\sum_{p=0}^{2} \binom{3r+i}{3q+2} \binom{3q+2}{q} p! f_p k^{q-(p-2)/3} + O(k^{q-1/3}) \right] \right\} . \qquad (3.4)$$

Using the known asymptotic relation[6]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \underset{z \to \infty}{\sim} z^{a-b} \left[1 + \frac{(a-b)(a+b-1)}{2z} + O(z^{-2}) \right], \qquad (3.5)$$

keeping the leading terms and exchanging the p and q summations in Eq. (3.4), we see the appearance of the combination:

$$p! f_p(l) k^{r+(i-p)/3} \sum_{q=0}^{r} (-1)^q \left\{ \binom{3r+i}{3q}\binom{3q}{p} - \binom{3r+i}{3q+1}\binom{3q+1}{p} + \binom{3r+1}{3q+2}\binom{3q+2}{p} \right\} , \qquad (3.6)$$

which vanishes identically according to property (2.12). This gives a direct proof of the cancellation of the leading divergent terms. A more elaborate study of this problem can be done and would involve lower order terms in the asymptotic expansions (3.3) and (3.5). Nevertheless, the procedure would still be the same.

IV. THE EIGENVALUE PROBLEM FOR ALL VALUES OF l

The discussion of Sec. III is incomplete in many respects. Unfortunately, we do not have a more elegant proof to show that the limit as k goes to infinity of $K_n^{(k)}(l)$ exists. Leaving aside this mathematical refinement, one is confident that the limit as $k \to \infty$ of Eq. (1.11) can be taken by collecting the finite parts of its right-hand side. In order to minimize the number of steps leading to the final answer, we find it convenient to once again write Eq. (1.11) in a different manner, using the discussions of Secs. II and III as a guide. We multiply and divide the right-hand side of Eq. (1.11) by the same factor $(3r+i)!$, and replace the factor $(3r+i)!$ in the denominator by its equivalent expression using the Γ function triplication formula. Then we put in evidence the three binomials

$$\binom{3r+i}{3q}, \quad \binom{3r+i}{3q+1}, \quad \text{and} \quad \binom{3r+i}{3q+2},$$

so that the resulting expression is

$$2(l+1)K_{3r+1}^{(k)}(l) = \frac{(1-\delta_{j0}\delta_{i0}-\delta_{j1}\delta_{i2}-\delta_{j2}\delta_{i1})(1-2\delta_{i2}\delta_{j2})(-1)^{j+r}[\Gamma(\frac{2}{3})]^2 a_0}{3^{2(r+i')+2(i+j-1)/3}\Gamma(2l'+(2j+2)/3)\Gamma(r+(i+1)/3)\Gamma(r+(i+2)/3)\Gamma(r+(i+3)/3)}$$

$$\times \sum_{q=0}^{r} \left\{ \frac{\Gamma(k+\frac{4}{3})\Gamma(k+2l'+(2j+5)/3)\Gamma(k+r+l'+1+(i+j)/3)}{\Gamma(k+l'+(j+3)/3)\Gamma(k+l'+(j+4)/3)\Gamma(k+l'+(j+5)/3)} \binom{3r+i}{3q} \right.$$

$$\times \frac{(-1)^q(3q)!}{3^{3q}}\sigma_{3l'+j}(k-q+1,\ 3q) - \frac{\Gamma(k+\frac{5}{3})(k+2l'+(2j+6)/3)\Gamma(k+r-q+l'+1+(i+j)/3)}{\Gamma(k+l'+(j+4)/3)\Gamma(k+l'+(j+5)/3)\Gamma(k+l'+(j+6)/3)}$$

$$\times \binom{3r+i}{3q+1} \frac{(-1)^q(3q+1)!}{3^{3q+1}}\sigma_{3l'+j}(k-q,\ 3q+1)$$

$$\times \frac{\Gamma(k+2)\Gamma(k+2l'+(2j+7)/3)\Gamma(k+r-q+l'+1+(i+j)/3)}{\Gamma(k+l'+(j+5)/3)\Gamma(k+l'+(j+6)/3)\Gamma(k+l'+(j+7)/3)} \binom{3r+i}{3q+2} \frac{(-1)^q(3q+2)!}{3^{3q+2}}\sigma_{3l'+j}(k-q,3q+2)$$

for $l=3l'+j$, l', $r=0,1,2,3,\ldots,\infty$, i, $j=0,1,2$. 　　　　(4.1)

The next step is to take the limit as k goes to infinity in Eq. (4.1) by collecting the finite part of its right-hand side. Recalling that $f_p(l)$ is the finite part of the combinatorics function $\sigma_l(n,p)$ in the limit as n becomes infinitely large, one easily finds

$$2(l+1)K_{3r+i}(l)$$

$$= \frac{(1-\delta_{i0}\delta_{j0}-\delta_{i1}\delta_{j2}-\delta_{i2}\delta_{j1})(1-2\delta_{i2}\delta_{j2})(-1)^{j+r}[\Gamma(\frac{2}{3})]^2 a_0}{3^{2r+2i'+2(i+j-1)/3}\Gamma((2l+2)/3)\Gamma(r+(i+1)/3)\Gamma(r+(i+2)/3)\Gamma(r+(i+3)/3)} \frac{(-1)^{3r+i}(3r+i)!}{3^{r+i}}f_{3r+i}(l).$$ 　　(4.2)

A special case of the above general formula is, of course, the case $l=0$ (or $l'=j=0$). According to Eq. (3.2), we have

$$[(-1)^{3r+i}(3r+1)!/3^{3r+i}]f_{3r+i}(0) = \Gamma(r+(i+3)/3).$$ 　　(4.3)

Thus, combining Eqs. (4.2) and (4.3), we recover the results of Sec. II, namely, the one stated in Eq. (2.13) and leading to Eq. (2.15).

Finally, combining Eqs. (1.8) and (4.2), we have at hand an explicit expression for the special function $H_l(l)$

$$H_l(l) = \sum_{p=0}^{\infty} \frac{(1-\delta_{i0}\delta_{j0}-\delta_{i1}\delta_{j2}-\delta_{i2}\delta_{j1})(1-2\delta_{i2}\delta_{j2})(-1)^{i+j}\Gamma(\frac{2}{3})}{3^{2r+2i'+2(i+j-1)/3}\Gamma((2l+2)/3)\Gamma(\frac{1}{3})} f_p(l)t^{p-1}$$ 　　(4.4)

for $l=3l'+j$, $p=3r+i$, $r,l'=0,1,\ldots,\infty$, $i,j=0,1,2$.

There is no numerical difficulty in evaluating the zeroes of $H_l(l)$ giving the energy eigenvalues, provided we are able to calculate $f_p(l)$. The very obscure connection between the energy eigenvalues and the combinatorics functions as first made in Ref. 3 has been neatly simplified, and the second objective of this paper has been reached.

V. NUMERICAL RESULTS

The aim of this section is to show that energy eigenvalues can be calculated by combining Eqs. (1.8) and (4.2). This is done by deriving the explicit n dependence of the combinatorics functions $\sigma_l(n,p)$ for each p, thus leading to the evaluation of the finite part $f_p(l)$. For the time being we do not have a general formula showing the explicit p and l dependence of $f_p(l)$. Nevertheless,

we developed a method based on a very simple residue technique to evaluate $f_p(l)$ for each value of p and for l belonging to the series of values $3l'$ and $3l'+2$. This has been worked out in detail in Ref. 13. Here we will simply state the following facts

(i) The recursion relation, Eq. (1.12), satisfied by the σ's involves $\gamma_l(i,1,p)$ which, in turn, is given in general by Eq. (1.10b) as the ratio of four Γ functions.

(ii) Finite series whose general term is the ratio of more than two Γ functions are not known to have closed form expressions. Thus, the only formula available to us so far is Eq. (1.13).

(iii) The expression of $\gamma_l(i, 1, p)$ can be reduced into a form that involves two Γ functions only for $l = 3l'$ and $l = 3l' + 2$.

To illustrate the method, we will just sketch out the developments for $l = 3l'$. In this case, we have

$\gamma_{3l'}(i, 1, p)$

$$= \sum_{q=0}^{2l'} \frac{(-1)^q (2l')!}{q!(2l'-q)!} \frac{\Gamma(q+\frac{1}{3})}{\Gamma\frac{1}{3}} \frac{\Gamma(i+p/3)}{\Gamma(i+q+(p+2)/3)}. \quad (5.1)$$

The i-dependent term in Eq. (5.1) involves the ratio of two functions, and the sum rule (1.13) is again at work. In this manner we have been able to separate out the divergent part of $\sigma_{3l'}(n, p)$ for various values of p, thus isolating the contribution $f_p(3l')$. Once again, the interested reader will find all the computational details in Ref. 13. Some of the simplest closed form expressions are $(l = 3)$[13]

$$f_0(3) = 1, \quad (5.2a)$$

$$f_1(3) = -(58/45)\Gamma(\tfrac{1}{3}), \quad (5.2b)$$

$$f_2(3) = (738/175)\Gamma(\tfrac{2}{3}) - (32/75)[\Gamma(\tfrac{1}{3})]^2. \quad (5.2c)$$

It is interesting to compare the new special function $H_3(t)$ for example with the Airy function $H_0(t)$, Eq. (2.16), by looking at their respective series expansions,

$$H_0(t) = a_0 \left[1 + 3^{1/3} \frac{\Gamma(\tfrac{2}{3})}{\Gamma(\tfrac{1}{3})} t - \frac{t^3}{6} + \cdots \right], \quad (5.3)$$

$$H_3(t) = a_0 \left[\frac{29}{75} + 0.1956 \times \frac{\Gamma(\tfrac{2}{3})}{\Gamma(\tfrac{1}{3})} t - 0.0069 t^3 + \cdots \right]. \quad (5.4)$$

If one terminates the series (5.3) to the first three terms and calculates the roots of $H_0(t)$, one finds only one real root

$$t = 2.56, \quad (5.5)$$

which is in good agreement with the first zero of $Ai(-t)$, (2.34). We thus expect the same to happen for Eq. (5.4). Cutting the series (5.4) to the first three terms, then calculating the roots of $H_3(t)$, one also finds one real root

$$t = 5.55. \quad (5.6)$$

Again this is in good agreement with the result that can be obtained by solving the Schrödinger equation numerically, namely, $t = 5.05$.[15]

VI. CONCLUSION

The energy eigenvalue problem associated with the linear potential is related to the study of a new special function, $H_l(t)$, that has been shown explicitly to reduce

to the Airy function $Ai(-t)$ for $l = 0$. The very obscure relationship between $H_l(t)$ and the combinatorics function $\sigma_l(n, p)$ has been clarified allowing the development of a systematic method for a direct numerical study of this function. The result as stated by Eq. (4.2) would not have been obtained without a careful study of the case $l = 0$ done in Sec. II and the analysis of Sec. III dealing with the cancellation of infinities.

A lot more is needed to complete the study of $H_l(t)$. Although we have been able to calculate a few lower order coefficients $f_p(l)$, we have been unsuccessful in deriving a general expression for these coefficients making explicit their dependence on p and l. Nevertheless, this paper has the merit of showing for the first time how combinatorics functions can be used constructively in problems of immediate concern where other analytical methods have failed.

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[6]See, for example, Handbook of Mathematical Functions, edited by M. Abramowitz and I.A. Stegun (National Bureau of Standards, Washington, D.C., 1964).
[7]A.F. Antippa, private communication that leads to the development of Appendix C of Ref. 3.
[8]Appendix C of Ref. 3.
[9]Conjugate combinatorics functions were also used to obtain the solutions of inverted inhomogeneous finite-difference equations, see Ref. 2.
[10]Throughout this paper we use the simplifying notation, $K_n^{(k)}(l)$, for $K_n^{(3k+1)}(l)$ as used in Ref. 3.
[11]As apparent from Eq. (D16) of Ref. 3, $\sigma_l(n, p)$ is a normalized conjugate combinatorics function of the first kind $[\sigma_l(n, 0) = 1]$, representing the partitions of the interval $(1, 3n + p - 2)$ into $(n + p - 1)$ parts, $(n - 1)$ of them of length 3, and p of them of length 1.
[12]L.B.W. Jolley, Summation of Series (Dover, New York, 1961), p. 10.
[13]A.J. Phares, Villanova preprint (Oct. 1977).
[14]It is interesting to remark that for $p \neq 0$, Eq. (2.2) yields $\sigma_0(0, p) = 0$. On the other hand, for $p = 0$, we recover the normalization condition $\sigma_0(n, 0) = 1$. We encountered a similar situation in discussing the inhomogeneous Legendre polynomial recursion relation,[2] namely Eqs. (A45a) and (A45b) of Ref. 2. A deeper understanding of this situation can be reached by looking at the interval associated with the combinatorics $\sigma_l(n, p)$ and the number of parts available for its partition.[1,11]
[15]A.F. Antippa, A.E. Everett, and A.J. Phares, "Families of Regge Trajectories in Hadronic Physics," Nuovo Cimento A 43, 347 (1978), Table I.